Distances between intuitionistic fuzzy sets

Eulalia Szmidt, Janusz Kacprzyk

Systems Research Institute, Polish Academy of Sciences, ul. Newelska 6, 01-447 Warsaw, Poland

Received May 1997; received in revised form June 1998

Abstract

A geometrical representation of an intuitionistic fuzzy set is a point of departure for our proposal of distances between intuitionistic fuzzy sets. New definitions are introduced and compared with the approach used for fuzzy sets. It is shown that all three parameters describing intuitionistic fuzzy sets should be taken into account while calculating those distances. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Fuzzy set; Intuitionistic fuzzy set; Distance between intuitionistic fuzzy sets

1. Introduction

In this paper we propose some new definitions of distances between intuitionistic fuzzy sets proposed by Atanassov [1–5]. These are basically meant to make it possible to reflect the fact that it may not always be certain that the degree of nonmembership of an element in a (intuitionistic) fuzzy set is just equal to 1 minus the degree of membership, but there may be some hesitation degree. By taking into account the three parameter characterization of intuitionistic fuzzy sets, and following the basic line of reasoning on which the definition of distances between fuzzy sets is based, we define the four basic distances between the intuitionistic fuzzy sets: the Hamming distance, the normalized Hamming distance, the Euclidean distance, and the normalized Euclidean distance. While deriving these distances a convenient geometric interpretation of intuitionistic fuzzy sets is employed. It is shown that the definitions proposed are consistent with their counterparts traditionally used for fuzzy sets.

2. Intuitionistic fuzzy sets – a geometrical interpretation, and comparison with fuzzy sets

Let us start with a short review of basic concepts related to intuitionistic fuzzy sets.

Definition (Zadeh [13]). A fuzzy set $A'$ in $X = \{x\}$ is given by [12]

$$A' = \{(x, \mu_{A'}(x)) \mid x \in X\},$$

(1)
where $\mu_{A'}: X \rightarrow [0, 1]$ is the membership function of the fuzzy set $A'$; $\mu_{A'}(x) \in [0, 1]$ is the membership of $x \in X$ in $A'$.

**Definition** (Atanassov [1]). An intuitionistic fuzzy set $A$ in $X$ is given by

$$A = \{(x, \mu_A(x), v_A(x)) \mid x \in X\},$$

where $\mu_A: X \rightarrow [0, 1], \quad v_A: X \rightarrow [0, 1]$

with the condition

$$0 \leq \mu_A(x) + v_A(x) \leq 1 \quad \forall x \in X.$$

The numbers $\mu_A(x), v_A(x) \in [0, 1]$ denote the degree of membership and non-membership of $x$ to $A$, respectively.

Obviously, every fuzzy set $A'$ corresponds to the following intuitionistic fuzzy set:

$$A = \{(x, \mu_{A'}(x), 1 - \mu_{A'}(x)) \mid x \in X\}.$$  *(3)*

For each intuitionistic fuzzy set in $X$, we will call

$$\pi_A(x) = 1 - \mu_A(x) - v_A(x)$$

the *intuitionistic index* of $x$ in $A$. It is hesitancy degree of $x$ to $A$ [1–5].

It is obvious that

$$0 \leq \pi_A(x) \leq 1 \quad \text{for each} \quad x \in X.$$

For each fuzzy set $A'$ in $X$, evidently,

$$\pi_A(x) = 1 - \mu_A(x) - [1 - \mu_A(x)] = 0 \quad \text{for each} \quad x \in X.$$

One of the convenient geometrical interpretations of the intuitionistic fuzzy sets [5] is shown in Fig. 1. Atanassov [5] considers a universe $E$ and subset $F$ in the Euclidean plane with the Cartesian coordinates.

For a fixed intuitionistic fuzzy set $A$, a function $f_A$ from $E$ to $F$ can be constructed, such that if $x \in E$, then

$$p = f_A(x) \in F,$$

and the point $p \in F$ has the coordinates $\langle a', b' \rangle$ for which

$$0 \leq a', \quad b' \leq 1,$$

where

$$a' = \mu_A(x), \quad b' = v_A(x).$$

The above geometrical interpretation can be used as an example when considering a situation at the beginning of negotiations – cf. Fig. 2 (applications of intuitionistic fuzzy sets for group decision making, negotiations and other real situations are presented in [8–12]).

