Borel complexity theory and classification problems

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Borel complexity theory

What is Borel complexity theory?

Borel complexity theory is a study of the relative complexity of classification problems in mathematics.

The theory allows one to formalize and investigate certain kinds of statements, such as:

- The classification of homeomorphisms of the unit interval is simpler than the classification of homeomorphisms of the unit square
- The classification of abelian separable C*-algebras is just as hard as the classification of all separable C*-algebras
- The ergodic measure-preserving transformations are not classifiable by reasonable invariants

At the heart of Borel complexity theory is an area of math called invariant descriptive set theory.

First, what is classical descriptive set theory?

Kechris, *Classical Descriptive Set Theory*:

> *Descriptive set theory is the area of mathematics concerned with the study of the structure of definable sets in Polish spaces. Beyond being a central part of contemporary set theory, the concepts and results of descriptive set theory are being used in diverse fields of mathematics, such as logic, combinatorics, togology, Banach space theory, real and harmonic analysis, potential theory, ergodic theory, operator algebras, and group represtnation theory.*

I'll add: it also studies the definable mappings on Polish spaces.

What is invariant descriptive set theory?

Gao, *Invariant descriptive set theory*:

> *Invariant descriptive set theory is a new branch of descriptive set theory that deals with the complexity of equivalence relations. On any Polish space there is always the identity equivalence relation, and any subset of a Polish space is an invariant set for this equivalence relation. Therefore on a rather fundamental level invariant descriptive set theory encompasses and strengthens the classical theory.*

I'll add: it also studies the definable, invariant mappings between Polish spaces with equivalence relations.

Classical result to invariant result

A fundamental result of descriptive set theory:

- Every uncountable Borel set has a perfect subset.

(thus the Borel sets satisfy the Continuum Hypothesis) (goes back to Cantor)

The less well-known, invariant version:

- Every Borel equivalence relation with uncountably many classes has perfectly many classes.

(so the Borel equivalence relations satisfy a Continuum Hypothesis too) (Silver)

Representing a classification problem

To study the classification of some objects up to some equivalence:

- The objects should be coded as elements of a Polish space X. (A group can be coded by its multiplication table, a graph by its incidence relation, etc.)
- The abstract equivalence is now an equivalence relation E on X. (Isomorphism, isometry, conjugacy, etc.)

Thesis

Any two reasonable ways of coding a given classification problem will be the same with respect to Borel complexity theory.

Note

The approach dates to the 1990s and has gained prominence thanks in part to an number of striking applications.

Key authors: Friedman–Stanley, Hjorth–Kechris, Becker, Dougherty, Harrington, Jackson, Louveau.

Borel reducibility

If E, F are equivalence relations on Polish spaces X, Y, we say E is Borel reducible to F (written $E \leq_B F$) if there is a Borel function $f : X \to Y$ satisfying

$$x \, E \, x' \iff f(x) \, F \, f(x')$$

Interpretation

The classification up to E-equivalence is no harder than the classification up to F-equivalence.

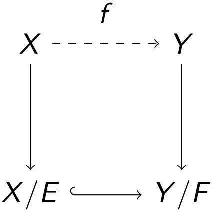
Indeed, if you have any set of invariants for F then after composing with f they may be used for E too.

Remark

It is necessary that we impose a definability constraint on the reduction functions. The slogan is Borel = Explicit.

Borel cardinality

A Borel reduction is equivalent to an injective mapping on the quotient spaces with a Borel lifting.



Thus $E \leq_B F$ is a definable version of a cardinality comparison $|X/E| \leq |Y/F|$.

Structure

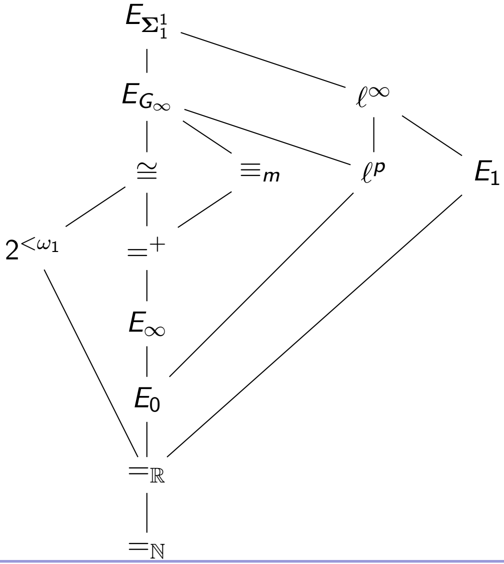
The Borel reducibility ordering is wild—there are just a few structure theorems and many non-structure theorems.

