A brief history of type theory (or, more honestly, implementing mathematical objects as sets)

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November 8, 2012 [notes revised later; this is the day the talk was given]

Abstract

Dr. Holmes will discuss the historical development of the simple type theory of sets as a foundation of mathematics, with side glances at the type system of the famous Principia Mathematica of Russell and Whitehead, at the history of the notion of ordered pair, and more generally at how mathematical ideas are implemented in type theory (or set theory generally) and at the end possibly explaining the famous (in narrow circles) open problem of the consistency of New Foundations.

Consumer Warning

That's the abstract I sent when I was first thinking about this talk. As I thought about it more, the topic mutated a bit.

This talk is really about the idea that mathematical objects are implemented as sets. The development of type theories is one of the strands in the history of this project in mathematics.

Mathematical objects are *implemented* as sets

though what is often said is that mathematical objects *are* sets. We will witness against the second, very usual way of putting it by pointing out that the same mathematical structures can be implemented in different ways. Which sets you take as the official implementation of some particular kind of mathematical object (say, natural numbers) is a matter of convention. But the fact that sets are flexible enough to implement any kind of mathematical structure that interests us is quite impressive.

What *is* a set?

A set is a collection determined by its elements. Finite sets are often written $\{a, b, c\}$ (for example), by listing their elements. Order does not matter and repeated items do not change the intended meaning.

The elements of the sets are *not* parts of the set. The set is not made by conglomerating its elements together. This is a common misunderstanding.

To see this it is enough to play with the notation. $\{x\}$ is not the same object as x: if a set were made up of its elements as parts, this would not make sense. If you don't believe this, look at $\{\{2, 3\}\}$: this is a set with one element, while its sole element is a set with two elements, so they are different.

Another way of seeing it is to notice that a relation of part to whole should be transitive. If a is part of b and b is part of c, then a is part of c. But notice that $2 \in \{2, 3\}$ and $\{2, 3\} \in \{\{2, 3\}\}$, but $2 \notin \{\{2, 3\}\}$.

Infinite sets are generally presented by giving a property we use to choose their members. For example the set of even numbers can be presented as the set of all natural numbers n such that there is a natural number m such that $2m = n$. In fancy notation, $\{n \in \mathbb{N} \mid (\exists m \in \mathbb{N} \mid 2m = n)\}$.

It is worth noting that the finite list set notation can also be presented in this way: $\{2, 3\} = \{x \mid x = 2 \vee x = 3\}$.

Sets do have parts – the parts of a set are its *subsets*. A is a subset of B ($A \subseteq B$) iff every element x of A is also in B: $(\forall x.x \in A \rightarrow x \in B)$.

An extended example

We begin by working through an extended example, the implementation of the real number line completely in terms of sets. It's a process that goes through several stages. [I made some changes in these definitions from the talk].

Initially, we are given the familiar counting numbers $1, 2, 3, \ldots$ and we will not make any attempt to explain them as sets. Yet.

Rational numbers > 1

We implement a rational number greater than 1 (which we call a large rational number, for convenience) as a two-element set $\{m, n\}$ where m and n are counting numbers: this represents the fraction $\frac{\mathtt{max}(m,n)}{\mathtt{min}(m,n)}$. In order to make sure that we have only one set representing each large rational number, we require that m and n be relatively prime, so that the fraction is in simplest form. So $\{3, 2\}$ represents a large rational, but $\{6, 4\}$ does not.

Notice that we can define the usual order relation on large rationals coded in this way, using the equivalence of $\frac{p}{q} < \frac{r}{s}$ and $ps < qr$: $\{p, q\} < \{r, s\}$ is defined as $\mathtt{max}(p, q)\mathtt{min}(r, s) < \mathtt{max}(r, s)\mathtt{max}(p, q)$.

Real numbers ≥ 1

The idea of our implementation of a real number $r \geq 1$ (which we will call a large real number for convenience) is that we will represent it as the set of all large rationals q with $q > r$. [I changed this definition from what I gave in the talk]

To put it this way is cheating, since we supposedly aren't familiar with real numbers yet. However, its not that difficult.

Our definition of a *large real number* is “a nonempty set R of large rational numbers such that for any large rationals p, q, if $p \in R$ and $q > p$ then $q \in R$ (R is upward closed), and also for any $p \in R$, there is $q < p$ which is in R (R has no smallest element)”. [again, definition changed]. A set of positive rationals of this form is the intersection of the set of rationals with an interval of the form (r, ∞) for some real number $r \geq 1$ (we use the least upper bound [actually greatest lower bound] property of the real numbers to show this), so this should give a coding for reals ≥ 1. We exclude intervals with a smallest element because this would give us two different sets coding each large rational considered as a real. Notice also that the large real 2 (for example) is the set of all large rationals greater than than the large rational 2 (coded by the set $\{1, 2\}$); it is not the same object as the large rational 2 itself.