Each expert $i$ is represented as a point having coordinates $\langle \mu_i, v_i, \pi_i \rangle$. Expert A: $\langle 1, 0, 0 \rangle$ – fully accepts a discussed idea. Expert B: $\langle 0, 1, 0 \rangle$ – fully rejects it. The experts placed on the segment AB fixed their points
Fig. 1. A geometrical interpretation of an intuitionistic fuzzy set. Fig. 2. An orthogonal projection of the real (three-dimension) representation (triangle ABD in Fig. 3) of an intuitionistic fuzzy set.

Fig. 3. A three-dimension representation of an intuitionistic fuzzy set.

of view (their hesitation margins equal zero for segment AB, so each expert is convinced to the extent $\mu_i$, is against to the extent $\nu_i$, and $\mu_i + \nu_i = 1$; segment AB represents a fuzzy set). Expert C: $(0, 0, 1)$ is absolutely hesitant i.e. undecided – he or she is the most open to the influence of the arguments presented.

A line parallel to AB describes a set of experts with the same level of hesitancy. For example, in Fig. 2, two sets are presented with intuitionistic indices equal to $\pi_m$, and $\pi_n$, where: $\pi_n > \pi_m$.

In other words, Fig. 2 (the triangle ABC) is an orthogonal projection of the real situation (the triangle ABD) presented in Fig. 3.
An element of an intuitionistic fuzzy set has three coordinates \((\mu, \nu, \pi)\), cf. (4), hence the most natural representation of an intuitionistic fuzzy set is to draw a cube (with edge length equal to 1), and because of (4), the triangle ABD (Fig. 3) represents an intuitionistic fuzzy set. As before (Fig. 2), the triangle ABC is the orthogonal projection of ABD.

This representation of an intuitionistic fuzzy set (Fig. 3) will be a point of departure for considering the intuitionistic distances, and entropy of intuitionistic fuzzy sets.

3. Distances in fuzzy sets and intuitionistic fuzzy sets

In many theoretical and practical issues we face the following problem. Having two fuzzy sets in the same universe, we want to calculate a difference between them represented by a distance.

In this section we will first reconsider some better known distances for the fuzzy sets in an intuitionistic setting, and then extend those distances to the intuitionistic fuzzy sets.

3.1. Distances for fuzzy sets

The most widely used distances for fuzzy sets \(A, B\) in \(X = \{x_1, x_2, \ldots, x_n\}\) are [6]

- the Hamming distance \(d(A, B)\):
  \[
  d(A, B) = \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|,
  \]
  (5)

- the normalized Hamming distance \(l(A, B)\):
  \[
  l(A, B) = \frac{1}{n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|,
  \]
  (6)

- the Euclidean distance \(e(A, B)\):
  \[
  e(A, B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2},
  \]
  (7)

- the normalized Euclidean distance \(q(A, B)\):
  \[
  q(A, B) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2}.
  \]
  (8)

In all the above formulas (5)–(8), only the membership functions are present. This is due to the fact that for a fuzzy set, \(\mu(x_i) + \nu(x_i) = 1\).

As we may remember from Section 2, we can represent a fuzzy set \(A'\) in \(X\) in an equivalent intuitionistic-type representation (3) given as

\[
A = \{ (x, \mu_{A'}(x), 1 - \mu_{A'}(x)) \mid x \in X \}
\]

and we will employ such a representation while rewriting the distances (5)–(8).
So, first, taking into account an intuitionistic-type representation of a fuzzy set, we can express the very essence of the Hamming distance by putting

\[ d'(A, B) = \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|) \]

\[ = \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |1 - \mu_A(x_i) - 1 + \mu_B(x_i)|) \]

\[ = 2 \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| = 2d(A, B), \quad (9) \]

i.e. it is twice as large as the Hamming distance of a fuzzy set (5).

