RESEARCH STATEMENT

ZACH TEITLER

My area of research is commutative algebra and algebraic geometry (MSC 13 and 14).

Together with a number of collaborators, I have worked on broad range of problems, often with a combinatorial or computational flavor. These projects have involved a variety of topics including secant varieties and Waring rank, multiplier ideals, computational experimentation, arrangements of points and hyperplanes, and more. My contributions have been recognized with the award of a Collaboration Grant for Mathematicians from the Simons Foundation (award #354574, 2015–2020, $35,000) which supports travel and visitors. I look forward to continuing this work, as well as pursuing future collaborations in new directions.

1. Waring rank

The *Waring rank* of a homogeneous form $F = F(x_1, \dots, x_n)$ of degree d is the least r such that $F = c_1\ell_1^d + \cdots + c_r\ell_r^d$ for some linear forms ℓ_i and scalars c_i. For example,

$$xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2,$$

so the rank $r(xy) \leq 2$; and $r(xy) \geq 2$ as xy is not equal to a square of a linear form.

Waring ranks and related questions have been studied since the mid-19th century (for example [76], see also [41, 51] and discussions of applications in [24, 56, 57]) but the last 6–7 years have seen explosive activity and rapid advances.

For example, Carlini, Catalisano, and Geramita have determined the Waring rank of a monomial [22]: $r(x_1^{a_1} \cdots x_n^{a_n}) = (a_2+1)\cdots(a_n+1)$, when $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ (see also [14]). It is remarkable that this was determined in 2011, after more than 150 years of study of Waring ranks. They also show that the Waring rank of a sum of monomials in separate variables is equal to the sum of the Waring ranks of the separate monomials. It is conjectured that, in general, $r(F(x)+G(y)) = r(F)+r(G)$; this remains open after decades of work, but building on the 2011 result for sums of monomials, progress has been made by Carlini–Catalisano–Chiantini [21] and Carlini–Catalisano–Chiantini–Geramita–Woo [18].

Another noteworthy line of inquiry is motivated by geometric complexity theory. We may regard ℓ^d as "simple" and $r(F)$ as a measure of complexity. Strassen showed that tensor rank is related to circuit complexity; Raz showed that a family of tensors with rank growing super-linearly would imply super-polynomial formula size complexity. A family as described by Raz would be of interest by analogy to the P vs. NP problem. This gives a reason to study lower bounds on ranks.

The primary tools for studying Waring rank arise from commutative algebra. For $F \in S = \mathbb{C}[x_1, \dots, x_n]$, the **dual ring** is $T = \mathbb{C}[\partial_1, \dots, \partial_n]$, acting on S by having each ∂_i act as $\partial/\partial x_i$; this action is the **apolarity pairing**, giving a perfect pairing $T_d \times S_d \to \mathbb{C}$ for each d (and $T_a \times S_d \to S_{d-a}$ for all d, a). Equivalently, the commutative algebra notion of **catalecticant map** of a given form $F \in S_d$ is the linear map $C_F^a : T_a \to S_{d-a}$, for $0 \leq a \leq d$, sending differential operators to their evaluations on F.

My contributions include the following.

1.1. **An upper bound for rank and generalized rank.** Let $X \subset \mathbb{P}^N$ be a nondegenerate projective variety. The X-rank of a point q is the least number of points on X linearly spanning a space containing q. This includes Waring rank as the case X is a Veronese variety. It also includes ordinary tensor rank (see for example [56]) as the case X is a Segre variety. Many other natural notions of rank can be expressed as X-rank for an appropriately chosen X.

Grigoriy Blekherman and I have given the following upper bound for X-rank, for any irreducible nondegenerate variety X over a closed field in any characteristic:

Theorem 1 ([7])**.** *Let r_g be the generic rank with respect to X, i.e., the rank of a general point. Then for all $q \in \mathbb{P}^N$, the X-rank of q is at most $2r_g$.*

This is a significant improvement for a number of previously studied cases, most notably Waring rank. A further improvement is shown in the case that X has some higher secant variety of codimension 1. We also give the following result for real rank.

Theorem 2 ([7])**.** *Let $X \subset \mathbb{RP}^N$ be a real nondegenerate projective variety such that the real points of X are dense in X. Let r_0 be the smallest typical real rank with respect to X and let r_g be the generic rank with respect to the complexification $X_{\mathbb{C}} = X \otimes \mathbb{C}$. Then $r_0 = r_g$ and for all $q \in \mathbb{RP}^N$, the X-rank of q is at most $2r_0$.*

Future work. I will study homogeneous forms and tensors which are believed to have high rank, in order to narrow down the range of possible values for the maximum rank.

Previous upper bounds for Waring rank are asymptotically not as good as the above, but they still raise intriguing questions. For example, the papers [5, 53, 2] actually give an upper bound for "open Waring rank," which is greater than ordinary Waring rank; I will investigate the value of open Waring rank for a general form.

1.2. **Power sum decompositions of monomials.** This is joint work with Weronika Buczyńska and Jarosław Buczyński.

Theorem 3 ([14])**.** *Let $F = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial, $0 < a_1 \leq \cdots \leq a_n$, and let $r = r(F) = (a_2+1)\cdots(a_n+1)$. Let $\ell_1, \ldots, \ell_r$ be linear forms and let $I \subset \mathbb{C}[y_1, \ldots, y_n]$ be the homogeneous defining ideal of the set of projective points $\{[\ell_1], \ldots, [\ell_r]\}$. Then $F = c_1\ell_1^d + \cdots + c_r\ell_r^d$ for some scalars c_i if and only if I is a complete intersection of type $(a_1+1, \ldots, a_n+1)$, generated by*

$$y_1^{a_1+1} - \phi_1 y_0^{a_0+1}, \ldots, y_n^{a_n+1} - \phi_n y_0^{a_0+1}$$

for some forms ϕ_i of degree $a_i - a_0$.