Notice that the large real 1 is the set of all large rationals.

Real numbers

A real number will be coded by a set $\{n, R\}$ where n is a natural number and R is a large real (it is important that no natural number is a large real). The motivation is that this set codes the real number $R - n$. To ensure that we have unique codes, we require that either $n = 1$ or $R < 2$. (2 here means the large real 2, which is the set of all large rationals less than the large rational 2; the relation $\leq$ on large reals is just the subset relation).

Any real number r will be coded by $\{R, n\}$ where n is the smallest counting number such that $r + n \geq 1$ and R is the set coding $r + n$.

I'm relying on your knowledge about the systems of numbers we are trying to represent to convince you that these representations will work. It is also possible to define familiar relations and operations on these representations of numbers and prove that they have the properties that we expect.

A little history

This is similar in spirit to the first set-theoretical constructions of the real line. It is quite different in details; I actually created this particular implementation for this talk, with some malicious intent. There is an important mathematical idea that I am carefully avoiding here for simplicity.

Can anyone tell me what it is?

Ordered pairs

A very important mathematical construction is the *ordered pair*. The ordered pair (a, b) has a and b as components, but also has an intended order on them. $\{a, b\} = \{b, a\}$: the two element set is an unordered pair. But $(a, b) = (b, a)$ only if $a = b$.

Ordered pairs can be represented as sets. The representation that is usually used now is $(a, b) = \{\{a\}, \{a, b\}\}$. A quick way to show that this works as a definition of ordered pair is to show that there is a way to pick out the first component and the second component. The first component a is the unique object which belongs to all elements of $(a, b) = \{\{a\}, \{a, b\}\}$, and the second component b is the unique object which belongs to exactly one of these elements. To see that this is useful, observe that it works for pairs $(a, a) = \{\{a\}, \{a, a\}\} = \{\{a\}\}$ as well.

There are other ways to implement the pair. Historically, the first definition of a pair as a set was given by Wiener in 1914, defining (a, b) as $\{\{\{a\}, \emptyset\}, \{\{b\}\}\}$.

It is important to notice that once you have given a definition of (a, b) and shown that for any a and b you can construct (a, b) and from any pair (a, b) you can determine the first projection and the second projection, you should never need to look at the details of the definition of the pair again.

Things we avoided in our construction

It is usual to represent fractions $\frac{m}{n}$ by ordered pairs (m, n) in the construction above (which allows us to represent all positive rationals rather than the positive rationals greater than 1).

It is also usual to use the important mathematical device of *equivalence classes*. We made sure that we had a single object coding $\frac{m}{n}$ by requiring $\texttt{gcd}(m, n) = 1$: another approach (and the more usual one) is to define $(m, n) \sim (p, q)$ as holding when $mq = np$ and represent $\frac{m}{n}$ as $\{(p, q) \mid (p, q) \sim (m, n)\}$.

We are rather pleased with the economy of our approach.

Uses of the ordered pair

Once you have the ordered pair you can for example represent points in the plane as pairs of real numbers.

You can represent functions and relations as sets of ordered pairs: a familiar function like $y = x^2$ can be coded as the set of pairs of real numbers (x, x^2), which can be viewed as simply the familiar *graph* of this function.

So, what are we assuming about sets here?

If we are going to implement mathematical objects as sets and prove that these implementations have the correct properties, we must have some basic knowledge about sets!

The criterion of identity of two sets is simple. Two sets are the same if and only if they have the same elements.

For any sets A and B, $A = B$ if and only if for every x, x is an element of A if and only if x is an element of B.

In logical notation, $(\forall AB.\mathtt{set}(A) \wedge \mathtt{set}(B) \rightarrow (A = B \leftrightarrow (\forall x.x \in A \leftrightarrow x \in B)))$

This is called the axiom of extensionality.

The axiom of comprehension

The other basic axiom allows us to construct sets.

For any property we can express about objects, there is a set whose elements are exactly the objects with that property.

For any property $P(x)$ of objects x, there is a set A such that for any x, $x \in A \leftrightarrow P(x)$.

Even more formally, for any sentence ϕ of our language in which the variable A does not appear, $(\exists A.\mathtt{set}(A) \land (\forall x.x \in A \leftrightarrow \phi))$ is an axiom. The set A is generally written $\{x \mid \phi\}$.