And similarly, the normalized Hamming distance \( l'(A, B) \) taking into account an intuitionistic-type representation of a fuzzy set, is in turn equal to

\[ l'(A, B) = \frac{1}{n} d'(A, B) = \frac{2}{n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|, \quad (10) \]

i.e. the result of (10) is two times multiplied as compared to (6).

Then, by the same line of reasoning, the Euclidean distance, taking into account an intuitionistic-type representation of a fuzzy set, is equal to

\[ e'(A, B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2} \]

\[ = \sqrt{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (1 - \mu_A(x_i) - 1 + \mu_B(x_i))^2} \]

\[ = \sqrt{2 \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2}, \quad (11) \]

i.e. it is multiplied by \( \sqrt{2} \) as compared to the Euclidean distance for the usual representation of fuzzy sets given by (7).

The normalized Euclidean distance \( q'(A, B) \), taking into account an intuitionistic-type representation of a fuzzy set, is then equal to

\[ q'(A, B) = \sqrt{\frac{1}{n} e'(A, B)} = \sqrt{\frac{2}{n} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2} \quad (12) \]

so again the result of (11) is multiplied by \( \sqrt{2} \) as compared to (8).

**Example 1.** For simplicity, let us consider “degenerated” fuzzy sets \( A, B, L, M, N \) in \( X = \{1\} \). A full description of each of them is \( A = (\mu_A, \nu_A)/1 \) exemplified by

\[ A = (1, 0)/1, \quad B = (0, 1)/1, \quad L = (1/3, 2/3)/1, \quad N = (2/3, 1/3)/1, \quad M = (1/2, 1/2)/1. \]

The geometrical interpretation (in the sense of Fig. 2) of these one-element fuzzy sets is shown in Fig. 4.
Let us calculate the Euclidean distances between the fuzzy sets using the “normal” representation by (7):

\[ e(L,N) = \sqrt{\left(\frac{1}{3} - \frac{2}{3}\right)^2} = \frac{1}{3}, \]
\[ e(L,M) = \sqrt{\left(\frac{1}{3} - \frac{1}{2}\right)^2} = \frac{1}{6}, \]
\[ e(N,M) = \sqrt{\left(\frac{2}{3} - \frac{1}{2}\right)^2} = \frac{1}{6}, \]
\[ e(L,A) = \sqrt{\left(\frac{1}{3} - 1\right)^2} = \frac{2}{3}, \]
\[ e(M,A) = \sqrt{\left(1 - \frac{1}{2}\right)^2} = \frac{1}{2}, \]
\[ e(B,M) = \sqrt{\left(0 - \frac{1}{2}\right)^2} = \frac{1}{2}, \]
\[ e(B,A) = \sqrt{1^2} = 1. \]

Now let us calculate the same Euclidean distances using the intuitionistic-type representation of fuzzy sets (11):

\[ e'(L,N) = \sqrt{\left(\frac{1}{3} - \frac{2}{3}\right)^2 + \left(\frac{2}{3} - \frac{1}{3}\right)^2} = \frac{\sqrt{2}}{3}, \]
\[ e'(L,M) = \sqrt{\left(\frac{1}{3} - \frac{1}{2}\right)^2 + \left(\frac{2}{3} - \frac{1}{2}\right)^2} = \frac{\sqrt{2}}{6}, \]
\[ e'(N,M) = \sqrt{\left(\frac{2}{3} - \frac{1}{2}\right)^2 + \left(\frac{1}{3} - \frac{1}{2}\right)^2} = \frac{\sqrt{2}}{6}. \]
Example 2. Let us consider two fuzzy sets $A, B$ in $X = \{1, 2, 3, 4, 5, 6, 7\}$. The intuitionistic-type representation of them is $A = (\mu_A, \nu_A)/1$, exemplified by

\[
A = (0.7, 0.3)/1 + (0.2, 0.8)/2 + (0.6, 0.4)/4 + (0.5, 0.5)/5 + (1, 0)/6,
\]

\[
B = (0.2, 0.8)/1 + (0.6, 0.4)/4 + (0.8, 0.2)/5 + (1, 0)/7.
\]

