López-Escobar's theorem and metric structures

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The space of countable structures

Definition

If $\mathcal{L}$ is a countable relational language with symbols R_i of arity n_i, then we define the space of countable $\mathcal{L}$-structures

$$\mathrm{Mod}(\mathcal{L}) = \prod \mathcal{P}(\mathbb{N}^{n_i}).$$

Definition

The Polish group S_∞ of permutations of $\mathbb{N}$ acts naturally on $\mathrm{Mod}(\mathcal{L})$ by translating the subsets of $\mathbb{N}^{n_i}$; we call this the logic action.

The orbits of the logic action are precisely the isomorphism equivalence classes.

López-Escobar's theorem

First we observe the following

Fact

Given any $\mathcal{L}$-theory T, the subset $\mathrm{Mod}(T) \subset \mathrm{Mod}(\mathcal{L})$ *consisting of the models of T is Borel. The same is true of* $\mathrm{Mod}(\phi)$, *where ϕ is a sentence of $\mathcal{L}_{\omega_1\omega}$.*

Definition

Here if $\mathcal{L}$ is any language, $\mathcal{L}_{\omega_1\omega}$ denotes the extension of first-order logic in which countable conjunctions and disjunctions are allowed. (We require formulas to have finitely many free variables.)

López-Escobar's theorem

If $X \subset \mathrm{Mod}(\mathcal{L})$ is Borel and isomorphism-closed then there exists a sentence ϕ of $\mathcal{L}_{\omega_1\omega}$ such that $X = \mathrm{Mod}(\phi)$.

Dynamical proof of López-Escobar

López-Escobar's theorem

If $X \subset \mathrm{Mod}(\mathcal{L})$ is Borel and isomorphism-closed then there exists a sentence ϕ of $\mathcal{L}_{\omega_1\omega}$ such that $X = \mathrm{Mod}(\phi)$.

Proof idea

If $X \subset \mathrm{Mod}(\mathcal{L})$ lies in the Borel hierarchy then X is approximated by simpler sets. Unfortunately the simpler sets will not be isomorphism-closed. We thus look for a stronger statement which applies even to sets X which are not isomorphism-closed.

Definition

If $X \subset \mathrm{Mod}(\mathcal{L})$ and $\bar{a} \in (\mathbb{N})^k$ then the Vaught transform $X^{*\bar{a}}$ is the set $\{\, M \mid \forall^* g \in S_\infty(\bar{a} \subset g \implies gM \in X)\,\}$.

Theorem

If $X \subset \mathrm{Mod}(\mathcal{L})$ is Borel and $k \in \mathbb{N}$, then there is a formula ϕ of $\mathcal{L}_{\omega_1\omega}$ with k free variables such that $M \in X^{*\bar{a}} \iff \phi^M(\bar{a})$.

Vaught's conjecture

VC, the Vaught conjecture for $\mathcal{L}_{\omega_1\omega}$

For any sentence ϕ of $\mathcal{L}_{\omega_1\omega}$, the subset $\mathrm{Mod}(\phi) \subset \mathrm{Mod}(\mathcal{L})$ consisting of the models of ϕ has either countably many or perfectly many isomorphism classes.

The role of the logic action leads to the dynamical variant of Vaught's conjecture:

TVC(S_∞), the topological Vaught conjecture for S_∞

Any standard Borel S_∞-space has countably many or perfectly many orbits.

Theorem

VC is equivalent to TVC(S_∞).

Proof of the equivalence using López-Escobar

Theorem

VC is equivalent to $\mathrm{TVC}(S_\infty)$.

Proof.

($\Leftarrow$) This is simply because $\mathrm{Mod}(\phi)$ is Borel.

($\Rightarrow$) Let X be a standard Borel S_∞-space.

- By Becker–Kechris, there exists $\mathcal{L}$ and a Borel S_∞-embedding $i\colon X \hookrightarrow \mathrm{Mod}(\mathcal{L})$. Note that $i(X)$ is Borel and isomorphism-closed.
- By López-Escobar's theorem there exists a sentence ϕ of $\mathcal{L}_{\omega_1\omega}$ such that $i(X) = \mathrm{Mod}(\phi)$.
- By the $\mathcal{L}_{\omega_1\omega}$-VC, the image $i(X)$ has countably or perfectly many isomorphism types.
- Hence X has countably many or perfectly many orbits. □

Metric structures

We now seek analogs of López-Escobar's theorem and its applications within the beautiful theory of metric structures and continuous logic.