This theorem, together with some additional reductions, allows the computation of the dimension of the variety parametrizing power sum decompositions:

$$\Big\{\{[\ell_1], \ldots, [\ell_r]\} : F = c_1\ell_1^d + \cdots + c_r\ell_r^d, \text{some } c_i\Big\}. \tag{1}$$

In particular let $(\mathbb{C}^*)^n$ act by scaling the variables and let $T \subset (\mathbb{C}^*)^n$ be the subtorus that fixes F.

Corollary 4 ([14])**.** *The induced action of T on the variety in* (1) *is transitive if and only if $a_1 = \cdots = a_n$.*

Thus the minimum length power sum decomposition of a monomial is unique up to scaling variables if and only if the monomial is of the form $(x_1 \cdots x_n)^k$.

Future work. Carlini–Catalisano–Geramita, in addition to determining Waring ranks of monomials, showed that the Waring rank of a sum of pairwise coprime monomials (equivalently: monomials in independent variables) is equal to the sum of the Waring ranks of the separate monomials. (This is conjectured to hold for all sums of forms in independent variables, see [21, 18].) An obvious conjecture is that every Waring decomposition (= minimum length power sum decomposition) of a sum of pairwise coprime monomials must be given by a concatenation of separate Waring decompositions of the separate monomials. This is open, even for 2-term sums.

I will study the generalization of the above theorem to other forms with complete intersection apolar annihilating ideals, such as defining equations of reflection arrangements of hyperplanes.

1.3. **Sub-generality of ranks of monomials.** Not many homogeneous forms are known to have higher than generic rank; it has not even been shown that forms with higher than generic rank exist in all degrees and numbers of variables (although it would be very surprising to learn otherwise).

Following the determination by Carlini–Catalisano–Geramita of the ranks of monomials and sums of pairwise coprime monomials, it is natural to ask if any of them have higher than generic rank. And indeed, Carlini–Catalisano–Geramita observed that infinitely many monomials in three variables have higher than generic rank, but at most finitely many in n variables do so, for each $n \geq 4$. But they did not determine how many such monomials there were, or give any examples.

This was completed in joint work with three undergraduate students Erik Holmes, Paul Plummer, and Jeremy Siegert.[1]

Theorem 5 ([50]). *Every monomial in* 4 *or more variables has Waring rank strictly less than the generic Waring rank. Furthermore, in* 4 *or more variables, every homogeneous sum of pairwise coprime monomials has Waring rank strictly less than the generic Waring rank, with exactly three exceptions:* $x_1x_2^2+x_3x_4^2$ *has Waring rank* 6*, strictly greater than the generic Waring rank of forms of degree* 3 *in* 4 *variables, which is* 5*;* $x_1x_2^2+x_3^3+x_4^3$ *has Waring rank* 5 *and* $x_1x_2x_3+x_4^2$ *has Waring rank* 5*, equal to the generic Waring rank.*

I proved the statement on monomials and conjectured the statement on sums of monomials, which was then proved by the three undergraduates.

Future work. It is not known what is the maximum Waring rank attained by sums of pairwise coprime monomials (of a given degree, in a given number of variables); the result above is shown by giving an upper bound which is less than the generic Waring rank, but the upper bound is not actually attained. A number of examples found by the three undergraduate students show that the rank of such a sum is not maximized by making "greedy" choices, i.e., choosing each term to be the monomial of largest rank possible with the number of remaining variables.

This is a suitable question for a group of undergraduate students. Small cases should be amenable to computer exploration. It will be a good opportunity for new researchers to use a combination of computer programming and exhaustive search, pattern recognition, and looking for a proof idea.

[1] All three graduated from Boise State in 2014 and have started Ph.D. programs: Erik Holmes at University of Hawaii, Paul Plummer at University of Oklahoma, and Jeremy Siegert at George Washington University.

1.4. **A lower bound for ranks of invariant forms.** This is joint work with Harm Derksen. In the following statement, for any polynomial P, let $\operatorname{Diff}(P)$ be the vector spanned by the derivatives of P of all orders, including P itself (zeroth order derivative):

Theorem 6 ([30])**.** *Let G be a connected group with an irreducible representation V and let F be an invariant form on V. Then the Waring rank of F is bounded below by*

$$\dim \operatorname{Diff}(F) - \dim \operatorname{Diff}(\partial F/\partial x)$$

for any nonzero $x \in V$.

More generally, the main result of our paper gives a lower bound for simultaneous Waring rank of an invariant linear series of forms, and allows reducible representations (then, roughly speaking, $x \in V$ must simply be chosen not lying in any proper subrepresentations). The same result is shown to hold for arbitrary forms (or linear series), not necessarily invariant under any group action, when $x \in V$ is general. These bounds are actually lower bounds for cactus rank, which is the following modification of Waring rank. The cactus rank $cr(F)$ is the least r for which there exists a zero-dimensional scheme Z of length r such that F lies in the linear span of the degree d Veronese re-embedding of Z. Waring rank is the case when Z is also required to be reduced, $Z = \{[\ell_1], \dots, [\ell_r]\}$.