Instead there are several dividing lines used in the theory:

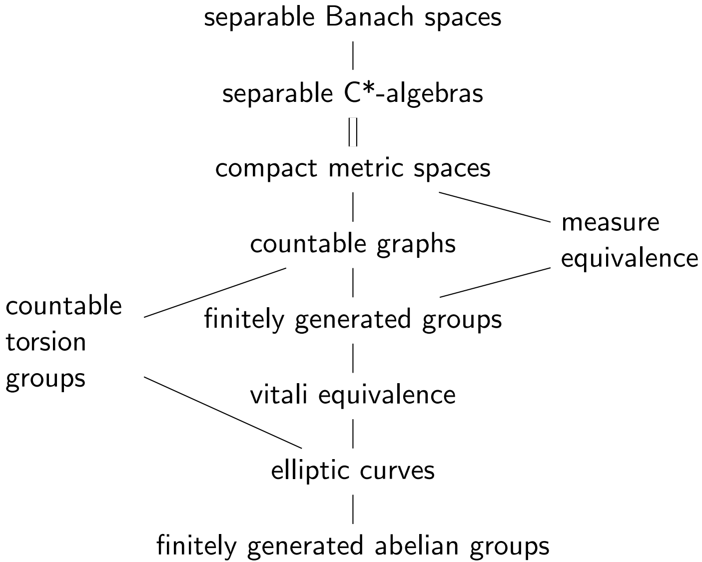
- Borel/Not Borel — asks whether E is Borel as a subset of $X \times X$. For instance the isomorphism relation on finitely generated countable groups is Borel, while the isomorphism relation on all countable groups is not Borel.
- Light side/Dark side — asks whether E corresponds to the orbits of a Polish group action. For instance the isometry relation on separable Banach spaces is on the light side, while the homeomorphism relation is on the dark side.

Moreover some benchmark equivalence relations occur frequently, are well-studied, and provide a battery of comparisons for a given equivalence relation.

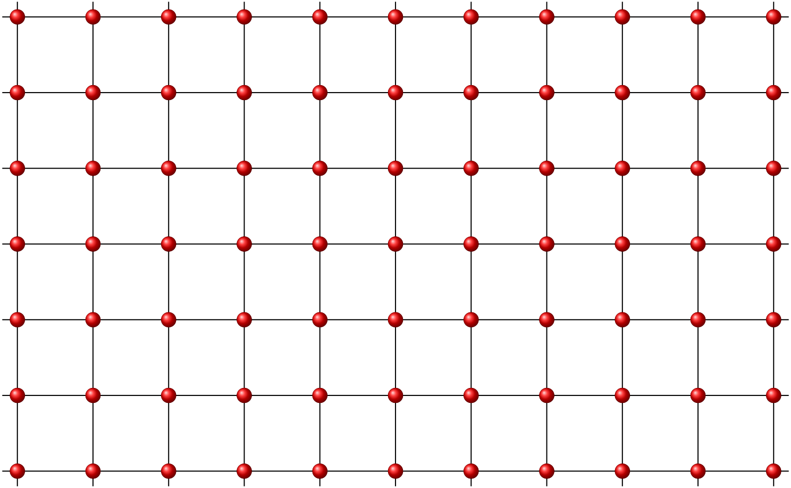
Map of some benchmark complexities



Map of some naturally occurring classifications



Classification problems



Completely classifiable

If E is an equivalence relation on a Polish space X, we say E is smooth if $E \leq_B =_{\mathbb{R}}$, that is, there is a Borel function $f\colon X \to \mathbb{R}$ satisfying

$$x \mathrel{E} x' \iff f(x) = f(x')$$

Remark

This means the E-classification is completely classifiable, since f provides an explicit computation of complete invariants.

We think of the smooth equivalence relations as very low in complexity; the perfect set theorem mentioned above makes this precise:

Theorem (Silver dichotomy)

If E is a Borel equivalence relation then either $E \leq_B =_{\mathbb{N}}$ or else $=_{\mathbb{R}} \leq_B E$.

Complete classifications in mathematics

Example

The complex matrices are classified up to conjugacy by the Jordan normal form.

Example

The elliptic curves are classified up to isomorphism by the j-invariant.

Example

Bernoulli shifts are classified up to measure-preserving equivariant isomorphism by the entropy.

Example

The countable divisible groups G are classified up to isomorphism by $f(G) =$ the sequence which gives the number of copies of $\mathbb{Q}$ and $\mathbb{Z}/p^\infty\mathbb{Z}$ for all p.

Hyperfinite equivalence relations

Definition

An equivalence relation E is hyperfinite if it can be expressed as the increasing union $E = \bigcup E_n$ of equivalence relations E_n with every equivalence class finite.

Fact

There is a universal hyperfinite equivalence relation E_0 defined on $2^{\mathbb{N}}$ by:

$$x \, E_0 \, y \iff x(n) = y(n) \text{ for all but finitely many } n$$

Theorem (Harrington–Kechris–Louveau)

If E is a Borel equivalence relation then either $E \leq_B =_{\mathbb{R}}$ or else $E_0 \leq_B E$.

Thus E_0 is the only obstruction to a complete classification*.

