Computational Problems Using Riemann Theta Functions in Sage

Chris Swierczewski

University of Washington
Department of Applied Mathematics
cswiercz@uw.edu

2011 AMS Western Fall Sectional
Salt Lake City, Utah

23 October 2011
1. Sage and the Riemann Theta Function
   - The Riemann Theta Function
   - Sage: Open-Source Mathematics Software
   - Implementation of Riemann Theta in Sage

2. Genus Three Solutions to the Kadomtsev–Petviashvili Equation
   - The Kadomtsev–Petviashvili Equation
   - Constructing Genus $g$ Solutions

3. Determinantal Representations
   - Definition and Spectrahedra
   - The Helton–Vinnikov Theorem
   - Implementation

4. (Optional) Genus 3 Algorithm and Implementation
1. Sage and the Riemann Theta Function
   - The Riemann Theta Function
   - Sage: Open-Source Mathematics Software
   - Implementation of Riemann Theta in Sage

2. Genus Three Solutions to the Kadomtsev–Petviashvili Equation
   - The Kadomtsev–Petviashvili Equation
   - Constructing Genus $g$ Solutions

3. Determinantal Representations
   - Definition and Spectrahedra
   - The Helton–Vinnikov Theorem
   - Implementation

4. (Optional) Genus 3 Algorithm and Implementation
Let $\mathcal{H}_g$ denote the space of “Riemann matrices”: set of all symmetric $\Omega \in \mathbb{C}^{g \times g}$ such that $\text{Im}(\Omega)$ is positive definite.

**Riemann Theta Function**

$$\theta : \mathbb{C}^g \times \mathcal{H}_g \rightarrow \mathbb{C}$$

$$\theta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i \left( \frac{1}{2} n \cdot \Omega n + z \cdot n \right)}$$

- Converges absolutely and uniformly on compact sets in $\mathbb{C}^g \times \mathcal{H}_g$.
- Quasiperiodic: integer period. Doubly exponential growth in the columns of $\Omega$. For $m_1, m_2 \in \mathbb{Z}^g$,

$$\theta(z + m_1 + \Omega m_2, \Omega) = e^{-2\pi i \left( \frac{1}{2} m_2 \cdot \Omega m_2 + z \cdot m_2 \right)} \theta(z, \Omega)$$
Slight generalization: Riemann theta functions with characteristic $[\alpha, \beta]$: let $\alpha, \beta \in [0, 1)^g$. Define

$$\theta[\alpha, \beta](z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i \left( \frac{1}{2} (n+\alpha) \cdot \Omega (n+\alpha) + (z+\beta) \cdot (n+\alpha) \right)}$$

 Addition formulas. One application: rewrite $\theta(z, \Omega)$ in terms of $\theta[\alpha, \beta](0, \Omega)$ for various $\alpha, \beta \in \{0, 1/2\}^g$.  

Chris Swierczewski (cswiercz@uw.edu)

Comp. Problems Using Riemann Theta Functions in Sage
A Special Way to Construct Riemann Matrices

Let \( f \in \mathbb{C}[x, y] \), possibly with singularities, with genus \( g \).

1. Desingularize, compactify and determine corresponding Riemann surface \( \Gamma \).
2. Determine basis for homology \( \{a_1, \ldots, a_g, b_1, \ldots, b_g\} \) and basis for cohomology \( \{\omega_1, \ldots, \omega_g\} \). (Basis of holomorphic differentials.)
3. Form the matrices \( A, B \) and \( \Omega \):
   \[
   \Omega = A^{-1}B \quad \text{where} \quad (A)_{ij} = \int_{a_j} \omega_i \quad \text{and} \quad (B)_{ij} = \int_{b_j} \omega_i.
   \]
4. Claim: \( \Omega \) is a Riemann matrix.
The Schottky Problem

Classifying which Riemann matrices come from algebraic curves:

- Dimension of $g \times g$ Riemann matrices:
  \[
  \frac{g(g+1)}{2}
  \]

- Dimension of $g \times g$ Riemann matrices derived from algebraic curves:
  \[
  3g - 3
  \]

This problem was solved by Shiota: $\Omega$ is a Riemann matrix derived from an algebraic curve if and only if

\[
u(x, y, t) = 2\partial_x^2 \log \theta(\bar{k}x + \bar{l}y + \bar{\omega}t, \Omega)
\]

is a solution to the Kadomtsev–Petviashvili Equation.
What is Sage?

Sage is a free open-source mathematics software system licensed under the GPL. It combines the power of many existing open-source packages into a common Python-based interface.

Mission: *Creating a viable free open-source alternative to Magma, Maple, Mathematica, and Matlab.*

Website: [http://www.sagemath.org](http://www.sagemath.org)
Goal: to provide functionality in Sage for working with Riemann theta functions.