For example, the generic $n \times n$ determinant $\det_n = \det((x_{i,j})_{1 \le i,j \le n})$ is an invariant form under left and right multiplication by SL_n and we get $r(\det_n) \ge \binom{2n}{n} - \binom{2n-2}{n-1}$. This is currently the best known lower bound for rank of the determinant.

Future work. I will study apolarity and Waring ranks of forms invariant under finite groups, such as symmetric polynomials, and also forms invariant under disconnected groups, such as the generic permanent. The permanent in particular is related to a version of P versus NP in geometric complexity theory.

I will also study upper bounds for the rank of determinant, building on previous work of Derksen.

1.5. **Reflection multiarrangements.** Alex Woo and I determined the Waring rank of many defining equations of reflection multiarrangements, i.e., fundamental skew invariants of finite complex reflection groups.

Theorem 7 ([84])**.** *Let G be a finite complex reflection group on $\mathbb{C}^n$ with degrees $1 \le d_1 \le \cdots \le d_n$. We do not assume G acts essentially. Let f_G be the defining equation of the reflection arrangement of G (equivalently: the fundamental skew invariant of G, or the Jacobian of the fundamental invariants of G). Then the cactus rank $cr(f_G)$ is*

$$cr(f_G) = \frac{|G|}{d_n} = d_1 \cdots d_{n-1}$$

and the Waring rank $r(f_G)$ is bounded by

$$\frac{|G|}{d_n} \le r(f_G) \le \frac{|G|}{D},$$

where D is the greatest regular number of G. The upper bound for Waring rank is given by an explicit power sum decomposition.

Note that $d_n = D$ is a regular number for many reflection groups of interest, including all irreducible real reflection groups and many irreducible complex reflection groups. For example, the fundamental skew invariant of the symmetric group S_n is the classical Vandermonde determinant

$$f_{S_n} = \prod_{i<j} x_j - x_i,$$

and we obtain $r(f_{S_n}) = (n-1)!$.

Future work. I will study other complex reflection groups. Perhaps the simplest example of a complex reflection group whose greatest degree is not a regular number is the group $G = \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_n\mathbb{Z}$ with not all a_i equal. In this case the fundamental skew invariant is the monomial $x_1^{a_1} \cdots x_n^{a_n}$, whose rank was determined by Carlini–Catalisano–Geramita using commutative algebra. The same algebraic technique should be tried for other examples.

For instance, one may consider the reducible reflection group given by a product of symmetric groups $G = S_{a_1} \times \cdots \times S_{a_n}$ with not all a_i equal. In this case the fundamental skew invariant is a product of classical Vandermonde determinants in separate variables.

I will work with Stefan Tohaneanu and Alex Woo to describe the apolar annihilating ideals of general hyperplane arrangements, in particular the Hilbert functions, graded Betti numbers, and minimal generators of those ideals, and bound or determine the Waring ranks of such arrangements. It might be possible to also describe the minimal free resolution of the annihilating ideals but this is more speculative. At this point we are able to produce candidate generators of the ideal via Proposition 3.3 of [81] which relates apolarity of F to singularities of the hypersurface $V(F)$: our candidate generators correspond naturally to the 0-skeleton of the (projectivized) hyperplane arrangement.

1.6. **Apolarity and direct sum decompositions.** In joint work with Weronika Buczyńska, Jarosław Buczyński, and Johannes Kleppe, the theory of apolarity is used to study decompositions of a form F as a sum of forms depending on linearly independent sets of variables, possibly after a linear change of coordinates. For example, it is clear that $xy \neq G(x) + H(y)$, but $xy = G(\ell_1) + H(\ell_2)$, namely,

$$xy = \frac{1}{4}\ell_1^2 - \frac{1}{4}\ell_2^2$$

for $\ell_1 = x + y$, $\ell_2 = x - y$. In general, for $F = F(x_1, \ldots, x_n)$, we ask whether there is an expression of the form

$$F = G(\ell_1, \ldots, \ell_k) + H(\ell_{k+1}, \ldots, \ell_n)$$

with the ℓ_i linearly independent linear forms. Such an expression is called a **direct sum decomposition** of F. For another example, consider the generic determinant $\det_n = \det((x_{i,j})_{1 \le i,j \le n})$. Then $\det_2 = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$ is visibly decomposed as a sum of forms in separate variables. It is easy to see that for $n > 2$, $\det_n \neq G(x_{1,1}, \ldots) + H(\ldots, x_{n,n})$, but it is less obvious whether after a linear change of coordinates (in n^2 variables!) $\det_n$ is so decomposable. We find a negative answer:

Theorem 8 ([13])**.** *Let $F \in \mathbb{C}[x_1, \ldots, x_n]$ be a homogeneous form of degree d. Let $F^\perp \subset T = \mathbb{C}[\partial_1, \ldots, \partial_n]$ be the homogeneous ideal of differential operators D such that $DF = 0$.*

(1) *If F is decomposable as a direct sum then $F^\perp$ has a minimal generator of degree d.*
(2) *If $F^\perp$ has a minimal generator of degree d then F is a limit of forms which are decomposable as direct sums.*

(3) *Suppose that F cannot be written as a form in fewer variables, even after a linear change of coordinates. Then F is a limit of forms which are decomposable as direct sums if and only if $F^\perp$ has a minimal generator of degree d.*

In particular, due to Shafiei's proof of my conjecture that $(\det_n)^\perp$ is generated in degree 2 [70], while $\det_n$ has degree n, it follows that $\det_n$ is indecomposable as a direct sum for $n > 2$. Shafiei's further results regarding permanents, pfaffians, symmetric determinants, symmetric permanents, etc., have similar implications for indecomposability of those forms [70, 69].

