Worksheet 1.2: Working with Vectors

From the Toolbox (what you need from previous classes)

□ Trigonometry: Pythagorean theorem. Polar coordinates \((x, y) = (r \cos \theta, r \sin \theta)\).

□ Vectors: Be able to find the component form of a vector given two points.

Vectors have magnitude (length) and direction. Direction can be expressed using the unit vector (vector with length equal to one unit) that has the same direction as the vector. Two vectors can be added (vector addition), and a vector can be multiplied by a scalar (scalar multiplication).

In this worksheet, you will:

□ Practice computing vector addition, scalar multiplication, and the magnitude of vectors.

□ Use scalar multiplication to change the magnitude of vectors. In particular, combine finding the magnitude of a vector with scalar multiplication to find the unit vector that has the same direction as a given vector.

□ Explore the geometric interpretations of vector addition, scalar multiplication, and unit vectors.

□ Express vectors both in terms of unit basis vectors \(\hat{i}, \hat{j}, \hat{k}\).

Definitions

Vector Addition in \(\mathbb{R}^2\): Given two vectors \(\mathbf{v} = \langle v_1, v_2 \rangle\), \(\mathbf{w} = \langle w_1, w_2 \rangle\):

\[
\mathbf{v} + \mathbf{w} = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle = \langle v_1 + w_1, v_2 + w_2 \rangle
\]

Scalar Multiplication in \(\mathbb{R}^2\): Given a vector \(\mathbf{v} = \langle v_1, v_2 \rangle\) and a scalar \(c \in \mathbb{R}\):

\[
c \mathbf{v} = c \langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle
\]

Magnitude: Denoted \(|\mathbf{v}|\) or \(\|\mathbf{v}\|\), the magnitude of a vector is the distance between its initial and terminal points. (You will find the formula for computing magnitude in Model 1.)

Unit Vector: A unit vector is a vector with magnitude equal to one unit: \(|\mathbf{v}| = 1\). Unit vectors are often denoted using a “hat”: \(\hat{\mathbf{v}}\).
Model 1: Vector Magnitude

DIAGRAM 1A:

Diagram 1A(i)

Diagram 1A(ii)

DIAGRAM 1B:

Diagram 1B(i)

Diagram 1B(ii)

Diagram 1B(iii)

Critical Thinking Questions

In this section, you will derive the formulas for computing the magnitude of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, and use them to compute magnitudes of vectors.

(Q1) Finding a formula for computing the magnitude of a 2-d vector $v = \langle a, b \rangle$:

(a) From Diagram 1A(i): Write the algebraic form for the vector $v$.

$v = \langle _____, _____ \rangle$

(b) The vector $v$ in Diagram 1A(i) forms the hypotenuse of a right triangle. The magnitude of $v$ is the length of this hypotenuse. Compute the magnitude of this vector.

$|v| =$ 

(c) From Diagram 1A(ii): Write the algebraic form for the vector \( \mathbf{v} \). \((a\ \text{and}\ b\ \text{are\ arbitrary\ numbers})\).

\[ \mathbf{v} = \langle \quad, \quad \rangle \]

(d) **The formula for computing the magnitude of a 2-d vector** \( \mathbf{v} = \langle a, b \rangle \) **is:**

\[ |\mathbf{v}| = \quad \]

(Q2) Finding a formula for computing the magnitude of a 3-d vector \( \mathbf{v} = \langle a, b, c \rangle \):

(a) In Diagram 1B(i): Look at the vector \( \mathbf{v} \). It forms the diagonal of a box, with the initial point at the origin and the terminal point at the corner farthest away from the origin. Write the algebraic form for this vector \( \mathbf{v} \). \((a,\ b,\ \text{and}\ c\ \text{are\ arbitrary\ numbers})\).

\[ \mathbf{v} = \quad \]