Hyperfinite and orbit equivalence relations

Fact

The hyperfinite equivalence relations are precisely the orbits corresponding to an action of $\mathbb{Z}$.

Theorem (Gao–Jackson)

The hyperfinite equivalence relations are precisely the orbits corresponding to an action of a countable abelian group.

Question

It is not known whether countable amenable groups can induce non-hyperfinite equivalence relations.

Torsion-free abelian groups

Definition

A group A is torsion-free abelian if it is isomorphic to a subgroup of some power $\mathbb{Q}^n$.

The rank of A is the minimum possible such n (it may be infinite).

Theorem (Baer, 1937)

The torsion-free abelian groups of rank 1 are classified by the sequence of p-heights of one of its elements (up to a finite error).

Remark

In our setting, this can be viewed as an E_0 classification.

Torsion-free abelian groups of rank ≥ 2

Kurosh (1937) and Mal'cev (1938) independently gave a classification for rank ≥ 2. But the classification was regarded as unsatisfactory, because the invariants were themselves complicated.

Question

Is there a satisfactory classification of torsion-free abelian groups of finite rank?

Theorem (Hjorth 1998, Thomas 2002)

The classification problem for torsion-free abelian groups increases strictly in Borel complexity with the rank:

$$E_0 \sim_B R_1 <_B R_2 <_B R_3 <_B \cdots$$

The role of ergodic rigidity

The Kurosh–Mal'cev invariants can be used to show that R_n contains a copy of the ergodic measure-preserving action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{PQ}_p^n$.

This latter action is superrigid, which means loosely that an orbit-preserving mapping to another free action must come from an action-preserving mapping. Examples of superrigid actions include:

- Actions of lattices in higher-rank semisimple Lie groups (Margulis–Zimmer)
- Profinite actions of property (T) groups (Ioana)
- Bernoulli actions of property (T) groups (Popa)

Either of the first two can be used to obtain a contradiction from $R_{n+1} \leq_B R_n$.

Classifiable by countable structures

Definition

A countable structure is a countable family of functions and relations on $\mathbb{N}$.

Examples

- countable trees and graphs
- countable groups and semigroups
- countable fields and rings
- countable linear orders, and partial orders, ...

Definition

An equivalence relation E is said to be classifiable by countable structures if E is Borel reducible to the isomorphism equivalence relation on some space of countable structures.

Countable structures and completeness

Example

The (nondecreasing) autohomeomorphisms of $[0,1]$ are classified up to conjugacy by "bump structure", a countable linear order with a unary relation.

Definition

A classification is called complete for countable structures if it is Borel bireducible with isomorphism $\cong$ on all countable structures.

Examples

The classification of linear orders is complete; the classification of groups is complete.

Question

It is not known whether the classification R_ω of all countable torsion-free abelian groups is complete.

Turbulence and unclassifiability

Definition

A continuous action of a Polish group G on X is turbulent if its orbits are meager and dense, and its local orbits are somewhere dense.

Definition

The local orbit corresponding to $x_0 \in X$ and open neighborhoods $x_0 \in U \subset X$ and $1 \in V \subset G$:

$$\mathcal{O} = \{\, x_n \in U \mid \exists x_i \in U, g_i \in V \text{ such that } x_{i+1} = g_i x_i \,\}$$

Theorem (Hjorth)

If E corresponds to the orbits of a turbulent Polish group action, then E is not classifiable by countable structures.

Unclassifiable thanks to turbulence

Example

Autohomeomorphisms of $[0,1]^2$ up to conjugacy (Hjorth)

Example

Unitary operators on ℓ^2 up to conjugacy (Kechris–Sofronidis)

Example

Separable, nuclear C*-aglebras up to homeomorphism (Farah–Toms–Törnquist)

Example

Ergodic measure-preserving transformations up to conjugacy (Foreman–Weiss)

Classifiable by Polish group orbits

Definition

An equivalence relation E on X is said to be classifiable by Polish group orbits (light side) if E is Borel reducible to an orbit equivalence relation induced by a Polish group action.

Remark

This is equivalent to classifiability by metric structures, a generalization of classical models encompassing analytic objects.

Example

The classification of separable abelian C*-algebras (compact metric spaces up to homeomorphism) is classifiable by Polish group orbits, and complete among such.

Unclassifiable by Polish group orbits

Example

The classification of separable Banach spaces up to homeomorphism is not classifiable by Polish group orbits. In fact it is complete among $\mathbf{\Sigma}^1_1$ equivalence relations. (Rosendal)

Example

The equivalence relation E_1 defined on $\mathbb{R}^\omega$ as follows is not classifiable by Polish group orbits. (Kechris–Louveau)

$$x \mathrel{E_1} y \iff x(n) = y(n) \text{ for all but finitely many } n$$

Question

Recalling that E_0 is the unique obstruction to complete classification, it is unknown whether E_1 is the unique obstruction to classification by Polish group orbits.

Thank you!