Many more results are given in [13], such as a lower bound on the greatest degree of a minimal generator of $F^\perp$.

Theorem 9 ([13])**.** *Let F be a form of degree d in n variables and suppose that $F^\perp$ is generated in degrees less than or equal to δ. Then $d \leq (\delta - 1)n$.*

Future work. I will study generalizations to local rings (such as formal power series) in place of polynomial rings. I will also study forms that make the lower bound sharp, $d = (\delta - 1)n$.

1.7. **A lower bound for Waring rank.** This is joint work with J.M. Landsberg. We give a geometric lower bound for Waring rank, specifically a lower bound for $r(F)$ in terms of the singularities of the hypersurface defined by F.

Theorem 10 ([58])**.** *Let F be a form of degree d. Fix an integer $0 < a < d$. Let C_F^a be the ath catalecticant map of F. Suppose F cannot be written as a form in fewer variables, even after a linear change of coordinates (this is equivalent to the easily checked condition that C_F^1 be injective). Let $\Sigma_a(F)$ be the projective variety defined by the vanishing of the forms in the image of C_F^a, i.e., the set of a-th derivatives of F, so $\Sigma_a(F)$ is the variety of points at which F vanishes with multiplicity at least $a+1$. Then $r(F) > \operatorname{rank} C_F^a + \dim \Sigma_a(F)$.*

This is notable for being the only currently known lower bound for Waring rank which is not actually a lower bound for cactus rank.

Future work. Examples in [58] suggest approaches to search for improvements to the above bound, by incorporating more information about the singularities, beyond just the dimension of the singular locus.

1.8. **Geometric lower bounds for generalized rank.** I have generalized the lower bound for Waring rank (and cactus rank) discovered by Ranestad and Schreyer [67], and the lower bound for Waring rank discovered by Landsberg and myself [58], to lower bounds for ranks of linear series (simultaneous Waring rank), ranks of multihomogeneous forms, and ranks with respect to nondegenerate projective varieties X [81]. As a simple example, it is shown that there are linear forms ℓ_i and m_i, and scalars c_i, such that

$$x_1 \cdots x_a y_1 \cdots y_b = \sum_{i=1}^{r} c_i \ell_i(x)^a m_i(y)^b$$

if and only if $r \geq 2^{a+b-2}$.

Future work. The generalizations to X-rank are incomplete; a number of questions are posed in [81]. For example, for a form F in n variables and any $1 \leq k \leq n$ let $r_k(F)$ be the least number of terms in an expression for F as a sum of forms each depending on k or fewer variables (possibly after a linear change of coordinates; i.e., the extension of a form on a k-dimensional subspace), see [19, 20]. Waring rank is the case $k = 1$. If $0 < d_1 \leq \cdots \leq d_n$, then

$$(d_1 + 1) \cdots (d_{n-k} + 1) \leq r_k(x_1^{d_1} \cdots x_n^{d_n}) \leq (d_2 + 1) \cdots (d_{n-k+1} + 1)$$

and the right hand inequality is conjectured to be equality, see [81, Conjecture 5.18].

I will study and hopefully determine r_k of a monomial. This is very close to the determination of classical Waring rank of a monomial. The commutative algebra techniques used by Carlini–Catalisano–Geramita for that problem should be applicable here. In particular I developed an apolarity lemma for this setting (generalizing work of Carlini). (Note that there is no known analogue of the apolarity lemma in the confusingly similar situation of writing a kt-form as a sum of kth powers of t-forms, and it is a wide open question what is the rank of a monomial in that setting, see [23]. Having an apolarity lemma should make r_k(monomial) tractable.)

Many, many more questions are suggested in [81], including a number of questions that would be good projects for students.

1.9. **Example of forms of high rank.** This is joint work with Jarosław Buczyński.

It is an open question what is the greatest Waring rank for homogeneous forms in a given number n of variables and of a given degree d. It is known that the maximum rank is at least the generic rank (as found by Alexander–Hirschowitz) and at most twice the generic rank (by my joint work with Blekherman [7]). For a handful of cases the maximum rank is known and is known to be strictly between these bounds. For a few additional cases it is known that the maximum rank must be strictly greater than the generic rank. In particular, monomials in $n = 3$ variables have greater than the maximum rank (asymptotically 3/2 times the generic rank). However in almost all cases it is an open question whether there even exist forms of greater than generic rank.

We developed a lower bound for Waring rank [15] which is a slight improvement (by +1) of one of the bounds used by Carlini, et al. We used this bound and some ad hoc methods to give examples of forms of high rank, higher than previously known examples.

Theorem 11 ([15])**.** *In three variables and odd degrees there exist forms of strictly greater rank than monomials (the forms with the previously highest known ranks in three variables).*

In particular the maximum Waring rank of quintics ($d = 5$) in 3 variables is 10, completing work of De Paris who had shown that the maximum rank must be 9 or 10 [27].

Theorem 12 ([15])**.** *In four variables and odd degree there exist forms with strictly greater than generic rank.*

Future work. Even though it is expected that forms of strictly greater than generic rank exist in all degrees for $n \geq 3$ variables, it remains open for $n \geq 5$ and for $n = 4$ in even degrees. Beyond just showing that forms of high rank exist we would like to understand how many of them there are: the dimension and geometry of the locus of forms of higher than generic rank. The forms of high rank described above form positive-dimensional families (although, we did not compute the precise dimension of the families).

At the end of [15] we give a suggestion for improving the lower bound further. We are also interested in diffferent ways to construct families of forms with high rank.