1. Evaluating Riemann theta functions:
   - Given a Riemann matrix, $\Omega \in \mathbb{C}^{g \times g}$, and $z \in \mathbb{C}^g$; compute $\theta(z, \Omega)$ and its derivatives.
     - Arbitrary (user-specified) precision.
     - Sage implementation of the technique of Deconinck, Heil, Bobenko, van Hoeij, and Schmies [1].
   - 8-20 times faster than Maple’s implementation.
   - Code submission needs peer-review: Sage Trac Ticket #6371. (http://trac.sagemath.org/sage_trac/ticket/6371)
(2) Given an algebraic curve in $\mathbb{C}[x, y]$, compute a corresponding Riemann matrix. Required components:

- Puiseux series,
- integral basis of $\overline{\mathbb{C}[x]}$ in $\mathbb{C}(x, y)$,
- singularities of a plane algebraic curve: branching numbers, multiplicities, etc.,
- genus,
- monodromy,
- homology basis,
- cohomology basis,
- period matrix.
Implementation

**Advertisement:** If you have any students, this project is a good way for them to learn these concepts.

Bill Thurston:

“The standard of correctness and completeness necessary to get a computer program to work at all is a couple of orders of magnitude higher than the mathematical community’s standard of valid proofs.”
Table of Contents

1. Sage and the Riemann Theta Function
   - The Riemann Theta Function
   - Sage: Open-Source Mathematics Software
   - Implementation of Riemann Theta in Sage

2. Genus Three Solutions to the Kadomtsev–Petviashvili Equation
   - The Kadomtsev–Petviashvili Equation
   - Constructing Genus $g$ Solutions

3. Determinantal Representations
   - Definition and Spectrahedra
   - The Helton–Vinnikov Theorem
   - Implementation

4. (Optional) Genus 3 Algorithm and Implementation
The KdV–P Equation

Find a solution \( u(x, y, t) \) to the non-linear PDE

\[
\frac{3}{4} u_{yy} = \frac{\partial}{\partial x} \left( u_t - \frac{1}{4} \left( 6uu_x + u_{xxx} \right) \right).
\]

- Integrable: can be written as the compatibility condition of a Lax-pair.
- Describes 2D shallow water wave-propagation.
- 2D counterpart to the Korteweg – de Vries equation

\[
4u_t = 6uu_x + u_{xxx}.
\]
Periodic Solutions to KP

KP admits a large family of solutions of the form

\[ u(x, y, t) = 2\partial_x^2 \log \theta(z, \Omega) \]

(up to a constant shift) where \( \theta \) is the Riemann theta function, the phase variable \( z = (z_1, \ldots, z_g) \) is defined as

\[ z = \bar{k}x + \bar{l}y + \bar{\omega}t + \bar{\phi}, \quad \bar{k}, \bar{l}, \bar{\omega}, \bar{\phi} \in \mathbb{C}^g \]

and \( \Omega \in \mathbb{C}^{g \times g} \) is a Riemann matrix derived from a particular algebraic curve.

- so-called, “genus \( g \) solution to KP”
- physically, \( \bar{k} \) and \( \bar{l} \) are vectors of wave numbers, \( \bar{\omega} \) is a vector of frequencies, and \( \bar{\phi} \) is a phase shift
Example genus 1, 2, and 3 solutions to KP. (Courtesy Dubrovin, Flickinger, and Segur.)
Constructing Genus $g$ Solutions

**Theorem**

For each Riemann surface $\Gamma$ of genus $g$ and each point $Q \in \Gamma$ we can construct in a neighborhood of $Q$ a family of solutions to the KP equation. These are parameterized by the non-special divisors of degree $g$ on $\Gamma$.

(Very brief) Sketch of construction:

- Pick your favorite Riemann surface and a point $Q$ on that surface.
- Choose a non-special divisor of degree $g$ and construct a corresponding Baker-Akhiezer function $\psi$ from the polynomial, $q(k) = kx + k^2y + k^3t$, where $k^{-1}$ is a local parameter at $Q$.
- Integrate Abelian differentials of the second kind with double, triple, and quadruple poles at infinity to obtain $\bar{k}, \bar{l}$, and $\bar{\omega}$, respectively.
Genus $g = 3$ Solutions

When considering only genus $g \leq 3$ solutions the process for generating solutions is greatly simplified, as the Schottky problem is not an issue. Outline (in the interest of time):

- Substitute $2\partial_x^2 \log \theta(z, \Omega)$ into the KP equation to obtain DE in terms of $\theta$.