Comprehension is sufficient for our constructions

All the sets we have constructed can be built using comprehension. It is useful to notice that $\emptyset$, the empty set, is $\{x \mid x \neq x\}$.

The unordered pair $\{a, b\}$ can be defined as $\{x \mid x = a \vee x = b\}$, the set of all x such that either $x = a$ or $x = b$.

The set $\mathbb{Q}_{\mathtt{large}}$ of large rationals can be defined as $\{\{m, n\} \mid m \in \mathbb{N} \wedge n \in \mathbb{N} \wedge m \neq n \wedge \mathtt{gcd}(m, n) = 1\}$, where $\mathbb{N}$ is the set of positive natural numbers.

And, to give a long example, the set $\mathbb{R}_{\mathtt{large}}$ of large reals that I defined above is $\{R \mid R \subseteq \mathbb{Q}_{\mathtt{large}} \wedge R \neq \emptyset \wedge (\forall p \in R.(\forall q > p.q \in R)) \wedge (\forall p \in R.(\exists q \in R.q < p))\}$

Comprehension (as we have stated it) is false! Russell's paradox

It took some time to notice that the apparently very nice scheme of set theory that I have just defined is nonsense. It is all the more surprising because the proof that it is nonsense is very short.

Consider $R = \{x \mid x \notin x\}$, the set of all sets which are not elements of themselves.

For any x, $x \in R$ iff $x \notin x$.

so in particular $R \in R$ iff $R \notin R$. Oops.

This was a considerable scandal. It's usually called a "paradox", which is rather portentous, as if it represents some essential problem with human reason. I prefer to simply call it a mistake.

An easy way of presenting the mistake is this: we don't really think of collections of completely arbitrary objects. We think instead of collections of objects of a particular kind. For example a graph of a real function is a set of points in the plane. Or a large real is a particular kind of set of large rationals.

A way of codifying this entirely in terms of sets which leads to a very standard treatment of set theory is to replace the axiom of comprehension, asserting that $\{x \mid \phi\}$ exists, with the more restricted axiom which asserts that $\{x \in A \mid \phi\}$ exists: the collection of all things with a given property which belong to a previously given set A exists.

This is called the axiom of separation, and it is a basic axiom in the system of set theory proposed by Zermelo in 1908, which is the direct ancestor of the system of set theory ZFC which is described in a chapter 0 in many modern mathematical texts.

The simple theory of types

I'll present a different solution to the same mistake, which is historically a bit older than Zermelo's set theory (that is, it was suggested by Bertrand Russell in 1903) but due to a technical problem which I may explain later was not actually presented until at least 1914, and possibly not really spelled out in the form I give until about 1930.

The underlying idea is as I said above that when we define a set, it is always a set of objects of a particular kind. In Zermelo's approach, the kinds of object are sets themselves. In the theory of types, the kinds are handled by the grammar of our language.

The kinds of object are indexed by the natural numbers, and called types. Type 0 is called the type of "individuals", about which we know nothing (we might like to assume that there are infinitely many of them). When type i has been defined, we define type $i+1$ as the type inhabited by collections of type i objects.

This is enforced grammatically by providing that each of our variables has a number associated with it (its type) and that a sentence $x = y$ is grammatical iff the type of x and the type of y are the same, while a sentence $x \in y$ is grammatical iff the type of y is the successor of the type of x.

The grammar doesn't really need numbers

The reference to numbers in the description of TST is not essential, just a matter of convenience. Philosophers have criticized the theory of types on the grounds that it supposedly presupposes knowledge of numbers in the type scheme: this just isn't true.

$\mathbf{x}$ is an individual variable. If y is an individual variable, so is y^*.

An individual variable is a variable. If y is a variable, so is y'.

If x and y are individual variables, $x = y$ and $x \in y'$ are well-formed sentences.

If $u = v$ is a well-formed sentence, so is $u' = v'$. If $u \in v$ is a well-formed sentence, so is $u' \in v'$.

A variable consisting of **x** followed by m stars and then n primes can be written x_m^n, where n indicates its type. The sentences generated by the grammar above are exactly the sentences allowed by our typing rules, and with no reference to numbers. But it's *much* more convenient to talk about the numbered types!

The axioms of the theory of types

are exactly the same as the axioms of our broken set theory above! The only difference is that we are restricted by our grammar in what sentences we write.

A sentence of the shape

$$(\forall AB.(A = B \leftrightarrow (\forall x.x \in A \leftrightarrow x \in B)))$$

is an axiom as long as it is grammatical: that is, if x is of type n, A and B must be of type $n+1$. We left out the explicit condition that A and B are sets because any object represented by a variable with type $n+1$ is a set of type n objects.