1.10. **Product ranks of the 3×3 permanent and determinant.** This is joint work with Nathan Ilten. The 3×3 determinant can be written as a sum of 5 terms which each factor as products of linear forms. The usual expression as a sum of monomials uses 6 such terms; an expression with 5 terms was found by Derksen [29]. Similarly, the 3×3 permanent involves 6 monomials, but Glynn [43] found an expression using 4 terms (that factor as products of linear forms). The *product rank* of a form F is the least number of terms summing to F with each term factoring as a product of linear forms. Derksen and Glynn showed that the 3×3 determinant and permanent have product rank at most 5 and 4, respectively.

We showed that Derksen's and Glynn's expressions are minimal:

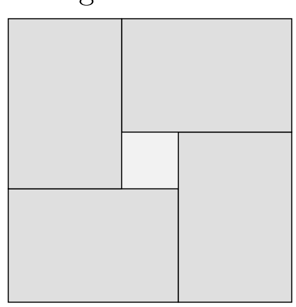
Theorem 13 ([52])**.** *The 3×3 permanent has product rank 4 and the 3×3 determinant has product rank 5. In fact the $n \times n$ permanent has border product rank at least $n+1$, meaning that the permanent of a generic $n \times n$ matrix cannot be written as a limit of any family of forms of product rank less than $n+1$.*

The border product rank of the determinant remains mysterious (but [38] gives a bound for the border Waring rank of the determinant).

This result rests on consideration of the Fano schemes of the permanent and determinant, i.e., the schemes parametrizing linear subspaces contained in the hypersurfaces defined by permanents and determinants. These have not been used before in any work on ranks (Waring or product). The introduction of this new technique should lead to further advances.

Future work. There is more to be learned from the study of Fano schemes related to product rank and Waring rank. It suggests a number of questions, for example, if X is an irreducible variety then when can the secant variety of X contain a plane, or a line other than a secant line to X? It opens the door to studying rank via tools other than commutative algebra and apolarity.

1.11. **Interlude.** The power sum decomposition $xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$ can be rewritten as $(x+y)^2 = 4xy + (x-y)^2$, and indeed, for $x > y > 0$, four $x \times y$ rectangles and an $(x-y) \times (x-y)$ square can be fitted together to form an $(x+y) \times (x+y)$ square:



Now $r(xyz) = 4$ and a power sum decomposition of length 4 is given by

$$xyz = \frac{1}{24}\Big((x+y+z)^3 - (x+y-z)^3 - (x-y+z)^3 - (-x+y+z)^3\Big),$$

which can be rearranged to

$$(x+y+z)^3 = 24xyz + (x+y-z)^3 + (x-y+z)^3 + (-x+y+z)^3.$$

So it is natural to ask: if $x, y, z > 0$ are the sides of a triangle, then can 24 $x \times y \times z$ "bricks", plus three cubes as indicated, fit together to form a cube of side $x + y + z$?

Following Igor Pak's suggestion, I obtained 24 bricks and 3 cubes of appropriate sizes created using a 3D printer, and engaged in experimental mathematics. I was able to show that the answer is *no* — for general x, y, z there is no such stacking. A simple counterexample is given by $(x, y, z) = (11, 13, 17)$. (For these values, I showed that each face of the big cube can only be covered by having exactly one small cube touching it, and the small cube must be centered on the face of the big cube; but there are 6 faces and only 3 small cubes, so it is impossible.)

2. Multiplier ideals

Multiplier ideals have been applied to a number of problems in algebraic geometry in recent years, most spectacularly in recent major advances in the minimal model program [45, 6] that built on earlier work showing the deformation invariance of plurigenera [72]. Other applications include several results on singularities and linear series [59], [36], a bound for symbolic powers [35], and applications to algebraic statistics [89], [90], [31, Chapter 5]. New applications of multiplier ideals continue to emerge in topics such as Chow stability [60] and singularities in generic liaison [66]. With broad and growing interest in multiplier ideals, it is increasingly valuable to compute examples.

For a thorough introduction to multiplier ideals see [59]. There are a number of equivalent characterizations of multiplier ideals, in terms of jet spaces [34], D-modules [17], test ideals for tight closure [73], and local integrability [54, 55, 64]. Here is a definition of multiplier ideals in terms of resolution of singularities. Suppose X is a smooth variety over a field $\Bbbk$ (we may assume X is affine, or even just $\Bbbk^n$, since we are primarily interested in local issues), $I \subset \mathcal{O}_X$ is a nonzero ideal sheaf, and $\mu : Y \to X$ is a log resolution of I, so that the total transform $I\mathcal{O}_Y$ defines a divisor F with simple normal crossings support, $F = \sum a_i E_i$, where the E_i are distinct reduced components of F. Then for each real number $c \geq 0$, the c'th multiplier ideal is defined by $\mathcal{J}(I^c) = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor c \cdot F \rfloor)$ where $K_{Y/X}$ is the relative canonical divisor of Y over X, defined locally by the vanishing of the determinant of the Jacobian $d\mu$, and $\lfloor c \cdot F \rfloor$ denotes the component-wise round-down of the $\mathbb{R}$-divisor $c \cdot F$, given by $\lfloor c \cdot F \rfloor = \sum \lfloor c a_i \rfloor E_i$.

My contributions include the following.