(b) In Diagram 1B(ii): The vector \( \mathbf{v} \) is the same. Its projection onto the \( xy \)-plane forms the hypotenuse of a right triangle in the \( xy \)-plane. In this diagram, we are calling the length of this hypotenuse is \( d \). Label the other two sides of this triangle with their lengths, then use the Pythagorean Theorem to find \( d^2 \) in terms of the lengths of the sides of this triangle. \( \text{(If you look at this\ worksheet\ online,\ the\ triangle\ is\ in\ red.)} \)

\[ d^2 = \quad \]

(c) In Diagram 1B(iii): There is a second triangle where the vector \( \mathbf{v} \) is the hypotenuse. One of the sides of this triangle is the hypotenuse of the triangle from 1B(ii), and the other is parallel to the \( z \)-axis. Label the side parallel to the \( z \)-axis with its length, then use the Pythagorean Theorem to express the quantity \( |\mathbf{v}|^2 \) in terms of the lengths of the sides of this triangle. \( \text{Your\ final\ answer\ should\ be\ in\ terms\ of\ } a,\ b,\ \text{and}\ c \).

\[ |\mathbf{v}|^2 = d^2 + \quad \]

\[ = \quad + \quad + \quad \]

(d) **The formula for computing the magnitude of a 3-d vector** \( \mathbf{v} = \langle a, b, c \rangle \ **is:**

\[ |\mathbf{v}| = \quad \]

(Q3) Compute the magnitude of the vector \( \mathbf{w} = \langle 2, -1, 5 \rangle \).
Model 2: Vector Addition

**Critical Thinking Questions**

In this section, you will work on vector addition (adding two vectors).

(Q4) On Diagram 2: The two copies of the vectors \( \mathbf{v} \) and \( \mathbf{w} \) form a parallelogram. On the diagram, sketch the vector \( \mathbf{D} \) that forms the diagonal of this parallelogram, with initial point at the origin. Write down the algebraic form of this vector:

\[
\mathbf{D} = \langle \quad \rangle
\]

(Q5) Using the definition of vector addition from the front page of the worksheet, compute the sums \( \mathbf{v} + \mathbf{w} \) and \( \mathbf{w} + \mathbf{v} \) of the vectors pictured in Diagram 2:

\[
\mathbf{v} + \mathbf{w} = \langle \quad \rangle \quad \quad \quad \mathbf{w} + \mathbf{v} = \langle \quad \rangle
\]

(Q6) True or False: \( \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \).

(Q7) Compare the vector sum of \( \mathbf{v} \) and \( \mathbf{w} \) from (Q5) to the vector \( \mathbf{D} \) from (Q4). Explain why the geometric interpretation of vector addition is sometimes called the **parallelogram law**.

(Q8) On the diagram at the right: Use the parallelogram law to sketch \( \mathbf{a} + \mathbf{b} \) and \( \mathbf{a} - \mathbf{b} \).

*To sketch \( \mathbf{a} - \mathbf{b} \), use the fact that \( \mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) \). Recall from last class: \( -\mathbf{b} \) has the same magnitude but opposite direction as \( \mathbf{b} \).*
(Q9) Vector addition for vectors in $\mathbb{R}^3$ follows the same pattern as for vectors in $\mathbb{R}^2$. Write down the definition for $\mathbf{v} + \mathbf{w}$ if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$:

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$$

(Q10) Compute the vector sum $\mathbf{a} + \mathbf{b}$ if $\mathbf{a} = \langle 2, -1, 0.6 \rangle$ and $\mathbf{b} = \langle -5, 3, 1 \rangle$.

$$\mathbf{a} + \mathbf{b} = \langle 2 - 5, -1 + 3, 0.6 + 1 \rangle$$

**Model 3: Scalar Multiplication**

![Diagram 3:](image)

**Critical Thinking Questions**

In this section, you will work on scalar multiplication (multiplying a vector by a scalar).

(Q11) Use the definition of scalar multiplication from the front page of the worksheet to compute $c\mathbf{v}$ for the vector $\mathbf{v}$ pictured in Diagram 3 and each of the following values of $c$. Then, sketch and label these vectors on Diagram 3.