For any sentence ϕ of our language in which the variable A does not appear, $(\exists A.(\forall x.x \in A \leftrightarrow \phi))$ is an axiom. The set A is generally written $\{x \mid \phi\}$ (this notation being of type one higher than that of x; not only variables but any names must have types). Here the type of A must be one higher than the type of x and the sentence ϕ must satisfy the rules of our grammar.

For example, there is no way to make sense of $R = \{x \mid x \notin x\}$, because we would need x to be of a type which was its own successor, which does not happen.

Developing the natural numbers in the theory of types

Frege proposed a definition of the natural number n in general which might seem circular but turns out not to be. It works in the naive set theory originally proposed above (which breaks as we have seen) but it also works in the theory of types, though there is something odd about it which I will point out in due course.

The basic idea is to define n as the set of all sets with n elements. This certainly *sounds* circular, but watch me.

We define the number 0 as $\{\emptyset\}$ (the set of all sets with 0 elements!). We can define $\emptyset$ as $\{x \mid x \neq x\}$ – we should really call this something like $\emptyset^{i+1}$ where i is the type of the variable x. We can define $\{x\}$ as $\{y \mid y = x\}$: notice that if x is of type i, so is y, and $\{x\}$ must be assigned type $i + 1$. The empty set $\emptyset$ can be defined in each type $i+1$ and its singleton 0 is defined in each type $i+2$. We do not say that the empty sets or zeroes in different types are the same or different – because our grammar doesn't allow us to!

For any set A, define $A + 1$ as $\{a \cup \{x\} \mid a \in A \wedge x \notin a\}$. Notice that if x is of type i, a must be of type $i + 1$ and A must be of type $i + 2$ – and $A + 1$ is of the same type $i + 2$. $A + 1$ is the collection of all sets which can be obtained by adding one new element to an element of A.

So clearly $0 + 1$, which we will call 1, is the collection of all sets with one element, 1+1, which we will call 2, is the collection of all sets with exactly two elements, and so forth.

We define $\mathbb{N}$, the set of natural numbers, as $\{n \mid (\forall I.(0 \in I \wedge (\forall m \in I.m + 1 \in I)) \rightarrow n \in I)\}$. That is, n is a natural number iff it belongs to every inductive set.

If we replace 0 with 1 in the definition of $\mathbb{N}$, we get the set of counting numbers (positive natural numbers) which we used in our construction above.

Notice that each natural number can be constructed in each type $i + 2$, and the set of natural numbers in any type $i + 3$. The type $i+2$ number three for example is the type $i+2$ set of all type $i+1$ sets with exactly three type i elements.

Importing our other number constructions

If m and n are positive natural numbers of type i, we can construct the large rational $\{m, n\}$ of type $i+1$ as above. Large rationals of type $i+1$ can be collected into type $i+2$ sets, some of which will be large reals. A technical modification is needed for the definition of reals: we define a large real as a type $i+3$ unordered pair of a type $i+2$ large real and a double singleton $\{\{n\}\}$ of a type i positive natural number. We need to apply the singleton operation to shift the type of the natural number upward so that it can be paired with the large real.

If a and b are type i objects, the pair $(a, b) = \{\{a\}, \{a, b\}\}$ is of type $i+2$. Thus a pair of the reals just constructed would be constructible in type $i+5$ and the plane $\mathbb{R}^2$ containing these pairs would be constructed at type $i+6$ (which is also where the graph of the real function $y = x^2$ would be found).

A little history

I did promise history. So far I have given some flavor of why the theory of types is a medium in which we can develop implementations of mathematical concepts. Russell suggested basically the same scheme of types that I presented here in his *Principles of Mathematics* in 1903, but when he formally developed mathematics from logic in *Principia Mathematica* (with Whitehead) in 1910-1913 he used a much more complicated system of types. This is because he had a technical problem: he knew he had to represent functions and relations in mathematics, and he even knew that a function or relation was a set of ordered pairs, but he did not know how to implement an ordered pair as a set: this was discovered by Wiener in 1914. Zermelo had the same problem in his 1908 presentation of set theory.