Let us calculate the Hamming distance $d(A, B)$ taking into account the membership functions (5) only, which yields

\[
d(A, B) = |0.7 - 0.2| + |0.2 - 0| + |0.6 - 0.6| + |0.5 - 0.8| + |1 - 0| + |0 - 1| = 3
\]

while the normalized distance (6) $l(A, B)$ is equal to

\[
l(A, B) = \frac{1}{7}d(A, B) = \frac{3}{7} = 0.43.
\]

When both the membership and non-membership functions are taken into account [cf. (9)], we have

\[
d'(A, B) = |0.7 - 0.2| + |0.3 - 0.8| + |0.2 - 0| + |0.8 - 1| + |0.6 - 0.6| + |0.4 - 0.4| + |0.5 - 0.8|
+ |0.5 - 0.2| + |1 - 0| + |0 - 1| + |0 - 1| + |1 - 0| = 6,
\]

i.e. two times the value obtained from (5), while the normalized distance (10) is equal to

\[
l'(A, B) = \frac{1}{n}d'(A, B) = \frac{6}{7} = 0.86.
\]

Let us compare the Euclidean distances calculated by (7) and (11). From (7) we have

\[
e(A, B) = \sqrt{(0.7 - 0.2)^2 + (0.2 - 0)^2 + (0.6 - 0.6)^2 + (0.5 - 0.8)^2 + (1 - 0.2)^2 + (0 - 1)^2} \\
= \sqrt{2.38} = 1.54,
\]

while the normalized Euclidean distance (8) is

\[
q(A, B) = \frac{1}{n}e(A, B) = \sqrt{\frac{2.38}{7}} = 0.58.
\]
From (11) we have the Euclidean distance, taking into account the intuitionistic-type representation of fuzzy sets, equal to

\[ e'(A, B) = ((0.7 - 0.2)^2 + (0.3 - 0.8)^2 + (0.2 - 0)^2 + (0.8 - 1)^2 \\
+ (0.6 - 0.6)^2 + (0.4 - 0.4)^2 + (0.5 - 0.8)^2 + (0.5 - 0.2)^2 + (1 - 0)^2 \\
+ (0 - 0)^2 + (0 - 1)^2 + (1 - 0)^2)^{1/2} = \sqrt{4.76} = 2.18 \]  

(35)

while the normalized Euclidean distance (12), taking into account the intuitionistic-type representation of fuzzy sets, is equal to

\[ q'(A, B) = \sqrt{\frac{4.76}{7}} = 0.83. \]  

(36)

Let us modify the fuzzy set \( B \) a little bit (make it closer to \( A \)). These fuzzy sets are now

\[ A = (0.7, 0.3)/1 + (0.2, 0.8)/2 + (0.6, 0.4)/4 + (0.5, 0.5)/5 + (1, 0)/6, \]  

\[ B = (0.2, 0.8)/1 + (0.6, 0.4)/4 + (0.8, 0.2)/5 + (0.4, 0.6)/6 + (1, 0)/7. \]  

(37)

(38)

The Hamming distance calculated by (5) is

\[ d(A, B) = |0.7 - 0.2| + |0.2 - 0| + |0.6 - 0.6| + |0.5 - 0.8| + |1 - 0.4| + |0 - 1| = 2.6 \]  

(39)

and the normalized Hamming distance (6) is

\[ l(A, B) = \frac{1}{4} d(A, B) = \frac{2.6}{4} = 0.37. \]  

(40)

From (9) we obtain the Hamming distance, taking into account the intuitionistic-type representation of fuzzy sets, equal to

\[ d'(A, B) = |0.7 - 0.2| + |0.3 - 0.8| + |0.2 - 0| + |0.8 - 1| + |0.6 - 0.6| + |0.4 - 0.4| + |0.5 - 0.8| \\
+ |0.5 - 0.2| + |1 - 0.4| + |0 - 0.6| + |0 - 1| + |1 - 0| = 5.2 \]  