Definition

A relational metric structure consists of:

- A complete metric space (M, d) of diameter 1
- Relations $R_i \colon M^{n_i} \to [0, 1]$, each uniformly continuous (the modulus of continuity is specified in the language)

Motivation

The R_i are grey sets. If $R_i(\bar{a}) = 0$ then $\bar{a}$ is surely in R_i, and if $R_i(\bar{a}) > 0$ then its value measures the failure.

The space of separable metric structures

We will confine ourselves to metric structures whose underlying metric space is the Urysohn sphere $\mathbb{U}$, that is, the universal ultrahomogeneous separable metric space of diameter 1.

Definition

If $\mathcal{L}$ is a countable metric language with symbols R_i of arity n_i and modulus Δ_i, then we define the space of separable $\mathcal{L}$-structures

$$\mathrm{MMod}(\mathcal{L}) = \prod \mathrm{Unif}_{\Delta_i}(\mathbb{U}^{n_i}, [0,1]).$$

Here $\mathrm{Unif}_{\Delta}(X, Y)$ denotes the space of Δ-uniformly continuous functions from X to Y with the topology of pointwise convergence.

Remark

The Polish group $\mathrm{Iso}(\mathbb{U})$ of isometric bijections of $\mathbb{U}$ acts naturally on $\mathrm{MMod}(\mathcal{L})$, and its orbits are the isomorphism classes.

López-Escobar's theorem for metric structures

The formulas of continuous logic consist of relational symbols, continuous combinations, and the quantifiers $\sup_x$ and $\inf_x$. The formulas of continuous $\mathcal{L}_{\omega_1\omega}$ additionally consist of $\sup_n$ and $\inf_n$.

Fact

For any sentence ϕ of $\mathcal{L}_{\omega_1\omega}$, the evaluation map $M \mapsto \phi^M$ is Borel.

Theorem (López-Escobar for MMod)

If $X \colon \mathrm{MMod}(\mathcal{L}) \to [0,1]$ is a Borel and isomorphism-invariant grey set, then there exists a sentence ϕ of $\mathcal{L}_{\omega_1\omega}$ such that for all $M \in \mathrm{MMod}(\mathcal{L})$ we have $X(M) = \phi^M$.

Remark

If X is $0,1$-valued we can additionally ensure that ϕ is $0,1$-valued. It follows that if X is a Borel and invariant true subset of $\mathrm{MMod}(\mathcal{L})$ then X is axiomatized by a sentence of $\mathcal{L}_{\omega_1\omega}$.

Grey transforms

As in Vaught's proof of López-Escobar's theorem, we isolate a stronger version which applies to grey sets that are not necessarily invariant.

Vaught transform

$X^{*\bar{a}} = \{\, M \mid \forall^* g \in S_\infty (\bar{a} \subset g \implies gM \in X) \,\}$

Grey transform

$X^{*\bar{a}}(M) = \sup^*_g \left[X(gM) - d(\bar{e}, g\bar{a}) \right]$

where $e_1, e_2, \ldots$ is a fixed dense sequence in $\mathbb{U}$.

Idea of proof of López-Escobar for metric structures

To prove classical López-Escobar, we used the strengthening:

Theorem

If $X \subset \mathrm{Mod}(\mathcal{L})$ is Borel and $k \in \mathbb{N}$, then there is a formula ϕ of $\mathcal{L}_{\omega_1\omega}$ with k free variables such that for all $\bar{a} \in (\mathbb{N})^k$ we have $M \in X^{*\bar{a}} \iff \phi^M(\bar{a})$.

With the grey transform in hand, we can state the analogous strengthening for metric structures:

Theorem

If $X \colon \mathrm{MMod}(\mathcal{L}) \to [0,1]$ is a Borel grey set and $k \in \mathbb{N}$, then there is a formula ϕ of continuous $\mathcal{L}_{\omega_1\omega}$ with k free variables such that for all $\bar{a} \in (\mathbb{U})^k$ we have $X^{*\bar{a}}(M) = \phi^M(\bar{a})$.

Vaught's conjecture, again

Continuous VC

Vaught's conjecture for subclasses of $\mathrm{MMod}(\mathcal{L})$ axiomatized by sentence of continuous $\mathcal{L}_{\omega_1\omega}$

TVC

For any Polish group G, Vaught's conjecture holds for all standard Borel G-spaces.

Corollary

The continuous VC is equivalent to the TVC.

Proof idea

In the previous proof we used Becker–Kechris plus López-Escobar's theorem. In this proof we use an analog of Becker–Kechris plus López-Escobar's theorem for metric structures.

Thank you!