Constructing automorphisms of corona algebras

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C*-algebras

Definition (Abstract)

A C*-algebra is a Banach *-algebra satisfying $\|a^*a\| = \|a\|^2$.

Fact

Any abelian C-algebra can be represented as $C_0(X)$ (X LCH).*

Fact

A general C-algebra can be represented as an algebra of operators on a Hilbert space:*

- $\mathcal{H}$ *denotes a separable complex Hilbert space.*
- $B(\mathcal{H})$ *denotes the set of bounded (continuous) linear mappings* $\mathcal{H} \to \mathcal{H}$.
- *Any (separable) C*-algebra is isomorphic to a* $\|\cdot\|$*-closed *-subalgebra of* $B(\mathcal{H})$.

The Calkin algebra

The Calkin algebra plays a prominent role in the classification of operators (for instance in the so-called BDF theory).

Definition

- The algebra of compact operators on $\mathcal{H}$, denoted $\mathcal{K}(\mathcal{H})$, is the subalgebra of $B(\mathcal{H})$ generated by the finite rank operators.
- The Calkin algebra is the quotient algebra $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Question (BDF)

Does the Calkin algebra have any outer automorphisms?

(Here, an automorphism is said to be outer iff it is not of the form $x \mapsto u^*xu$ for u unitary.)

Analogies in set theory

Question (BDF)

Does $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ have any outer automorphisms?

Remark

The question is reminiscent of set-theoretic questions regarding "nontrivial" automorphisms of quotient structures.

Examples

- Does the Stone–Cech remainder $\beta\omega \setminus \omega$ have any automorphisms other than permutations of ω?
- Does the Boolean algebra $P(\omega)/\mathrm{Fin}$ have any automorphisms other than permutations ω?
- What about other quotients $P(\omega)/J$?

Dichotomies

The answer to these questions almost follow the template below, an observation that has led to an extensive program of research.

- CH implies there is a nontrivial automorphism.
- Forcing axioms imply all automorphisms are trivial.

Examples

- $P(\omega)/\mathrm{Fin}$: W. Rudin, Shelah
- $P(\omega)/J$: Many!
- $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$: Phillips–Weaver, Farah

Corona

The corona construction generalizes both the Calkin algebra construction and the Stone–Čech remainder.

Definition

- If A is a separable C*-subalgebra of $B(\mathcal{H})$, then the multiplier algebra of A, denoted $M(A)$, is defined by

$$M(A) = \{m \in B(\mathcal{H}) \mid mA, Am \subset A\}$$

- The corona of A, denoted $C(A)$, is the quotient $M(A)/A$.

Examples

- The corona of $C_0(X)$ is exactly $C_0(\beta X \setminus X)$.
- The corona of $\mathcal{K}(\mathcal{H})$ is exactly $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

CH gives many automorphisms

We addressed the CH side of the coin for a wide variety of corona algebras:

Theorem (C.–Farah)

Suppose that A is separable, and either simple or stable. Then CH implies that the corona of A has $2^{2^{\aleph_0}}$ many automorphisms (and hence plenty of nontrivial ones).

The proof generalizes Farah's proof of the Phillips–Weaver result in the case of the Calkin algebra.

For this talk we add the following hypothesis:

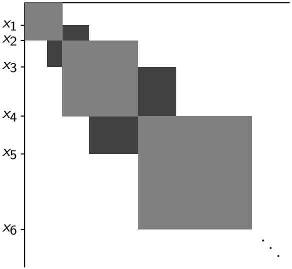
- Assume A has an increasing sequence of projections p_i such that $p_i \to 1$ and $(p_{i+1} - p_i)A(p_{j+1} - p_j) \neq 0$ for al i, j. (For instance if $A = \mathcal{K}(\mathcal{H})$, one can take $p_i =$ the projection onto $\langle e_1, \ldots, e_i \rangle$.)

Stratifying the corona

Definition

For each $X \subset P(\omega)$ let $C_X(A) =$

$$\left\{ \dot{m} \;\middle|\; \begin{array}{l} (p_{x_{i+1}} - p_{x_i}) m (p_{x_{j+1}} - p_{x_j}) = 0 \\ \text{whenever } \; |i-j| \geq 2 \end{array} \right\}$$



Lemma

Let $X_\xi \in P(\omega)$ for $\xi \in \omega_1$ be such that:

- *X_ξ is $\subset^*$-decreasing; and*
- *the increasing enumerations of the X_ξ form a dominating family.*

Then the sets $C_{X_\xi}(A)$ stratify the corona of A.

Coherence

Definition

- For $\alpha \in \mathbb{T}^{\mathbb{N}}$ define a multiplier $u_\alpha = \sum \alpha(i)(p_{i+1} - p_i)$.
- A sequence $\alpha_\xi \in \mathbb{T}^{\mathbb{N}}$ is said to be coherent iff the corresponding conjugation automorphisms Ad $\dot{u}_{\alpha_\xi}$ extend Ad $\dot{u}_{\alpha_\eta} \upharpoonright C_{X_\eta}(A)$ whenever $\eta < \xi$.

Remark

Every coherent ω_1-sequence gives rise to an automorphism of the corona $C(A)$.

Lemma

The sequence α_ξ is coherent iff whenever $\eta < \xi$,

$$\lim_{j\to\infty} \operatorname{diam}\left[\operatorname{range}\left(\alpha_\xi\alpha_\eta^{-1} \upharpoonright_{[X_\eta(j), X_\eta(j+2))}\right)\right] = 0$$

Conclusion: Splitting

Theorem

If A is separable and simple or stable, then CH implies that the corona $C(A)$ has $2^{\aleph_1}$ many automorphisms.

Proof idea.

Build a binary tree of height $\aleph_1$ consisting of elements $\alpha_s \in \mathbb{T}^{\mathbb{N}}$ for $s \in 2^{<\omega_1}$ satisfying:

- each branch $b \in 2^{\omega_1}$ gives rise to a coherent sequence $\alpha_{b\restriction\xi}$; and
- each splitting pair $\alpha_{s\frown 0}, \alpha_{s\frown 1}$ is incompatible. □