(41)

while the normalized Hamming distance (10), taking into account the intuitionistic-type representation of fuzzy sets, equal to

\[ l'(A, B) = \frac{1}{4} 5.2 = 0.74. \]  

(42)

Now let us calculate the Euclidean distances. From (7) we have

\[ e(A, B) = \sqrt{(0.7 - 0.2)^2 + (0.2 - 0)^2 + (0.6 - 0.6)^2 + (0.5 - 0.8)^2 + (1 - 0.4)^2 + (0 - 1)^2} \]

\[ = \sqrt{1.74} = 1.32. \]  

(43)

while from (8) we obtain the normalized Euclidean distance

\[ q(A, B) = \sqrt{\frac{1.74}{7}} = 0.5. \]  

(44)

From (11) we have the Euclidean distance, taking into account the intuitionistic-type representation of fuzzy sets, equal to

\[ e'(A, B) = ((0.7 - 0.2)^2 + (0.3 - 0.8)^2 + (0.2 - 0)^2 + (0.8 - 1)^2 + (0.6 - 0.6)^2 + (0.4 - 0.4)^2 \\
+ (0.5 - 0.8)^2 + (0.5 - 0.2)^2 + (1 - 0.4)^2 + (0 - 0.6)^2 + (0 - 1)^2 + (1 - 0)^2)^{1/2} \]

\[ = \sqrt{3.48} = 1.87, \]  

(45)
while the normalized Euclidean distance (12), taking into account the intuitionistic-type representation of fuzzy sets, is equal to

\[ q'(A, B) = \sqrt{\frac{1}{2}3.48} = 0.705. \] 

(46)

The results obtained in Examples 1 and 2 (see also Fig. 4) show that:

- for the distances calculated between any fuzzy sets \( A \) and \( B \), when taking into account the membership functions (5)–(8) only, we have

\[
\begin{align*}
0 & \leq d(A, B) \leq n, \\
0 & \leq l(A, B) \leq 1, \\
0 & \leq e(A, B) \leq \sqrt{n}, \\
0 & \leq q(A, B) \leq 1,
\end{align*}
\]

(47)–(50)

- for the distances calculated for any fuzzy sets \( A \) and \( B \), taking into account the intuitionistic-type representation of fuzzy sets (9)–(12), we have

\[
\begin{align*}
0 & \leq d'(A, B) \leq 2n, \\
0 & \leq l'(A, B) \leq 2, \\
0 & \leq e'(A, B) \leq \sqrt{2n}, \\
0 & \leq q'(A, B) \leq \sqrt{2}.
\end{align*}
\]

(51)–(54)

It is worth noticing that it is not our purpose to introduce a new way of calculating distances for fuzzy sets. Conversely, we have shown that the intuitionistic-type representation of fuzzy sets results in multiplying the distances by constant values only. But similar reasoning in a case of intuitionistic fuzzy sets (i.e. omitting one of the three parameters) would lead to incorrect results. It is discussed in the next chapter.

3.2. Distances for intuitionistic fuzzy sets

We will now extend the concepts of distances presented in Section 3.1 to the case of intuitionistic fuzzy sets.

Following the line of reasoning presented in Section 3.1, for two intuitionistic fuzzy sets \( A \) and \( B \) in \( X = \{x_1, x_2, \ldots, x_n\} \) the Hamming distance is equal to

\[
d_{IFS}(A, B) = \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|).
\]

(55)

Taking into account that

\[
\begin{align*}
\pi_A(x_i) &= 1 - \mu_A(x_i) - \nu_A(x_i) \quad \text{and} \quad \pi_B(x_i) = 1 - \mu_B(x_i) - \nu_B(x_i)
\end{align*}
\]

we have

\[
|\pi_A(x_i) - \pi_B(x_i)| = |1 - \mu_A(x_i) - \nu_A(x_i) - 1 + \mu_B(x_i) + \nu_B(x_i)|
\]

\[
\leq |\mu_B(x_i) - \mu_A(x_i)| + |\nu_B(x_i) - \nu_A(x_i)|.
\]