<table>
<thead>
<tr>
<th>$c$</th>
<th>$c\mathbf{v}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbf{v}$</td>
</tr>
<tr>
<td>2</td>
<td>$2\mathbf{v}$</td>
</tr>
<tr>
<td>1/2</td>
<td>$1/2\mathbf{v}$</td>
</tr>
<tr>
<td>0</td>
<td>$0\mathbf{v}$</td>
</tr>
<tr>
<td>-1</td>
<td>$-\mathbf{v}$</td>
</tr>
<tr>
<td>-2</td>
<td>$-2\mathbf{v}$</td>
</tr>
</tbody>
</table>
(Q12) Use your sketches of the vectors from (Q11) to answer the following:

a) If the absolute value of the scalar $c$ is greater than 1, then the magnitude of the vector $cv$ is [less than || equal to || greater than] the magnitude of the vector $v$.

b) If the absolute value of the scalar $c$ is less than 1, then the magnitude of the vector $cv$ is [less than || equal to || greater than] the magnitude of the vector $v$.

c) For which two values of $c$ does scalar multiplication $cv$ not change the magnitude of the vector $v$?

d) If the scalar $c$ is negative, describe the relationship between the directions of the vectors $v$ and $cv$.

e) If you look at the terminal points of all vectors $cv$, what “shape” do you get?

(Q13) **Parallel vectors:** Two non-zero vectors are parallel if they span parallel lines (that is, the lines drawn through the initial and terminal points are parallel). Algebraically, two non-zero vectors are parallel if their algebraic forms are scalar multiples of each other.

Which of the following vectors (if any) are parallel to $w = \langle 2, -1 \rangle$?

$\vec{v}_1 = \langle 4, -2 \rangle \quad \vec{v}_2 = \langle -6, 3 \rangle \quad \vec{v}_3 = \langle 4, 3 \rangle \quad \vec{v}_4 = \left\langle -1, \frac{1}{2} \right\rangle \quad \vec{0} = \langle 0, 0 \rangle$

(Q14) Scalar multiplication for 3-d vectors in 3-space follows the same pattern as for 2-d vectors in the plane. Write down the definition for $cv$ if $v = \langle v_1, v_2, v_3 \rangle$ and $c \in \mathbb{R}$:

$$cv = \dots$$

(⊕ Q15) (a) Compute $|\langle 4, 2, 10 \rangle|$. Then, compute $2|\langle 2, 1, 5 \rangle|$. You should get the same answer for both.

(b) Compute $|\langle -4, -2, -10 \rangle|$. Then, compute $-2|\langle 2, 1, 5 \rangle|$. You should get *almost* the same answer for both. What’s different?

(c) **Useful computational fact:** For $c \in \mathbb{R}$ a scalar and $v$ a vector, you can multiply a vector by a scalar first and then compute the magnitude, or you can compute the magnitude of the vector first, then multiply the magnitude by the absolute value of the scalar.

There’s something missing from this equation — fix it so it is correct: $|cv| = c|v|$
Model 4: Unit Vectors

**Critical Thinking Questions**

A unit vector is a vector with magnitude equal to one: $|\mathbf{v}| = 1$. In this section, you will work with unit vectors.

(Q16) In Diagram 4: Compute the magnitudes of the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\hat{\mathbf{v}}$, and add them to Diagram 4. Which one is a unit vector (that is, which one has magnitude equal to 1)?

| $\mathbf{v}_1$ | $|\mathbf{v}_1|$ |
|---------------|-----------------|
| $\langle 4, 3 \rangle$ | ________ |

| $\mathbf{v}_2$ | $|\mathbf{v}_2|$ |
|---------------|-----------------|
| $\langle 2, \frac{3}{2} \rangle$ | ________ |

| $\hat{\mathbf{v}}$ | $|\hat{\mathbf{v}}|$ |
|-------------------|-----------------|
| $\langle \frac{4}{5}, \frac{3}{5} \rangle$ | ________ |

Notation: Unit vectors are often denoted using a “hat”, for example: $\hat{\mathbf{v}}$.

(Q17) If $\mathbf{v}$ is a vector, how can you find a unit vector that has the same direction as $\mathbf{v}$?

