SET THEORY AND CLASSIFICATION PROBLEMS

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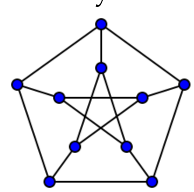
§1. CLASSIFICATION PROBLEMS

- A "classification problem" is a problem of the following form: given a mathematical object and a notion of equivalence, figure out which object you have up to equivalence.

- For example, a graph is a collection of vertices and edges between them. Two graphs are isomorphic iff there is an edge-preserving bijection between their vertices. But the same graph may be given in many different ways: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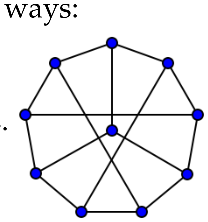

- Similarly, if one is trying to distinguish groups up to isomorphism, one ecounters the problem that one may have two very different presentations:

$$\langle a, b \mid aba = bab \rangle \text{ vs. } \langle x, y \mid x^2 = y^3 \rangle$$

- The question is, given a graph or group or whatever, can we write down enough information about its properties to determine its isomorphism type uniquely?

§2. Standard Borel spaces

- To study this question rigorously, we must put some limits on what things we are allowed to "write down". Indeed any collection can be classified uniquely up to equivalence by the equivalence classes themselves.

- The field known as *descriptive set theory* provides us with the tools to decide what is sufficiently explicit to be reasonable and what is not.

- First, we confine ourselves to consider only classes of mathematical objects which can be reasonably coded e.g., by a countable binary sequence (an element of the space $2^{\mathbb{N}}$).

- This may sound restrictive, but in fact this includes elements of any standard Borel space.

  Definition 2.1. A *standard Borel space* is a complete separable metric space equipped just with its σ-algebra of Borel sets.

- Examples include $\mathbb{R}$, $\mathbb{C}$, $[0,1]$, $\mathbb{N}^{\mathbb{N}}$, L^1, ℓ^2, $C[0,1]$, and Borel subsets of these. But as we have said, any standard Borel space is isomorphic to $2^{\mathbb{N}}$.

- Perhaps the most important example is the space of countable structures. A countable (first order) structure is simply a collection of relations on $\mathbb{N}$ (subsets of $\mathbb{N}^n$) and functions on $\mathbb{N}$ (subsets of $\mathbb{N}^{n+1}$). Countable graphs and groups are examples of countable structures.

- Any class of countable structures, such as the class of countable graphs or countable groups, can be realized as a Borel subset of the standard Borel space of all countable structures:
  $$\mathrm{Str} = \prod 2^{\mathbb{N}^{n_i}}$$
  Thus each of these classes can be coded as binary sequences too!

- We can actually go very far beyond countable structures, so long as we stick to objects that are in some sense countable determined. For example there are standard Borel spaces of
  - separable Banach spaces
  - measures on a compact separable space
  - measure-preserving actions of a fixed countable group

§3. Smooth equivalence relations

- Our classification problem asked, given a mathematical object and a notion of equivalence, how can we determine the object up to equivalence? In our theory the object corresponds to an element of a standard Borel space X. Hence the notion of equivalence corresponds to an *equivalence relation* E on X.

- Solving the classification problem for members of X up to E means assigning to each $x \in X$ an element $f(x) \in 2^{\mathbb{N}}$ which is a *complete invariant* in the sense that $x \; E \; x'$ iff $f(x) = f(x')$. In this case the classification problem is called *concretely classifiable* or *smooth*.

- With the axiom of choice, this can always be done trivially. Hence we actually require that the map f be explicitly given, for example, Borel measurable.

- Smooth vs. nonsmooth classification

  Example 3.1. The classification problem for permutations of $\mathbb{N}$ up to conjugacy is smooth. Here, σ, τ are conjugate iff they have the same cycle type, *i.e.*, the same number of cycles of each length. Hence they are classified by an element of the standard Borel space $(\mathbb{N} \cup \{\infty\})^{\mathbb{N} \cup \{\infty\}}$.

  Example 3.2. Let $=^*$ be the relation on $2^{\mathbb{N}}$ defined by $x =^* x'$ iff $x_n = x'_n$ for all but finitely many n. If $f \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ satisfies $x =^* x'$ iff $f(x) = f(x')$, then membership in the sets $f^{-1}\{x \mid x_n = i\}$ is independent of initial segments. The $0 - 1$ law then implies that f is constant on a conull set, a contradiction.

- Two examples of classes of groups.

  Example 3.3. An abelian group G is called *divisible* if for every $g \in G$ and $n \in \mathbb{N}$ we have $g/n \in G$. The classification of countable divisible groups is smooth. It is a classical fact that any divisible group can be expressed as a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$. Thus any such group is determined up to isomorphism by the sequence containing the number of factors of each type.

  Example 3.4. An abelian group G is said to be *torsion-free* if whenever $g \neq 1$ and $n \in \mathbb{N}$ we have $gn \neq 1$. A torsion-free group is said to have rank 1 if it is a subgroup of $\mathbb{Q}$. The classification of torsion-free groups of rank 1 is nonsmooth. Here if A is a torsion-free abelian group of rank 1 and $a \in A$, consider the set of p^n such that $p^n \mid a$. Then A is determined up to isomorphism by a tail of this sequence, and one can conclude as with $=^*$.

§4. Borel reducibility

- The relation $=^*$ plays a significant role in our theory.

Theorem 4.1. *$=^*$ is the canonical obstruction to smoothness. That is, every nonsmooth, Borel equivalence relation E contains a copy of $=^*$.*

- In the world of classification problems, this result is the first in an emerging geography of dividing lines and benchmark equivalence relations. In order to make this precise we have the following.

Definition 4.2. If E, F are equivalence relations on X, Y respectively, then E is said to be *Borel reducible* to F if there is a Borel function $f\colon X \to Y$ such that

$$x \; E \; x' \iff f(x) \; F \; f(x')$$

- For examples, divisible groups (smooth) lie below torsion-free groups of rank 1 ($=^*$), which in turn lies below all abelian groups, which lies below all countable groups.

§5. Classifiability by countable structures

- All of the equivalence relations which we have mentioned are examples of countable structures. More generally we have the following.

Definition 5.1. A classification problem is said to be *classifiable by countable structures* if it is Borel reducible to the isomorphism equivalence relation on Str.

Example 5.2. Let Aut$[0,1]$ be the class of order-preserving homeomorphisms of the unit interval, and let the notion of equivalence be conjugacy. Then an element is classified by its "bump" structure, that is, the linear ordering consisting of its bumps and fixed intervals, together with unary relations for the labels "up," "down," and "fixed."

- It is also possible to go beyond countable structures.

Example 5.3. Let Aut$[0,1]^2$ be the class of homeomorphisms of the unit square, and again consider them up to conjugacy. Hjorth showed that this is not classifiable by countable structures. That is, the natural generalization of the previous example (bump structure) to two dimensions (topographical structure) is a significant jump in complexity!