2.1. Multiplier ideals of hyperplane arrangements.

Mustață computed multiplier ideals of hyperplane arrangements using jet schemes [63]. I then gave a proof via resolution of singularities [78], using the De Concini–Procesi notion of wonderful models [26]. This argument allows one to simplify the formula for the multiplier ideals by eliminating a large number of redundant terms. My argument also remains in a finite-dimensional setting (unlike jet schemes, which are infinite-dimensional) and allows one to treat hyperplane arrangements with multiplicities.

Theorem 14 ([78])**.** *Let A be a hyperplane arrangement with nonnegative multiplicities in a vector space V and let $L(A)$ be the set of subspaces obtained as intersections of hyperplanes in A. For each $W \in L(A)$ let $r(A)$ be the codimension of W and let $s(W)$ be the sum of multiplicities of hyperplanes of A containing W. Let $\mathcal{G} \subset L(A)$ be any building set. Then* *the multiplier ideals of $I = I(A)$ are given by, for any $c \geq 0$,*

$$\mathcal{J}(I^c) = \bigcap_{W \in \mathcal{G}} I_W^{\lfloor c \cdot s(W) \rfloor - r(W) + 1},$$

where I_W is the ideal of W.

Mustaţă's result is the case $\mathcal{G} = L(A) \setminus \{V\}$, but smaller building sets exist, and simplify the formula by reducing the number of ideals being intersected. For example, for the braid arrangement $\mathcal{B}_n$ on $\mathbb{C}^n$, $|L(\mathcal{B}_n)|$ is the number of set partitions of $\{1, \ldots, n\}$, which is super-exponential; while $\mathcal{B}_n$ admits a building set (corresponding to so-called modular partitions) of size $2^n - n - 1$, i.e., exponential.

Future work. It remains unknown whether every member of a minimal building set is necessarily irredundant in the above formula, or whether a further reduction is possible.

2.2. **Multiplier ideals of line arrangements.** I computed the multiplier ideals of reduced unions of lines through the origin in $\mathbb{C}^3$ under certain hypotheses [85]. This is notable for being essentially the only computation of multiplier ideals carried out without special combinatorial or representation-theoretic structure. The computation was carried out using resolution of singularities. Thus, for example, for the ideal I_1 of three non-coplanar lines through the origin, $\mathcal{J}(I_1^c) = (1)$ for $0 \leq c < 3/2$, while for the ideal I_2 of three coplanar lines through the origin, $\mathcal{J}(I_2^c) = (1)$ for $0 \leq c < 5/3$. With my results, similar comparisons can be given in other examples: for instance, between 6 general lines through the origin and 6 lines through the origin lying on a quadratic cone.

Consideration of these line arrangements is motivated by the following example of Ein–Lazarsfeld–Smith [35]: If $Z \subset \mathbb{P}^2$ is a finite set of points, I is the homogeneous defining ideal of Z, $m > 0$ is a positive integer, and F is a homogeneous form vanishing to order at least $2m$ at each point of Z, then $F \in I^m$. That is, the symbolic power $I^{(2m)}$ is contained in the ordinary power: $I^{(2m)} \subseteq I^m$. Despite the elementary nature of the statement, the only known proof is the one given by Ein–Lazarsfeld–Smith, using the asymptotic multiplier ideals of the line arrangement corresponding to the points Z. So computing the multiplier ideals of line arrangements is a natural first step toward a deeper understanding of such containments of symbolic powers. See for example [10, 9] where these are studied intensively and elementary proofs are given for the case that Z is a general set of points; these Ein–Lazarsfeld–Smith and Bocci–Harbourne papers sparked a great deal of (ongoing!) activity, see for example [47, 32, 33, 44, 8].

Future work. For arrangements of lines satisfying a certain hypothesis, a key part of the work in [85] is to show that certain exceptional divisors arising in the resolution of singularities are redundant. Compare [74, 87] where this phenomenon is studied for singularities of curves on a surface. I will study multiplier ideals of arbitrary line arrangements with an eye toward determining which exceptional divisors are redundant or irredundant.

2.3. **Software for computing multiplier ideals.** In theory it is algorithmic to compute multiplier ideals by computing a resolution of singularities of I followed by a sheaf pushforward. In practice it is more difficult, see [39].

Shibuta's algorithm for computing Bernstein-Sato polynomials and multiplier ideals via Gröbner basis methods in Weyl algebras [71] (implemented by Shibuta in Risa/Asir) was refined and implemented in the Dmodules library for Macaulay2 by Berkesch and Leykin

[4]. The DMODULES library can compute multiplier ideals and jumping numbers of arbitrary ideals, but due to the difficulty of the computations, can only handle modestly sized examples.

I have developed a new software package named MULTIPLIERIDEALS, see [82], that computes multiplier ideals of special ideals including monomial ideals, ideals of monomial curves, generic determinantal ideals, and hyperplane arrangements via combinatorial methods, using the NORMALIZ software and interface to MACAULAY2 by Bruns, et al [11, 12]. The combinatorial methods allow computations of somewhat larger examples than can be handled by general methods.

The MULTIPLIERIDEALS package also computes *jumping numbers*, those values of c at which $\mathcal{J}(I^c)$ changes, and the *log canonical threshold* $\operatorname{lct}(I)$, the smallest positive jumping number.

The MULTIPLIERIDEALS package is available from my web site at `http://math.boisestate.edu/~zteitler/math/MultiplierIdealsSoftware.php`. It has been submitted to the Journal of Software for Algebra and Geometry; upon acceptance it will be distributed as part of MACAULAY2.

Future work. I will continue to add to the software as more algorithms for multiplier ideals are developed. I will also work on applications of this software to statistical computations and other applications, as described in [89].

