Page 49: Question 8.7c

Calculate the following products:

c) \( \prod_{k=1}^{n} \frac{(k+1)}{k} \), where \( n \) is a positive integer.

Answer:

When we begin to calculate the problem we see the following:

\[
2/1 * 3/2 * 4/3 * 5/4 * ... * (n-1)/n + 1/n - (n+1)/n = ?
\]

However, we quickly notice that the denominator of every number following the first, is the equivalent of the quotient of the previous fraction, in every case including the last. Therefore, we are able to cross cancel every term until we are left with the following expression(s):

\[
1/1 * (n+1)/1 = ?
\]

From this, we can easily multiply out the one and we are left with our solution:

\[ \text{Answer: n+1} \]

Page 49: Question 8.9

Prove that all of the following numbers are composite: 1000! + 2, 1000! + 3, 1000! + 4, ... , 1000! + 1002.

The point of this problem is to present a long list of consecutive numbers, all of which are composite.

Answer:

We are given a list of consecutive numbers, and are asked to prove that they are all composite.

To start with, the definition for a composite number is: "a positive integer \( a \) is called composite provided there is an integer \( b \) such that \( 1 < b < a \) and that \( b \mid a \)."
Following the guidelines of the definition, we can rewrite the first number in the list as follows:

\[ 1000! + 2 = (1000 \times 999 \times 998 \times 997 \times \ldots \times 3 \times 2 \times 1) + (2 \times 1). \]

Now we notice that we can pull out the common factor of 2 from both of the numbers as follows:

\[ 1000! + 2 = 2 \times (1000 \times 999 \times 998 \times 997 \times \ldots \times 3 \times 2 \times 1) + 1. \]

Since 2 is a factor of 1000! + 2, we have that 2 | 1000! + 2. Also, since 2 < 1000! + 2, we can conclude that 1000! + 2 is a composite number.

Moving onto the next number in the list, we attempt the same procedure as we did with the first number, as seen in the following steps:

\[ 1000! + 3 = (1000 \times 999 \times 998 \times 997 \times \ldots \times 3 \times 2 \times 1) + (3 \times 1). \]

\[ 1000! + 3 = 3 \times ((1000 \times 999 \times 998 \times 997 \times \ldots \times 2 \times 1) + 1). \]

Once again, we have a number (3) which is a factor of the number 1000! + 3, and therefore 3 | 1000! + 3. Also again, since 3 < 1000! + 3, we can conclude that 1000! + 3 is a composite number.

Because this method works for the entire list, up to 1000! + 1000, we can conclude that all of the number in the list from 1000! + 2 to 1000! + 1000 are composite. Now we must prove that the cases for which the added number is larger than 1000! are also composite numbers.

We can start with 1000! + 1001. If we rewrite the number using the definition of factorial, we have the following expression:

\[ 1000! + 1001 = (1000 \times 999 \times 998 \times 997 \times \ldots \times 7 \times 6 \times \ldots \times 2 \times 1) + (7 \times 143). \]

Since 1001 is an integer number, we can also write it as the product of 7 and 143. Now we remove any common factors, as we did in the previous examples.

\[ 1000! + 1001 = 7 \times ((1000 \times 999 \times 998 \times 997 \times \ldots \times 8 \times 6 \times \ldots \times 2 \times 1) + 143). \]

Once more, we have 7 as a factor of the number, therefore 7 | 1000! + 1001. Also, since 7 < 1000! + 1001, we can conclude that even for cases wherein the added number is higher than a multiple of 1000!, that 10000! + 1001 is a composite number.
For a final case, we will also show that our final element of our list is also composite, using the same steps as we did before:

\[1000! + 1002 = (1000 \times 999 \times 998 \times \ldots \times 501 \times 500 \times \ldots \times 2 \times 1) + (2 \times 501).\]

We can write that 1002 is equivalent to 2 \times 501 since 2 \times 501 = 1002.

Therefore, we can factor out the common multiple of 501 as follows:

\[1000! + 1002 = 501 \times ((1000 \times 999 \times 998 \times \ldots \times 502 \times 500 \times \ldots \times 2 \times 1) + 2)\]

Finally, since 501 is a factor of the expression, we have that 501 \mid 1000! + 1002. As well, since 501 < 1000! + 1002, we can make the conclusion that 1000! + 1002 is a composite number as well.

Now we are able to make our final conclusion for the problem, and using the data and information gained above, we can say that all of the numbers in the given list are all composite numbers.

Page 63: Question 10.4d

True of False: Please label each of the following sentences about integers as either true or false. (You do not need to prove your assertions.)

d. \exists x, \exists y, x + y = 0.

Answer: True

Page 74: Question 11.6

Suppose A, B, and C are pairwise disjoint sets. Prove or disprove: \(|A \cup B \cup C| = |A| + |B| + |C|\).

Answer:

Suppose that A, B, and C are pairwise disjoint sets. We are asked to prove that \(|A \cup B \cup C| = |A| + |B| + |C|\). We can start this problem by referencing the Addition principle, which is Corollary 11.8. It states that as long as A and B, are pairwise disjoint sets, that