Generalized Choquet games and spaces

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Motivation: DST in larger spaces

Definition

A Polish space is a separable, completely metrizable space.

Examples

2^ω, ω^ω, $\mathrm{Homeo}[0,1]$, the logic space of countable structures, *etc*.

Remark

There has been substantial work in the descriptive set theory of the spaces 2^κ, κ^κ, and the logic space of structures of size κ (with the $<\kappa$-supported product topology)

Question

How can these be unified by an appropriate analog of the Polish space?

Standing assumption

$\kappa^{<\kappa} = \kappa$

The Choquet game

Definition

The strong Choquet game in X is played as follows:

$$\begin{array}{lcccccc} \text{I} & U_0, x_0 & & U_1, x_1 & & \cdots & \\ \text{II} & & V_0 & & V_1 & & \cdots \end{array}$$

Rules: $U_n \supset V_n \supset U_{n+1}$ and $x_n \in U_n \cap V_n$

We say that II wins the play if $\bigcap U_n \neq \emptyset$.

Fact

If X is second countable and T_3, then X is Polish iff player II has a winning strategy in the Choquet game in X.

A longer Choquet game

Definition

The strong κ-Choquet game in X is played as follows:

$$\begin{array}{lccccccccc} \text{I} & U_0, x_0 & & U_1, x_1 & & \cdots & & U_\lambda, x_\lambda & & \cdots \\ \text{II} & & V_0 & & V_1 & & \cdots & & V_\lambda & & \cdots \end{array}$$

Each set is relatively open in the intersection of the sets so far.

We say that II wins the play if for all limit ordinals $\lambda \leq \kappa$, we have $\bigcap_{\alpha<\lambda} U_\alpha \neq \emptyset$.

Definition

If X has weight κ and is T_3, then we say X is strong κ-Choquet if player II has a winning strategy in this game in X.

Examples

Examples

- 2^κ, κ^κ with the $<\kappa$-supported product topology
- 2^κ, κ^κ with the lexicographic order topology
- The symmetric group S_κ of bijections of κ
- The logic space of structures of cardinality κ

Question

All of the above are isomorphic to each other by a κ-Borel function. Is this always the case?

Caveat

If $T \subset \kappa^{<\kappa}$ is a κ-Kurepa tree, then one can extend the dead branches of T trivially to obtain a tree T' such that $[T']$ is strong κ-Choquet but $[T']$ is not κ-Borel isomorphic to κ^κ.

Covering strong κ-Choquet spaces

Theorem

If X is strong κ-Choquet, then there is a continuous surjection $\kappa^\kappa \to X$ with κ-Borel right inverse.

Proof outline.

Let σ be a winning strategy for player II in the κ-Choquet game on X. We construct for $s \in \kappa^{<\kappa}$ plays U_s, x_s satisfying

(1) each branch $U_\emptyset, x_{s_0}, U_{s_0}, x_{s_0 s_1}, \ldots$ is a sequence of plays for *I* where *II* plays by σ

(2) the U_s shrink (use the T_3 property)

(3) for each s, $\{ U_{s^\frown\alpha} \mid \alpha < \kappa \}$ covers U_s

Then (1) guarantees that there is $f(b) \in \bigcap U_{b\restriction\alpha}$, (2) guarantees it is unique, and (3) guarantees f is surjective. □

A bijective version

We can improve the covering to find a bijective mapping on a subtree $[T]$.

Theorem

If X is strong κ-Choquet, then there is a subtree $T \subset \kappa^{<\kappa}$ and a continuous bijection $[T] \to X$ with κ-Borel right inverse.

Proof outline.

- In the above argument, make the U_s disjoint by letting $U'_{s^\frown\alpha} = U_{s^\frown\alpha} \setminus \bigcup_{\beta<\alpha} U_{s^\frown\beta}$.
- Whenever $U'_s = \emptyset$, toss s out of the tree T.
- Everything else is the same, but now the mapping f is additionally injective. □

Consistency of an isomorphism theorem

Theorem

Suppose the following holds:

(∗) *For any $T \subset \kappa^{<\kappa}$ such that $\kappa < |[T]|$, T has a perfect binary subtree.*

Then any two strong κ-Choquet spaces of size $> \kappa$ are isomorphic by a κ-Borel function.

Proof.

Given such a space X, the previous slide showed there is T such that $X \cong [T]$. And from the assumption (∗) we have

$$\kappa^\kappa \hookrightarrow 2^\kappa \hookrightarrow [T] \subset \kappa^\kappa$$

Thus $[T] \cong \kappa^\kappa$ by a Cantor–Schröder–Bernstein argument. □

Alternatives

The assumption $(*)$ on the previous slide has the consistency strength of an inaccessible cardinal. In fact it holds in Solovay's model without κ-Kurepa trees.

If we wish to obtain an absolute result, we may restrict the class of spaces X further:

Theorem

Any two κ-perfect strong κ-Choquet spaces of size $> \kappa$ are isomorphic by a κ-Borel function.

Question

Without using the assumption $(*)$, can we establish a Borel isomorphism theorem for a wider class of spaces?

Further

Using arguments of this type, Motto Ros and Schlicht have established a number of change-of-topology results, such as:

Theorem

If X is a strong κ-Choquet space and $B \subset X$ is κ-Borel, then there is a strong κ-Choquet topology on X (with the same κ-Borel sets) making B strong κ-Choquet.

Definition

A space X is *standard κ-Borel* iff its κ-Borel sets arise from a strong κ-Choquet topology on X.

Corollary

X is standard κ-Borel iff it is κ-Borel isomorphic to a closed subset of κ^κ.

A Question and thanks

Question

Is there a universal strong κ-Choquet space?

Thank you!