2.4. **Monodromy of hyperplane arrangements.** In joint work with Nero Budur and Mircea Mustață [16], we show that the Monodromy Conjecture holds for hyperplane arrangements and reduce a stronger version of the conjecture to a conjecture on Bernstein-Sato polynomials of hyperplane arrangements. We prove the latter conjecture for a class of hyperplane arrangements including generic arrangements and, using multiplier ideals, for arrangements of "moderate type" (a certain monotonicity condition on multiplicities in the intersection lattice of the arrangement).

2.5. **Asymptotic multiplier ideals of monomial ideals.** The note [79], originally written as an appendix for lecture notes of Brian Harbourne [46], provides an exposition of asymptotic multiplier ideals and their application to the uniform bounds for symbolic powers developed by Ein–Lazarsfeld–Smith [35]. I show that a certain improvement of the Ein–Lazarsfeld–Smith bound found by Takagi–Yoshida [77] using characteristic p methods can also be obtained by the Ein–Lazarsfeld–Smith approach using asymptotic multiplier ideals:

Theorem 15 ([77])**.** *Let R be a regular local ring of equal characteristic 0, $I \subseteq R$ a reduced ideal, e be the greatest height of an associated prime of I, and ℓ an integer, $0 \leq \ell < \operatorname{lct}(I^{(\bullet)})$ where $\operatorname{lct}(I^{(\bullet)})$ is the log canonical threshold of the graded system of symbolic powers of I. Then $I^{(m)} \subseteq I^r$ whenever $m \geq er - \ell$. More generally, for any $k \geq 0$, $I^{(m)} \subseteq (I^{(k+1)})^r$ whenever $m \geq er + kr - \ell$.*

In addition I elaborate on an idea of Mustață [62] to compute asymptotic multiplier ideals of several families of graded sequences of monomial ideals, including in particular the sequence of symbolic powers of a reduced (squarefree) monomial ideal. These are among the very few nontrivial examples of asymptotic multiplier ideals of symbolic powers that have been computed to date. I also compute asymptotic multiplier ideals of graded systems of hyperplane arrangements.

3. Experiments at the frontiers of reality in Schubert calculus

I am a member of a team led by Frank Sottile to test a number of conjectures on reality in Schubert calculus by means of massive computational experiments.

Schubert calculus concerns the number of linear subspaces of $\mathbb{C}^n$ having intersections of specified dimensions with some given flags (sequences of nested subspaces). For example, given 4 general planes in $\mathbb{C}^4$, one finds that there are exactly 2 planes having at least 1-dimensional intersection with each — correspondingly, given 4 general lines in $\mathbb{P}^3$, there are exactly 2 lines meeting each given line. When the given flags are *real*, the solution subspaces are real or conjugate pairs. "Reality in Schubert calculus" refers to the phenomenon that, for any given list of intersection conditions (called Schubert conditions), there exists some configuration of real flags for which the solution subspaces are all real. This was shown by Vakil [88] building on earlier work by Sottile [75]. Boris and Michael Shapiro conjectured, astonishingly, that an exceedingly simple recipe would produce flag configurations giving real solutions to all Schubert problems: namely, configurations of osculating flags to a fixed real moment curve. Interest in this conjecture, leading eventually to its proof by Mukhin–Tarasov–Varchenko [61], was spurred by massive computational experiments by Sottile and others, see for example [68].

I joined Sottile's team which consisted of postdocs and graduate students working with Sottile. As a postdoc on the team, I mentored graduate students. We developed software to test variations of the Shapiro–Shapiro conjecture (Mukhin–Tarasov–Varchenko theorem), notably the *Secant Conjecture* which replaced osculating flags with secant flags spanned by points along the moment curve in disjoint intervals; the *Monotone Conjecture* which generalized the Shapiro–Shapiro conjecture to the setting of flag manifolds instead of Grassmannians; and the *Monotone Secant Conjecture*. These all remain open. In addition to providing overwhelming evidence for all three conjectures, our experiments explored new territory and uncovered new phenomena by including computations of secant flags along non-disjoint intervals, and non-monotone configurations.

The results of our experiments are reported in [40, 48]. The running of the experiments is described in [49]. All the data generated by the experiments is publicly accessible through the project web site, `http://www.math.tamu.edu/~secant/`.

The computations were carried out primarily using Singular and Maple, as well as Macaulay2, CoCoA, and Sage. We used Perl and batch scheduling software to automate the computation of billions of examples. Results were stored in a database using MySQL and displayed on dynamically created web pages using PHP. I was involved in the development of every part of the codebase.

4. Arrangement, combinatorial, and determinantal problems

My study of multiplier ideals of line arrangements led me to look into arrangements of points, hyperplanes, and more generally arrangements of linear subspaces. As points in the plane are defined by the maximal minors of a Hilbert–Burch matrix, I also became interested in certain questions about determinantal ideals. My activities in these areas include the following.

4.1. **Hilbert functions of fat point schemes in the plane.** This is joint work with Susan Cooper and Brian Harbourne [25]. Let A be a fat point scheme in the plane, $A = m_1P_1 + \cdots + m_nP_n$, and suppose we are given certain subsets $S_1, \ldots, S_k \subseteq \{P_1, \ldots, P_n\}$ such that for each i, there exists a line L_i containing the points in S_i and no other points of A; but we do not assume knowledge of the lines L_i, the positions of the points along each L_i, actual coordinates of the P_j, etc. We also do not assume that the given list of S_i is complete, i.e., there may be additional subsets of collinear points, but we are limited to just the given S_i.