(57)

Inequality (57) means that the third parameter in (55) cannot be omitted as it was in the case of fuzzy sets, for which taking into account the second parameter would only result in the multiplication by a constant value.
A similar situation occurs for the Euclidean distance. Namely, for intuitionistic fuzzy sets $A$ and $B$ in $X = \{x_1, x_2, \ldots, x_n\}$, by following the line of reasoning as in Section 3.1, their Euclidean distance is equal to

$$e_{IFS}(A, B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2}. \quad (58)$$

Let us verify the effect of omitting the third parameter ($\pi$) in (58). Taking into account (56), we have

$$(\pi_A(x_i) - \pi_B(x_i))^2 = (1 - \mu_A(x_i) - v_A(x_i) - 1 + \mu_B(x_i) + v_B(x_i))^2$$

$$= (\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2 + 2(\mu_A(x_i) - \mu_B(x_i))(v_A(x_i) - v_B(x_i)) \quad (59)$$

which means that taking into account the third parameter $\pi$, when calculating the Euclidean distance for intuitionistic fuzzy sets does have an influence on the final result. It is obvious because a two-dimensional geometrical interpretation (Fig. 2) is an orthogonal projection of a real situation presented in Fig. 3.

Having in mind (51)–(54), in order to be more concordant with the mathematical notion of normalization, the following distances for two intuitionistic fuzzy sets $A$ and $B$ in $X = \{x_1, x_2, \ldots, x_n\}$ are proposed

- the Hamming distance:

$$d_{IFS}^1(A, B) = \frac{1}{2} \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|); \quad (60)$$

- the Euclidean distance:

$$e_{IFS}^1(A, B) = \frac{1}{2} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2; \quad (61)$$

- the normalized Hamming distance:

$$l_{IFS}^1(A, B) = \frac{1}{2n} \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|); \quad (62)$$

- the normalized Euclidean distance:

$$q_{IFS}^1(A, B) = \frac{1}{2n} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2. \quad (63)$$

Clearly these distances satisfy the conditions of the metric (cf. [7]).

**Example 3.** Let us consider for simplicity “degenerated” intuitionistic fuzzy sets $A, B, D, G, F$ in $X = \{1\}$. A full description of each intuitionistic fuzzy set, i.e. $A = (\mu_A, v_A, \pi_A)/1$, may be exemplified by

$$A = (1, 0, 0)/1, \quad B = (0, 1, 0)/1, \quad D = (0, 0, 1)/1, \quad G = (1/2, 1/2, 0)/1, \quad E = (1/4, 1/4, 1/2)/1 \quad (64)$$

and their geometrical interpretation is presented in Fig 5.

Let us calculate the Euclidean distances between the above intuitionistic fuzzy sets using the formula (i.e. omitting the third parameter):

$$e(A, B) = \sqrt{\frac{1}{2} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2}. \quad (65)$$
We obtain

\[ e(A, D) = \sqrt{\frac{1}{2}((1-0)^2 + 0^2)} = \frac{1}{2}, \]

\[ e(B, D) = \sqrt{\frac{1}{2}(0^2 + (0-1)^2)} = \frac{1}{2}, \]

\[ e(A, B) = \sqrt{\frac{1}{2}((1-0)^2 + (0-1)^2)} = 1, \]

\[ e(A, G) = \sqrt{\frac{1}{2}\left((1-1)^2 + (0-1)^2\right)} = \frac{1}{2}, \]

\[ e(B, G) = \sqrt{\frac{1}{2}\left((0-1)^2 + (1-1)^2\right)} = \frac{1}{2}, \]

\[ e(E, G) = \sqrt{\frac{1}{2}\left((1-1)^2 + (1-1)^2\right)} = \frac{1}{4}, \]

\[ e(D, G) = \sqrt{\frac{1}{2}\left((0-1)^2 + (1-1)^2\right)} = \frac{1}{4}. \] (66)