- Use Riemann theta addition formulas to rewrite in terms of theta with characteristics (and their derivatives) evaluated at $z = 0$.
  
  - 8 half-period characteristics $\rightarrow$ 8 equations.

- The coefficients of this system are polynomial functions of the components of $\vec{k}, \vec{l}$, and $\vec{\omega}$. Construct a linear system in terms of these coefficients and solve.

- (Time permitting, I will share the details at the end of the talk.)
Sage Implementation

This algorithm is implemented in Sage. It will be part of an “extended examples” documentation for RiemannTheta.

Key difference from DFS algorithm: computation of Riemann theta functions.
## Table of Contents

1. **Sage and the Riemann Theta Function**
   - The Riemann Theta Function
   - Sage: Open-Source Mathematics Software
   - Implementation of Riemann Theta in Sage

2. **Genus Three Solutions to the Kadomtsev–Petviashvili Equation**
   - The Kadomtsev–Petviashvili Equation
   - Constructing Genus $g$ Solutions

3. **Determinantal Representations**
   - Definition and Spectrahedra
   - The Helton–Vinnikov Theorem
   - Implementation

4. **(Optional) Genus 3 Algorithm and Implementation**
Writing homogenous polynomials as determinants of “Linear Matrix Representations”: “determinantal representations” of polynomials.

Theorem

Every homogenous polynomial in three variables can be written as

\[ f(x, y, z) = \det(Ax + By + Cz) \]

where \( A, B \) and \( C \) are symmetric matrices.

(Discussions with Daniel Plaumann, Bernd Sturmfels, and Cynthia Vinzant. Additional discussions with Rekha Thomas.)
Applications: Classifying Spectrahedra

Spectrahedron: the intersection of an affine subspace $K$ with the cone of positive semidefinite matrices $S^n_+$. Very loosely, an intersection of finitely many polynomial inequalities.

- Applications to semidefinite programming.

**Claim:** all two-dimensional spectrahedra are precisely the subsets of $\mathbb{R}^2$ bounded by rigidly convex algebraic curves, i.e. *Helton-Vinnikov curves.*
The algebraic curve has a maximal number of nested ovals. Namely, there are \( \lfloor d/2 \rfloor \) nested ovals where \( d = \deg f \). The innermost oval bounds a spectrahedron.

Plot in \( \mathbb{R}^2 \) of the curve \( f(x, y) = x^4 + x^2y^2 - 3x^2 + y^4 - 3y^2 + 2 \).
Theorem: (Helton–Vinnikov) Let $f \in \mathbb{R}[x, y, z]_d$ with $f(1, 0, 0) = 1$ and $\Gamma = \mathcal{V}_C(f) \subset \mathbb{P}^2$. Assume

1. $\Gamma$ is a non-rational (genus $> 0$) Helton–Vinnikov curve with the point $(1 : 0 : 0)$ inside its innermost oval.

2. The $d$ real intersection points of $\Gamma$ with the line $\{z = 0\}$ are distinct non-singular points $Q_1, \ldots, Q_d$ with coordinates $Q_i = (-\beta_j : 1 : 0)$ where $\beta_j \neq 0$.

Then, $f(x, y, z) = \det(I_{d \times d} x + B y + C z)$ where $B = \text{diag}(\beta_1, \ldots, \beta_d)$ and $C$ is real symmetric with diagonal entries

$$c_{ii} = \beta_i \frac{\partial z f(-\beta_i, 1, 0)}{\partial y f(-\beta_i, 1, 0)}$$

and off-diagonal entries of $C$ are...
\[ c_{jk} = \frac{\beta_k - \beta_j}{\theta[\delta](0)} \frac{\theta[\delta](A(Q_k) - A(Q_j))}{\theta[\epsilon](A(Q_k) - A(Q_j))} \sqrt{\frac{\omega \cdot \nabla \theta[\epsilon](0)}{-d(z/y)}} (Q_j) \sqrt{\frac{\omega \cdot \nabla \theta[\epsilon](0)}{-d(z/y)}} (Q_k) \]

where \( \epsilon \) is an arbitrary odd theta characteristic, \( \delta \) is an even theta characteristic such that \( \theta[\delta](0) \neq 0 \), and \( A : \Gamma \to \text{Jac}(\Gamma) \) is the Abel-Jacobi map.
Plaumann wrote a Maple script to compute determinantal representations:

www.math.uni-konstanz.de/~plaumann/theta.html

Compute times for the matrix $C$: (1.6 Ghz dual-core processor, 4GB RAM)

- $d = 4, g = 3$: approx. 5 minutes (longer to plot than compute!)
- $d = 5, g = 6$: approx. 4 hours
- $d = 6, g = 10$: ???
Since the ability to compute Riemann matrices is needed, a Sage implementation is not yet available. Also needed are:

- calculation of the Abel-Jacobi map $A : \Gamma \rightarrow \text{Jac}(\Gamma)$,
- calculation of Fay's prime form $E(Q_i, Q_j)$.