(a) In (Q16), you found $|\mathbf{v}_1|$, which is a scalar. Multiply the vector $\mathbf{v}_1 = \langle 4, 3 \rangle$ from Diagram 4 by the scalar $\frac{1}{|\mathbf{v}_1|}$, to show that $\frac{1}{|\mathbf{v}_1|}\mathbf{v}_1 = \hat{\mathbf{v}}$.

$$\frac{1}{|\mathbf{v}_1|}\mathbf{v}_1 = \frac{1}{4}\langle 4, 3 \rangle = \langle 4, 3 \rangle = \hat{\mathbf{v}}$$

(b) In (Q16) you found $|\mathbf{v}_2|$, which is a scalar. Multiply the vector $\mathbf{v}_2 = \langle 2, \frac{3}{2} \rangle$ from Diagram 4 by the scalar $\frac{1}{|\mathbf{v}_2|}$, to show that $\frac{1}{|\mathbf{v}_2|}\mathbf{v}_2 = \hat{\mathbf{v}}$.

$$\frac{1}{|\mathbf{v}_2|}\mathbf{v}_2 = \frac{1}{2}\langle 2, \frac{3}{2} \rangle = \langle 2, \frac{3}{2} \rangle = \hat{\mathbf{v}}$$

Notation: You will often see $\frac{\mathbf{v}_1}{|\mathbf{v}_1|}$ used instead of $\frac{1}{|\mathbf{v}_1|}\mathbf{v}_1$.

(Q18) Based on your answers to (Q17), complete the statement:

**Statement:** If $\mathbf{v}$ is a non-zero vector, then $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ is a ____________ vector.

This means the magnitude of $\mathbf{v}/|\mathbf{v}|$ is: $\left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| = ____$. 
(⊕ Q19) Find the two unit vectors parallel to \( \mathbf{v} = \langle 2, -5 \rangle \). (Hint: One points in the same direction as \( \mathbf{v} \), the other points in the opposite direction.)

(⊕ Q20) Every vector can be written as the product of a unit vector and a scalar.

(a) Returning to Diagram 4, find the scalar multiple \( c_1 \) so that \( \mathbf{v}_1 = \langle 4, 3 \rangle \) is a multiple of the unit vector \( \hat{\mathbf{v}} \). Compare the scalar \( c_1 \) with the magnitude \( |\mathbf{v}_1|\).

\[ \mathbf{v}_1 = c_1 \hat{\mathbf{v}} = \quad \hat{\mathbf{v}} \quad /\!/ \quad c_1 = \quad |\mathbf{v}_1| = \quad \]

(b) Returning to Diagram 4, find the scalar multiple \( c_2 \) so that \( \mathbf{v}_2 = \langle 2, 3/2 \rangle \) is a multiple of the unit vector \( \hat{\mathbf{v}} \). Compare the scalar \( c_2 \) with the magnitude \( |\mathbf{v}_2|\).

\[ \mathbf{v}_2 = c_2 \hat{\mathbf{v}} = \quad \hat{\mathbf{v}} \quad /\!/ \quad c_2 = \quad |\mathbf{v}_2| = \quad \]

(c) Returning to Diagram 4, find the vector \( \mathbf{v} \) that has magnitude \( |\mathbf{v}| = 10 \) and the same direction as \( \hat{\mathbf{v}} \).

\[ \mathbf{v} = \quad \hat{\mathbf{v}} = \langle \quad , \quad \rangle \]

(d) Returning to Diagram 4, find the vector \( \mathbf{v} \) that has magnitude \( |\mathbf{v}| = 10 \) and the opposite direction to \( \hat{\mathbf{v}} \).

\[ \mathbf{v} = \quad \hat{\mathbf{v}} = \langle \quad , \quad \rangle \]

(⊕ Q21) If you look at the terminal points of all 2-d unit vectors based at the origin in the plane, what “shape” do you get? What about all 3-d unit vectors based at the origin in \( \mathbb{R}^3 \)?

(⊕ Q22) A common mistake is to think the vector \( \langle 1, 1 \rangle \) is a unit vector.