So if you ever read the Principia of Russell and Whitehead, you will discover a weirdly complex system of types. There is a type of individuals 0. For any list of types $(\tau_1, \ldots, \tau_n)$ there is a type of n-ary relations between objects (taken in order) of the given types. If R were a type inhabited by real numbers (RR) would be the type of relations on real numbers. If $\mathcal{P}$ were the type of points on a plane, and $\mathcal{R}$ the type of real numbers, $(\mathcal{RPP})$ would be the type of the relation "d is the distance between P and Q", which we would be inclined to present as a set of ordered triples. Even worse, there is no notation for types in the Principia at all; later readers invented notation for the types.

Once the notation for the pair is introduced, there is no need for such a complex system. The simple linear hierarchy of individuals, sets of individuals, sets of sets of individuals and so forth is sufficient.

Contrast with Zermelo's set theory

In the set theory of Zermelo, there are no types. A sentence like $x \in x$ is perfectly grammatical.

The restriction of comprehension is that we do not assert the existence of a set of all things with a given property: if we are given a set A and a property $P(x)$, we can assert the existence of $\{x \in A \mid P(x)\}$, the set of all things in the set A which have the property $P(x)$. The bounding set A is playing a very similar role here, implementing the idea that when we build a set, we are not taking its elements from the whole universe but from a particular sort of object.

We need sets to work with: you do not get a set $\{x \in A \mid P(x)\}$ unless you already have a set A you got in some other way. In the system

of Zermelo, we are also told that we can build unordered pairs $\{a, b\}$,
power sets $\mathcal{P}(A) = \{B \mid B \subseteq A\}$ (the set of all subsets of A) and
union sets $\bigcup A = \{x \mid (\exists a \in A . x \in a)\}$, the set of all elements of elements of A.

If we define 0 as $\emptyset$ and define $A+1$ as $\{A\}$, we can define $\mathbb{N}$ as the intersection of all inductive sets in much the way we did above, though the resulting implementation of the natural numbers will look quite different. The existence of $\mathbb{N}$ is provided as an axiom.

The numbers in Zermelo set theory are $\emptyset, \{\emptyset\}, \{\{\emptyset\}\} \ldots$
This was Zermelo's definition. It is more usual now to use von Neumann's definition $A+1 = A \cup \{A\}$, which has the effect that $0 = \emptyset; 1 = \{0\}, 2 = \{0, 1\}, 3 = \{0, 1, 2\} \ldots$.

Zermelo set theory (and the more modern ZFC have the advantage that their language is unconstrained by the grammatical rules of typing.

They have the possible disadvantage that the list of set constructions looks rather arbitrary: the axioms of the theory of types are exactly the simple axioms of the original broken theory of sets, with a repair to our language. Historically, Zermelo set theory won out as the generally used foundation of mathematics, but type theory still has a niche of its own: more complex typed theories (with types inhabited by functions and relations as well as just sets) have applications in computer science, for example.

The non-paradox of Zermelo

What happens when we consider $R_A = \{x \in A \mid x \notin x\}$? We aren't prevented from talking about this as we are in the theory of types.

$x \in R_A$ iff $x \in A$ and $x \notin x$.

So $R_A \in R_A$ iff $R_A \in A \wedge R_A \notin R_A$.

Clearly if $R_A \in R_A$ we would have a contradiction.

Also if $R_A \in A \wedge R_A \notin R_A$ we would have a contradiction.

So the only way to avoid contradiction is for $R_A \notin A$ and $R_A \notin R_A$.

This shows that for every set A there is at least one set $R_A \notin A$: there cannot be a set which contains *every* object as an element. In Zermelo's theory, there can be no universal set.

New Foundations

And finally... Notice that every object we constructed we can actually build at every type above an arbitrarily chosen type i. If we are counting type i objects, we use type $i+2$ numbers; this same relativity applies to all structures in the theory of types. Further, whatever we can prove about a class of mathematical structures defined in type theory, we can prove about the analogously defined structures in each higher type.

The philosopher and logician W. v. O. Quine proposed a simplification of the theory of types in 1937. He suggested that perhaps the types are all the same. The theory he defined is called NF, for "New Foundations", a name taken from the title of the article he proposed it in.

New Foundations has no grammar distinctions. Any sentence you can write with equality and membership is grammatical. But the axioms of comprehension that are allowed are only those which would be grammatical in the theory of types with some assignment of types to the variables.

This avoids the Russell paradox and seems to avoid other known paradoxes, but it is not known whether this set theory is really safe to work with.

In this theory, the set 3 that we defined above turns out to be the actual set of *all* sets with three elements (so for example $\{1, 2, 3\} \in 3$) which is a bit dizzying... The set
$\{x \mid x = x\}$, which is the type $i + 1$ collection of all type i objects in the theory of types, is the actual universal set (it contains *everything* as an element) in New Foundations.

With that we will stop.