From this limited information we describe a simple recursive reduction procedure as follows: at each step, choose a subset S_i, record the total of multiplicities of points in S_i, then decrease each of those multiplicities by 1, discarding any point whose multiplicity reaches zero. The output is a vector of nonnegative integers, the totals of multiplicities recorded at each step. This is called a *reduction vector*.

Also for any vector d of nonnegative integers we define functions $f_d, F_d : \mathbb{Z} \to \mathbb{Z}$, and show:

Theorem 16 ([25])**.** *If d is a reduction vector of A then the Hilbert function h_A satisfies $f_d \leq h_A \leq F_d$. Furthermore, if d contains no zero entries and is non-increasing, then $f_d = F_d$, unless d contains a subsequence of the form $(\dots, a, a, a, \dots)$ or $(\dots, a, a, a+1, a+2, \dots, a+k-1, a+k, a+k, \dots)$, that is, three equal entries, or consecutive entries with the first and last repeated.*

Conversely if $f_d = F_d$ then d satisfies the condition, i.e., avoids the indicated subsequences. Both f_d and F_d are defined recursively but are also given explicit combinatorial descriptions. Interestingly, "greedy" choices (i.e., choosing at each step a subset S_i to have the largest possible sum of multiplicities) do not necessarily result in the best possible bounds, as we show by example.

In the case d avoids the indicated subsequences, so that $f_d = F_d$ and h_A is uniquely determined, we also give upper and lower bounds for the graded Betti numbers of the ideal I_A, and we show that those bounds coincide precisely when the reduction vector d is strictly decreasing.

We compute the Hilbert functions and graded Betti numbers for a number of interesting examples, such as star configurations with multiplicities.

4.2. **Complete bipartite subspace arrangements.** This is joint work with Douglas A. Torrance [83]. We consider arrangements of codimension 2 subspaces about which one has only the following partial information: the simple graph with one vertex for each subspace and links whenever the subspaces have codimension 3 intersection. This graph does not typically determine the Castelnuovo–Mumford regularity of the (ideal of the) arrangement, or whether the arrangement is arithmetically Cohen–Macaulay. However we show that in case the graph is a complete bipartite graph of type (a, b) with $a \leq b \leq 2$ or $2 \leq a \leq b \leq 3$ or $3 \leq a \leq b$, the regularity of the ideal is uniquely determined, in fact equal to $\max(a+1, b)$; and the arrangement A is arithmetically Cohen–Macaulay if and only if $b = a$ or $b = a + 1$.

4.3. **Decompositions of ideals of minors meeting a submatrix.** This is joint work with Kent M. Neuerburg [65]. We give primary decompositions of certain ideals generated by subsets of minors or Pfaffians of a generic matrix. First, let $X = (x_{i,j})$ be a generic matrix. It is well known that the ideal $I_t(X)$ of t-minors of X is prime. Now consider the ideal generated by those t-minors that involve at least t_i columns in the first c_i columns of X, for each i, for some given values t_i, c_i; and one can similarly impose row conditions. These ideals arise when, for example, one considers replacement of the $x_{i,j}$ by homogeneous forms of degree $b_j - a_i$; then the low-degree part of the resulting determinantal ideal is described by these sorts of row and column conditions. We give an explicit primary decomposition of these determinantal ideals, using the theory of algebras with straightening law. This generalizes one of the results of [1] concering the ideal generated by minors that contain a submatrix, i.e., involve all c_i of the first c_i columns of X (and all the rows).

We find similar results for minors in a generic symmetric matrix and Pfaffians in a generic skew-symmetric matrix.

5. Other

Not discussed here are my papers on the following:

- on nef cone volumes of generalized Del Pezzo surfaces (joint work with Ulrich Derenthal and Michael Joyce) [28];
- a report on recent developments and open problems in linear series [3] (I contributed a section on bounds for symbolic powers);
- on arithmetic forms of toric varieties, i.e., with nonstandard structures over nonclosed fields (joint work with Javier Elizondo, Paulo Lima-Filho, and Frank Sottile) [37];
- generalizing Chris Hammond's topological criterion for schlichtness of the domain of holomorphy of a function [80];
- on intersections of curves through a set of points in the plane [86].

Research Impact: Two problems or conjectures that I suggested have led to papers by other researchers.

- A problem which I suggested to Dan Erman was used as the basis for a successful research program by a group of graduate students at the MSRI Summer Graduate Workshop in Commutative Algebra in summer 2011. The problem was to find the Boij-Söderberg decomposition for the Betti table of a complete intersection. The group of graduate students under Dan's direction found partial results toward this problem, leading to the recent article [42]. Understanding this (very) special case is a first step toward understanding more interesting cases, and suggests more questions for future investigation.
- I conjectured that the apolar ideals of the generic determinant and permanent are generated in degree 2. This was shown by a recent graduate student of A. Iarrobino, Masoumeh (Sepideh) Shafiei, who determined the apolar annihilating ideals of the generic determinant, permanent, Pfaffian, Hafnian, symmetric determinant, and more. These results were the basis of her Ph.D. dissertation and led to her recent article [70] and preprint [69].

Other problems: I am interested in studying the sequences of regularities that occur for exact sequences of graded modules, and the "Betti tensor" obtained by stacking the Betti tables of graded modules in an exact sequence, generalizing Boij-Söderberg theory.

I would like to study jet schemes of monomial ideals and subspace arrangements, building on work of Mustaţă, Goward–Smith, and C. Yuen.

I would like to extend Jeremy Marin's slope varieties and picture varieties to higher dimension and codimension and other settings such as tropical geometry.

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