Jumps and the classification of scattered linear orders

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Scattered orders definition and examples

Definition

A linear order L is said to be scattered if there does not exist an embedding from $\mathbb{Q}$ to L.

Example

- α, for α an ordinal
- α^* (reverse), for α an ordinal
- $\mathbb{Z}$
- $\mathbb{Z}^k$, the lexicographic power
- $\mathbb{Z} \cdot (\mathbb{Z} + 2 + \mathbb{Z}) + \mathbb{Z} \cdot (\mathbb{Z} + 3 + \mathbb{Z}) + \mathbb{N}$, combinations using sums and products

Question

How complex is the classification of countable scattered linear orders?

Derivatives

Definition

The derivative of L is the quotient $L/{\sim}$ where:

$$x \sim y \iff \text{the interval between } x, y \text{ is finite}$$

Definition

The α-derivative of L is the quotient $L/{\sim_\alpha}$ where:

$$x \sim_{\beta+1} y \iff [x]_\beta \sim [y]_\beta$$
$$x \sim_\lambda y \iff (\exists \beta < \lambda)\, x \sim_\beta y$$

Proposition

L is scattered if and only if there exists α such that $L/{\sim_\alpha}$ is trivial (has just one equivalence class).

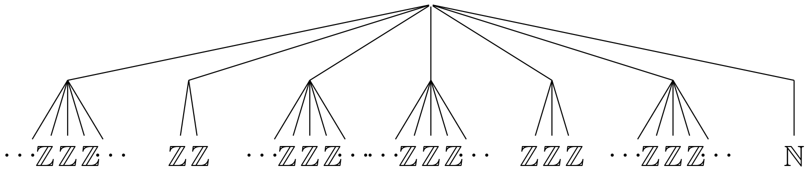
Derivatives and trees

Example

$$L = \mathbb{Z} \cdot (\mathbb{Z} + 2 + \mathbb{Z}) + \mathbb{Z} \cdot (\mathbb{Z} + 3 + \mathbb{Z}) + \mathbb{N}$$
$$L/{\sim} = \mathbb{Z} + 2 + \mathbb{Z} + \mathbb{Z} + 3 + \mathbb{Z} + 1$$
$$L/{\sim_2} = 7$$
$$L/{\sim_3} = 1$$

Remark

We can view L as a well-founded $\mathbb{Z}$-tree: each node carries a suborder of $\mathbb{Z}$ on its set of immediate successors.



Rank and complexity

Definition

The rank of L is the least α such that $L/{\sim_\alpha} = 1$. (Or, the rank of the corresponding tree is $1+\alpha$.)

Remark

Incrementing the rank results in up to $\mathbb{Z}$-many structures of the previous rank. This suggests that incrementing the rank results in a “jump” in complexity.

Classification and Borel complexity

Remark

If X is a class of countable linear orders, we may regard X as a subspace of $2^{\mathbb{N}\times\mathbb{N}}$ and the classification of X as an equivalence relation $\cong_X$. The complexity of the classification of X is measured by the position of $\cong_X$ in the Borel reducibility hierarchy:

Definition

$E \leq_B F$ iff there is a Borel function $f\colon \mathrm{dom}(E) \to \mathrm{dom}(Y)$ such that

$$x\, E\, x' \iff f(x)\, F\, f(x')$$

Jump operators

Definition

A proper jump operator on Borel equivalence relations is a mapping $E \mapsto J(E)$ which is:

- (monotone) $E \leq_B F$ implies $J(E) \leq_B J(F)$, and;
- (proper) $E <_B J(E)$ whenever E has at least two equivalence classes.

Remark

One may additionally impose a definability condition; our examples will all be suitably definable.

Examples of jumps

Example

If E is a Borel equivalence relation on X, the Friedman–Stanley jump of E is defined on X^ω by:

$$x \; E^+ \; y \iff \{[x(n)]_E : n \in \omega\} = \{[y(n)]_E : n \in \omega\}$$

Example

If E is a Borel equivalence relation on X, the Louveau jump of E with respect to the filter $\mathcal{F}$ is defined on X^ω by:

$$x \; E^{\mathcal{F}} \; y \iff \{n \in \omega : x(n) \; E \; y(n)\} \in \mathcal{F}$$

Bernoulli jumps

Definition

Let E be an equivalence relation on X, and let Γ be a countable group. The Γ-jump of E is the equivalence relation $E^{[\Gamma]}$ defined on X^Γ by

$$x \, E^{[\Gamma]} \, y \iff (\exists \gamma \in \Gamma)\,(\forall \alpha \in \Gamma)\; x(\gamma^{-1}\alpha) \; E \; y(\alpha)$$

Remark

- In words $E^{[\Gamma]}$ consists of Γ-many factors of E, modulo translation by Γ.
- If $\Delta(2)^{[\Gamma]}$ is simply the orbit equivalence relation of the Bernoulli action of Γ.
- The Γ-jump may be iterated through countable ordinals. We write $E^{[\Gamma]^\alpha}$ for the α-iterated Γ-jump.