The above results are not of the sort that one can agree with. As it was shown (Fig. 3), the triangle ABD (Fig. 5) has all edges equal to \( \sqrt{2} \) (as they are diagonals of squares with sides equal to 1). So we should obtain \( e(A, D) = e(B, D) = e(A, B) \). But our results show only that \( e(A, D) = e(B, D) \) [cf. (66)–(67)], but \( e(A, D) \neq e(A, B) \), and \( e(B, D) \neq e(A, B) \). Also \( e(E, G) \), which is half of the height of triangle ABD multiplied in (65) by \( \sqrt{\frac{1}{2}} \), is not the value we expect [it is too short, and the same concerns the height of \( e(D, G) \)].
Let us calculate the same Euclidean distances using (61). We obtain

\[
e_{IFS}^1(A, D) = \sqrt{\frac{1}{2}[(1 - 0)^2 + 0^2 + (0 - 1)^2]} = 1,
\]

\[
e_{IFS}^1(B, D) = \sqrt{\frac{1}{2}[(0^2 + (1 - 0)^2 + (0 - 1)^2]} = 1,
\]

\[
e_{IFS}^1(A, B) = \sqrt{\frac{1}{2}[(1 - 0)^2 + (0 - 1)^2 + 0^2]} = 1,
\]

\[
e_{IFS}^1(A, G) = \sqrt{\frac{1}{2}[(1 - \frac{1}{2})^2 + (0 - \frac{1}{2})^2 + 0^2]} = \frac{3}{2},
\]

\[
e_{IFS}^1(B, G) = \sqrt{\frac{1}{2}[(0 - \frac{1}{2})^2 + (1 - \frac{1}{2})^2 + 0^2]} = \frac{3}{2},
\]

\[
e_{IFS}^1(E, G) = \sqrt{\frac{1}{2}[(\frac{1}{4} - \frac{1}{2})^2 + (\frac{1}{4} - \frac{1}{2})^2 + (\frac{1}{2} - 0)^2]} = \sqrt{\frac{3}{4}},
\]

\[
e_{IFS}^1(D, G) = \sqrt{\frac{1}{2}[(0 - \frac{1}{2})^2 + (0 - \frac{1}{2})^2 + (1 - 0)^2]} = \sqrt{\frac{3}{2}}.
\]

Formula (61) gives the results we expect, i.e.

\[
e_{IFS}^1(A, D) = e_{IFS}^1(B, D) = e_{IFS}^1(A, B) = 2e_{IFS}^1(A, G) = 2e_{IFS}^1(B, G)
\]

and \(e_{IFS}^1(E, G)\) is equal to half of the height of a triangle with all edges equal to \(\sqrt{2}\) multiplied by \(\frac{1}{\sqrt{2}}\), i.e. \(\frac{\sqrt{2}}{4}\).

**Example 4.** Let us consider the following intuitionistic fuzzy sets \(A\) and \(B\) in \(X = \{1, 2, 3, 4, 5, 6, 7\}:

\[
A = (0.5, 0.3, 0.2)/1 + (0.2, 0.6, 0.2)/2 + (0.3, 0.2, 0.5)/4 + (0.2, 0.2, 0.6)/5 + (1, 0, 0)/6,
\]

\[
B = (0.2, 0.6, 0.2)/1 + (0.3, 0.2, 0.5)/4 + (0.5, 0.2, 0.3)/5 + (0.9, 0.1)/7.
\]

Then, the Hamming distance (60), taking into account all three parameters, is equal to

\[
d_{IFS}^1(A, B) = \frac{1}{2}\left[|0.5 - 0.2| + |0.3 - 0.6| + |0.2 - 0.2| + |0.2 - 0| + |0.6 - 1| + |0.2 - 0|
\]

\[
+ |0.3 - 0.3| + |0.2 - 0.2| + |0.5 - 0.5| + |0.2 - 0.5| + |0.2 - 0.2| + |0.6 - 0.3|
\]

\[
+ |1 - 0| + |0 - 1| + |0 - 0| + |0 - 0.9| + |1 - 0| + |0 - 0.1| \right] = 3.
\]

