REVIEW OF INVARIANT DESCRIPTIVE SET THEORY, BY SU GAO

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Su Gao. **Invariant descriptive set theory**. *Pure and applied mathematics*. Chapman & Hall/CRC, Boca Raton, 2009. xiv + 392 pp.

Invariant descriptive set theory is the rigorous foundational study of classification problems in mathematics. The subject is also frequently called "Borel equivalence relations," a misnomer that Gao takes aim at with the title of his book. (Indeed, many of the most important equivalence relations turn out not to be Borel.) An important figure in the field once proposed "quantum set theory" as a sexy alternative. All of these names marginalize the most attractive feature of the subject: the classification problems it studies are naturally occurring (often classical), and span dozens of areas of mathematics.

Formally, the principle objects of study are definable equivalence relations on *standard Borel spaces*, that is, sets equipped with the Borel structure of a complete separable metric space. This is an extremely large collection of spaces encompassing virtually all continua studied in mathematics. Just a few examples: The unitary group of a separable Hilbert space, the space of countable groups, the space of Borel actions of a fixed countable group, and the space of complete separable metric spaces. Numerous classification problems correspond to natural equivalence relations on a standard Borel space. For the examples above, some relations of central importance would be conjugacy, isomorphism, equivariant isomorphism, and isometry, respectively.

The key notion in the subject is a measure of complexity of an equivalence relation called *Borel reducibility*, which was first made precise by Friedman and Stanley about 30 years ago. Here, if E and F are equivalence relations on standard Borel spaces X and Y, then we say E is Borel reducible to F if there is a Borel function $f\colon X \to Y$ which satisfies

$$x \mathrel{E} x' \iff f(x) \mathrel{F} f(x') \ .$$

Borel reducibility measures the complexity of equivalence relations not as sets of pairs, but as *classification problems*. That is, if E is Borel reducible to F, then the classification of elements of X up to E is no harder than the classification of elements of Y up to F.

Date: 2011.

From this point of view, the simplest classification problems are those which admit a system of complete invariants from a standard Borel space I. Indeed, so long as the invariants can be computed in an explicit (*viz.*, Borel) fashion, the corresponding equivalence relation will be Borel reducible to the equality relation on I, and a classical dichotomy theorem of Silver implies that the latter relation is minimum among Borel equivalence relations with uncountably many classes. Beyond this level, the Borel reducibility hierarchy becomes a wild and rich universe. In part, the subject aims to explore the structure of this universe, and to establish well-understood relations as benchmarks against which naturally occurring problems can be measured.

We now turn to Gao's exposition, which is divided into four sections. The first gives all of the set-theoretic grounding needed to make the book self-contained. While some of the exposition here is a little hurried (indeed, it would be impossible to also give a complete course in descriptive set theory), most of the background material is interesting in its own right. Gao treats a number of elementary subjects which are of critical importance, but at the same time are not easily found elsewhere: effective descriptive set theory, the universal (Urysohn) Polish space, and the general theory of Polish group actions. The last is vital to the development, since some of the most important equivalence relations can be realized as the orbit equivalence relation of a Polish group action.

In the second section, Gao collects together the deepest and most important results from the recent literature. The highlight here is his presentation of the Glimm-Effros dichotomy: if a Borel equivalence relation is not completely classifiable, then it embeds a copy of E_0 (the almost equality relation for infinite binary sequences). Gao gives this result its own chapter, beginning with Effros's version for F_σ orbit equivalence relations, moving on to the version stated above due to Harrington-Kechris-Louveau, and culminating with Becker's generalization to orbit equivalence relations of certain Polish groups.

Such general results are rare in a subject where the principal objects of study come from such diverse areas of mathematics as group theory, model theory, functional analysis, measure theory, representation theory, and so on. But there are several, and Gao has done a good job working them into his narrative: the Silver and Glimm-Effros dichotomies mentioned above, a theorem of Dougherty-Jackson-Kechris characterizing the hyperfinite Borel equivalence relations, the Friedman jump operation, the Burgess trichotomy theorem for analytic relations, Hjorth's theory of turbulence, and many more.

The final two sections are dedicated to finding the complexity of real life classification problems. Though many examples appear throughout the text, the most interesting problems are held until after Gao has treated the general theory. The third section deals with examples from countable model theory, including the classification of countable groups, graphs, and linear orders. For this, Gao first explores in detail the close connection between countable model theory and actions of the permutation group S_∞ of the naturals. The fourth section gives one of the most important examples: the classification of Polish metric spaces. These results are likely to be of interest to any mathematician, and they serve as good motivators for the first two sections.

This book has numerous strengths which make it recommendable for several uses. It is an excellent research reference, thanks largely to its thorough style and streamlined proofs of key results. Moreover, the last chapter contains indispensable diagrams summarizing the state of the Borel hierarchy of equivalence relations. The book also serves as an introduction to the subject for the working mathematician. Such readers should look to the beginning of the second section for the motivation, the first section for necessary background, and the final sections for concrete examples.

On the back cover this book advertises itself as a course textbook, and this is also a reasonable use. A course based on this book would be fairly advanced; incoming students would benefit greatly from familiarity with Kechris's text "Classical Descriptive Set Theory." That being said, we add that Gao's exercises are wonderful: they vary in difficulty from quite reasonable to somewhat challenging, and they manage to simultaneously relate strongly to the chapter text and to expand the scope of the book.