Properties of Bernoulli jumps

Proposition

For any countable group Γ we have:

- $E \leq_B E^{[\Gamma]}$
- *If* $E \leq_B F$ *then* $E^{[\Gamma]} \leq_B F^{[\Gamma]}$
- *If* E *is Borel then* $E^{[\Gamma]}$ *is Borel*
- *If* E *is pinned then* $E^{[\Gamma]}$ *is pinned*
- *If* E_Λ *is the orbit equivalence relation of* $\Lambda \curvearrowright X$, *and* Γ *is any countable group, then* $(E_\Lambda)^{[\Gamma]}$ *is the orbit equivalence relation of* $(\Lambda \wr \Gamma) \curvearrowright X^\Gamma$
- *If* Λ *is a subgroup or quotient of* Γ, *then* $E^{[\Lambda]} \leq_B E^{[\Gamma]}$

The $\mathbb{Z}$-jump and scattered orders

Theorem

The isomorphism relation $\cong_{1+\alpha}$ on countable scattered linear orders of rank $1+\alpha$ is Borel bireducible with the αth iterated jump of the identity $\Delta(\mathbb{N})$ (that is, with $\Delta(\omega)^{[\mathbb{Z}]^\alpha}$).

Proof sketch.

- First we can confirm that $\cong_{1+\alpha}$ is Borel bireducible with isomorphism of $\mathbb{Z}$-trees of rank $2+\alpha$.
- Second we show that incrementing the rank of the $\mathbb{Z}$-trees corresponds with taking a $\mathbb{Z}$-jump. □

Remark

Thus once we know the $\mathbb{Z}$-jump is proper, we may conclude that the classification of countable scattered linear orders increases in complexity with the rank.

Not all Bernoulli jumps are proper

Theorem

Let Γ be a countable group with no infinite sequence of properly descending subgroups. Then the Γ-jump is not proper.

Proof sketch.

It is straightforward to show that for such a group Γ, we can find an explicit reduction:

$$\Delta(2)^{[\Gamma]^{\omega+1}} \leq_B \Delta(2)^{[\Gamma]^{\omega}}$$

□

Most Bernoulli jumps are proper

Theorem

Let Γ be a countable group such that $\mathbb{Z}$ or $\mathbb{Z}_p^{<\omega}$ for p prime is a quotient of a subgroup of Γ. Then the Γ-jump is proper.

Remark

It follows that the $\mathbb{Z}$-jump is proper, concluding our discussion of the countable scattered linear orders.

Remark

It remains open precisely which groups Γ give rise to a proper jump. An example of a group that doesn't satisfy the hypothesis of either theorem is $\Gamma = \bigoplus_{p \text{ prime}} \mathbb{Z}_p$.

Properness and essential complexity

Theorem

Let Γ be a countable group such that $\mathbb{Z}$ or $\mathbb{Z}_p^{<\omega}$ for p prime is a quotient of a subgroup of Γ. Then the Γ-jump is proper.

Remark

The proof is an adaptation of the proof of Louveau's theorem, and the Hjorth–Kechris–Louveau proof of Friedman–Stanley's theorem.

Definition

A family $\mathcal{F}$ of Borel equivalence relations has cofinal essential complexity if for every α there exists $E \in \mathcal{F}$ such that E is not Borel reducible to any equivalence relation in Π^0_α.

Theorem (Solecki)

If Γ is one of the groups $\mathbb{Z}$ or $\mathbb{Z}_p^{<\omega}$ for p a prime, then the family of Γ^ω-actions has cofinal essential complexity.

Sketch of proof of properness

Theorem

Let Γ be a countable group such that $\mathbb{Z}$ or $\mathbb{Z}_p^{<\omega}$ for p prime is a quotient of a subgroup of Γ. Then the Γ-jump is proper.

Proof sketch.

- We adapt the Hjorth–Kechris–Louveau machinery to prove: any orbit equivalence relation induced by an action of Γ^ω is Borel reducible to some iterate $\Delta(2)^{[\Gamma]^\alpha}$.
- Solecki's theorem: The family of Γ^ω-actions has cofinal essential complexity. Thus the family of iterates $\Delta(2)^{[\Gamma]^\alpha}$ has cofinal essential complexity.
- Now if $E^{[\Gamma]} \sim_B E$, then all iterates $\Delta(2)^{[\Gamma]^\alpha}$ are Borel reducible to E. Since the iterates have cofinal essential complexity, E is not Borel, a contradiction. □

Thank you!