Hence, the normalized Hamming distance (62), taking into account all three parameters, is equal to

\[
l_{IFS}^1(A, B) = \frac{3}{7} = 0.43.
\]
The Hamming distance taking into account only the two parameters is equal to
\[
d^1(A, B) = \frac{1}{2}(|0.5 - 0.2| + |0.3 - 0.6| + |0.2 - 0| + |0.6 - 1| + |0.3 - 0.3| + |0.2 - 0.2|
\]
\[+ |0.2 - 0.5| + |0.2 - 0.2| + |1 - 0| + |0 - 1| + |0 - 0.9| + |1 - 0|) = 2.7 \quad (84)
\]
and the normalized Hamming distance taking into account only two parameters is
\[
l^1(A, B) = \frac{1}{q} d(A, B) = \frac{2.7}{2} = 0.39. \quad (85)
\]

The Euclidean distance (61) taking into account all three parameters is equal to
\[
e_{IFS}^1(A, B) = 0.5^{0.5}(0.5 - 0.2)^2 + (0.3 - 0.6)^2 + (0.2 - 0.2)^2 + (0.2 - 0)^2 + (0.6 - 1)^2 + (0.2 - 0)^2
\]
\[+ (0.3 - 0.3)^2 + (0.2 - 0.2)^2 + (0.5 - 0.5)^2 + (0.2 - 0.5)^2 + (0.2 - 0.2)^2 + (0.6 - 0.3)^2
\]
\[+ (1 - 0)^2 + (0 - 1)^2 + 0^2 + (0 - 0.9)^2 + (1 - 0)^2 - (0 - 0.1)^2)^{0.5} = \sqrt{2.21} = 1.49, \quad (86)
\]
hence the normalized Euclidean distance taking into account all three parameters is
\[
q_{IFS}^1(A, B) = \frac{e(A, B)}{\sqrt{7}} = \sqrt{\frac{2.21}{7}} = 0.56. \quad (87)
\]
The Euclidean distance (65) taking into account only two parameters is equal to
\[
e^1(A, B) = 0.5^{0.5}(0.5 - 0.2)^2 + (0.3 - 0.6)^2 + (0.2 - 0)^2 + (0.6 - 1)^2 + (0.3 - 0.3)^2 + (0.2 - 0.2)^2
\]
\[+ (0.2 - 0.5)^2 + (0.2 - 0.2)^2 + (1 - 0)^2 + (0 - 1)^2 + (0 - 0.9)^2 - (1 - 0)^2)^{0.5}
\]
\[= \sqrt{2.14} = 1.46, \quad (88)
\]
hence the normalized Euclidean distance taking into account two parameters is
\[
q(A, B) = \sqrt{\frac{1}{q} e(A, B)} = \sqrt{\frac{2.14}{7}} = 0.55. \quad (89)
\]
The results obtained in Examples 3 and 4 confirm that distances in intuitionistic fuzzy sets should be calculated by taking into account all three parameters (membership degree, non-membership degree, and values of hesitancy margin). It is also easy to notice that for formulas (60)–(63) the following is valid:
\[
0 \leq d_{IFS}(A, B) \leq n, \quad (90)
\]
\[
0 \leq l_{IFS}(A, B) \leq 1, \quad (91)
\]
\[
0 \leq e_{IFS}(A, B) \leq \sqrt{n}, \quad (92)
\]
\[
0 \leq q_{IFS}(A, B) \leq 1. \quad (93)
\]

Using only two parameters gives values of distances which are orthogonal projection of the real distances (Fig. 3), and this implies that they are lower.

4. Conclusions

In this paper, we proposed new definitions of distances between intuitionistic fuzzy sets. It was shown that their definitions should be calculated by taking into account three parameters describing an intuitionistic fuzzy set. Taking into account all three parameters describing intuitionistic fuzzy sets when calculating distances ensures that the distances for fuzzy sets and intuitionistic fuzzy sets can be easily compared [cf. (47)–(50) and (90)–(93)].
References