The many "faces" of E_0

we begin with a puzzle...

<u>The setup</u> infinitely many people gather in a room wearing "racing" nametags #1, #2, #3, etc. In a moment they will all be given a fez, either red or blue. they ~~will see~~ will see each other's fez but not their own. they must guess their own fez color. they can ~~...~~ communicate now, but not once the fezes are given.

<u>question</u> can they come up with a strategy so they all guess correctly?

<u>A</u> no. give them all blue fezes. if they get that right then flip one to red. he will be wrong.

<u>question</u> can they come up with a strategy so all but <u>ONE</u> guess correctly?

<u>A</u> no. this argument is somewhat longer (?) but still works. (see next page)

<u>exercise</u> come up with an argument that shows no finite number of allowed exceptions can save these poor people.

* we can force two people to be wrong.

First, note that everybody must change their own guess when they see just a single hat change color.

↳ to see this, suppose #N guesses red both times even though #M changed from red to blue.
then give #N a blue hat, and leaving everything else fixed, give #M whichever color makes him wrong. //

Now, we'll show that all of this mind-changing will sink them anyway. focus on just three people, give them all blue hats. at least two of them will be right, say #1 and #2.

hats #1	hats #2	hats #3	guesses #1	guesses #2
B	B	B	B	B
B	B	R	R	R

Now #1 and #2 are wrong. □

so the real question is:

<u>question</u> can they make it so all but finitely many of them are right?

* the problem space, the space of hat assignments, (and the solution space of guesses) is $\{R, B\}^{\mathbb{N}}$, the space of infinite sequences of R's and B's.

* this can be naturally identified with <u>Cantor space</u> $2^{\mathbb{N}}$ of infinite sequences of 0's and 1's.

* topology: a basic open set has just finitely many bits specified:

$$B_s = \{ x \in 2^{\mathbb{N}} \mid x(0) = s_0 \wedge \cdots \wedge x(n) = s_n \}$$

(product topology)

* the error margin corresponds to an equivalence relation:

<u>definition</u> E_0 is the "almost equality" equivalence relation on $2^{\mathbb{N}}$, ie,

$$x \, E_0 \, y \iff \exists N \; \forall n \geq N \;\; x(n) = y(n).$$

* a strategy for this problem is a function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ satisfying $f(x) \, E_0 \, x$, plus some invariance condition.

* such functions exist by the axiom of choice!

Nonmeasurable sets.

* recall the Vitali construction of a non Lebesgue measurable subset of $[0,1]$.

    $\hookrightarrow$ consider the equivalence relation E_v on $[0,1]$ defined by $x E_v y \iff x-y \in \mathbb{Q}$.

    then let $N = T =$ a <u>transversal</u> for E_v, that is, a set which meets each E_v-equivalence class exactly once.

    ...

* A similar construction works for the Haar measure on $2^{\mathbb{N}}$.

    the measure: "coin-flipping" based on the probability of $\mathbb{N}$-many coin-flips $0/1$ landing in a given set.

$$\mu(\{x \mid x(0)=s_0 \wedge \dots \wedge x(n)=s_n\}) = \frac{1}{2^{n+1}}.$$

    this "premeasure" extends to a measure.

<u>Theorem</u> there exists a μ-nonmeasurable set.

proof: this time we look to E_0. let T be a transversal for E_0. then:

$$2^{\mathbb{N}} = \dot{\bigcup}_{\sigma \in 2^{<\mathbb{N}}} (\sigma + T).$$

since $\mu(T) = \mu(\sigma + T)$ for all $\sigma \in 2^{\mathbb{N}}$, we have

$$1 = \sum_{\sigma \in 2^{<\mathbb{N}}} \mu(T),$$ a contradiction $\square$

Returning to the puzzle,

* We outlined a strategy before. We can think of it this way:

  Let $F(x)$ = the unique element of $T \cap [x]_{E_0}$, where T is a transversal for E_0.

  Then as promised, $F(x) \, E_0 \, x$, and $F(x)(n)$ can be computed without looking at your own fez.

* This strategy requires coming up with a non-measurable set! We know this requires the axiom of choice.

<u>Exercise</u> show that any strategy, at its heart, somehow relies on a nonmeasurable set.

Classification: Baer's theorem.

* an abelian group is said to be torsion-free if $(\forall n)$ $(\forall a \in A)$ $a \neq 0 \Rightarrow na \neq 0$.

* among finitely generated abelian groups, they have the form $\mathbb{Z}^r \oplus$ torsion. So the only torsion-free examples are $\mathbb{Z}^r$.

* f.g., t-free ab. groups are classified by their rank.

* if we drop "f.g.", this is not true: there are other groups even in rank 1: e.g. $\mathbb{Q}$.

question classify the t-free abelian groups of rank 1, i.e., the subgroups of $\mathbb{Q}$.

* what kinds of examples are we talking about?

$\mathbb{Z}, \mathbb{Q}, \dots$?

also $\mathbb{Z}[\frac{1}{p}] \neq \mathbb{Z}$

$\mathbb{Z}[\frac{1}{p^n} : n \in \mathbb{N}]$

$\mathbb{Z}_{(p)}$

...

* idea: a group should be determined by how much you can divide by each prime.

<u>definition</u> let $A \leq \mathbb{Q}$ and (wlog) suppose $1 \in A$.

then $\text{type}_1(A) = \{ (p,n) \in \text{Primes} \times \mathbb{N} \mid \frac{1}{p^n} \in A \}$

* the space of types is (a subset of) $2^{\mathbb{P} \times \mathbb{N}}$

and so carries its own "version" of E_0:

$t \; E_0 \; s$ iff $|t \triangle s| < \infty$.

<u>lemma</u> the isomorphism equivalence relation on the space of subgroups of $\mathbb{Q}$ corresponds with the E_0 relation on the space of types.

<u>proof</u>. both are equivalent to the rational multiple relation:

* if $A \cong B$, say $1 \mapsto q$, it is not hard to see the isomorphism is just $a \mapsto qa$.

  and $B = qA$.

* also $\text{type}_1(A) \; E_0 \; \text{type}_1(B)$ corresponds to the existence of some $q = \prod p_i^{n_i} / \prod r_i^{m_i}$

  (where $(p_i, n_i) \in \text{type}_1(A) \setminus \text{type}_1(B)$
  and $(q_i, m_i) \in \text{type}_1(B) \setminus \text{type}_1(A)$)

  such that $B = qA$.

* thus $A \cong B$ iff $\text{type}_1(A) \; E_0 \; \text{type}_1(B)$ $\square$

<u>Conclusion</u>: we can classify the subgroups of $\mathbb{Q}$ (i.e., choose a "canonical" group in each isomorphism class) only with the help of a non-measurable set / axiom of choice!