Fast calculation of these two functions is a primary goal.
Acknowledgements

- My advisor, Bernard Deconinck, for all of his support and guidance.
- Bernd Sturmfels, Rekha Thomas, Cynthia Vinzant, and Daniel Plaumann for discussions and problems on bitangents and determinantal representations.
- Christian Klein and Harry Braden for advice and perspectives on the computational issues.
- William Stein for Sage implementation tips.
- National Science Foundation Grant No. DMS-0821725 for providing support for Sage development.
- American Mathematical Society for the travel grant.
Thank you
Table of Contents

1 Sage and the Riemann Theta Function
   - The Riemann Theta Function
   - Sage: Open-Source Mathematics Software
   - Implementation of Riemann Theta in Sage

2 Genus Three Solutions to the Kadomtsev–Petviashvili Equation
   - The Kadomtsev–Petviashvili Equation
   - Constructing Genus $g$ Solutions

3 Determinantal Representations
   - Definition and Spectrahedra
   - The Helton–Vinnikov Theorem
   - Implementation

4 (Optional) Genus 3 Algorithm and Implementation
Genus $g = 3$ Solutions: Algorithm

**Algorithm**

**Input:** a Riemann matrix $\Omega$ and $k_1, k_2, l_1$;

**Output:** $k_3, l_2, l_3, \bar{\omega}$ and the corresponding solution

$$u(x, y, t) = 2\partial_x^2 \log \theta(\bar{k}x + \bar{l}y + \bar{\omega}z, \Omega)$$

**(Step 1)** Choose arbitrary $k_1, k_2,$ and $l_1$. (Possible due to the Lie Symmetries of KP.)
Genus $g = 3$ Solutions: Algorithm

(Step 2) Construct the $7 \times 7$ matrix

$$
\begin{pmatrix}
\theta_{11}[m_1, 0] & \theta_{12}[m_1, 0] & \cdots & \theta_{33}[m_1, 0] & \theta[m_1, 0] \\
\theta_{11}[m_2, 0] & & \cdots & & \theta[m_2, 0] \\
\vdots & & & & \\
\theta_{11}[m_7, 0] & \cdots & & \theta[m_7, 0]
\end{pmatrix}
$$

where...
Genus $g = 3$ Solutions: Algorithm

- $[m_i, 0], m_i \in \{0, \frac{1}{2}\}^3$ are theta characteristics chosen such that the matrix is invertible,

- $\theta_{ij}[m, 0]$ is defined by

$$\theta_{ij}[m, 0] := \frac{\partial^2 \theta[m, 0](0, \Omega)}{\partial z_i \partial z_j},$$

- and $\partial^4_k \theta[m, 0]$ is defined by

$$\partial^4_k \theta[m, 0] := \sum_{1 \leq i, j, k, l \leq 3} k_i k_j k_k k_l \frac{\partial^4 \theta[m, 0](0, \Omega)}{\partial z_i \partial z_j \partial z_k \partial z_l}.$$
Genus $g = 3$ Solutions: Algorithm

(Step 3) Compute the inverse matrix

$$
\begin{pmatrix}
a_{11} & \cdots & a_{11} \\
a_{m_1} & \cdots & a_{m_7} \\
a_{12} & \cdots & a_{12} \\
\vdots & \cdots & \vdots \\
a_{33} & \cdots & a_{33} \\
a_{m_1} & \cdots & a_{m_7} \\
a_{m_1} & \cdots & a_{m_7}
\end{pmatrix}.
$$

(Step 4) Construct the following degree 4 and 6 polynomials:

$$Q_{ij}(k) := - \sum_{m \in \{m_1, \ldots, m_7\}} a_{m}^{ij} \partial_k^4 \theta[m, 0]$$

$$P_{ij}(k) := \frac{1}{3} \left[ k_i^2 Q_{jj}(k) - k_i k_j Q_{ij}(k) + k_j^2 Q_{ii}(k) \right]$$
(Step 5) Finally, find $k_3, l_2, l_3, \omega_1, \omega_2, \omega_3$ by computing the solutions to the following system of six equations

$$k_i l_j - k_j l_i = \sqrt{P_{ij}(k)}, \quad \text{for } 1 \leq i < j \leq 3$$

$$\omega_i = \frac{Q_{ii}(k) - 3l_i^2}{k_i}$$

for some choice of sign on the square roots. Every solution to this system gives rise to a solution to KP.