(a) Why do you think people make this mistake?

(b) How do you know that \( \langle 1, 1 \rangle \) is not a unit vector?
Model 5: The Basis Vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$

**Diagram 5:**

<table>
<thead>
<tr>
<th>Basis Vectors in $\mathbb{R}^2$</th>
<th>Basis Vectors in $\mathbb{R}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{i} = \langle 1, 0 \rangle$</td>
<td>$\mathbf{i} = \langle 1, 0, 0 \rangle$</td>
</tr>
<tr>
<td>$\mathbf{j} = \langle 0, 1 \rangle$</td>
<td>$\mathbf{j} = \langle 0, 1, 0 \rangle$</td>
</tr>
<tr>
<td>$\mathbf{k} = \langle 0, 0, 1 \rangle$</td>
<td>$\mathbf{k} = \langle 0, 0, 1 \rangle$</td>
</tr>
</tbody>
</table>

**Critical Thinking Questions**

In this section, you will write vectors using basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ instead of angle brackets.

(Q23) The basis vector $\mathbf{i}$ is the unit vector that points in the direction of the positive ____-axis. The basis vector $\mathbf{j}$ is the unit vector that points in the direction of the positive ____-axis. The basis vector $\mathbf{k}$ is the unit vector that points in the direction of the positive ____-axis.

(Q24) The $\mathbf{i}, \mathbf{j}, \mathbf{k}$ basis vectors can be used to represent vectors instead of angle brackets. For example, consider the vector $\mathbf{v} = \langle 2, -5, 7 \rangle$:

$$\mathbf{v} = \langle 2, -5, 7 \rangle$$
$$= \langle 2, 0, 0 \rangle + \langle 0, -5, 0 \rangle + \langle 0, 0, 7 \rangle$$
$$= 2 \mathbf{i} - 5 \mathbf{j} + 7 \mathbf{k}$$

(Q25) In general, the relationship between angle-bracket notation and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ notation is:

$$\langle a, b \rangle = \text{____} \mathbf{i} + \text{____} \mathbf{j}$$
$$\langle a, b, c \rangle = \text{____} \mathbf{i} + \text{____} \mathbf{j} + \text{____} \mathbf{k}$$

(Q26) Write the vector $\mathbf{v} = \langle 3, 0, -7 \rangle$ using $\mathbf{i}, \mathbf{j}, \mathbf{k}$-notation.

(Q27) Write the vector $\mathbf{v} = 3 \mathbf{j} - \mathbf{k}$ using angle-bracket notation.
Summary

The sum of two vectors is a \underline{______________}.

The vector $v + w$ is the diagonal of the \underline{______________} spanned by the vectors $v$ and $w$.

The product of a scalar and a vector is a \underline{______________}.

Scalar multiplication changes the magnitude of a vector. If the scalar is positive, scalar multiplication \underline{reverses / does not change} the direction of the vector. If the scalar is negative, scalar multiplication \underline{reverses / does not change} the direction of the vector.

If $c$ is a scalar and $v$ a vector, then $|cv| = \underline{______} |v|$.

Two non-zero vectors are parallel if one is the \underline{vector sum / scalar multiple} of the other.

A unit vector is a vector with magnitude equal to \underline{__________}. If $v \neq 0$ is any vector, the unit vector in the direction of $v$ is $\hat{v} = v/\underline{__________}$.

The basis vectors $\hat{i}, \hat{j}, \hat{k}$ are unit vectors that point in the direction of the positive coordinate axes. They can be used instead of angle-bracket notation:

\begin{align*}
\text{In } \mathbb{R}^2 : & \quad \langle a, b \rangle = \underline{______} \hat{i} + \underline{______} \hat{j} \\
\text{In } \mathbb{R}^3 : & \quad \langle a, b, c \rangle = \underline{______} \hat{i} + \underline{______} \hat{j} + \underline{______} \hat{k}
\end{align*}

If $v \neq \vec{0}$, it can also be represented in terms of magnitude (a scalar) and direction (a unit vector):

$$v = \underline{______} \hat{v}$$