1 Sample Space and Events

1.1 Definition

- **Experiment**: Some process which generates observations

- **Sample Space**: Collection of all possible outcomes of an experiment, denoted by $S$.
  
  Examples:
  
  - Toss a coin: $S = \{H, T\}$.
  - Tossing 2 coins: $S = \{HH, HT, TH, TT\}$.
  - Rolling 2 dices and calculating the sum: $S = \{2, 3, \cdots, 12\}$.

- **Event**: a collection of outcome in $S$. Or, a subset of $S$ and denoted by $A, B, \ldots$.

  Examples:
  
  - Tossing 2 coins:
    
    - $A =$ "the event of exactly one H" = $\{HT, TH\}$.
    - $B =$ "the event of at least one H" = $\{HT, TH, HH\}$.

- **Discrete Sample Space**: countable number of outcomes (eg. coin or die)

- **Continuous Sample Space**: uncountable number of outcomes (eg. the time I finish the exam).

- **Compound Event**: an event consisting of more than one outcomes

- **Simple Event**: an event consisting of only one outcome.
1.2 Connection to Set Theory

- **A and B are events, \( S \) is the sample space, and \( \emptyset \) is the empty set.**

- **Union:** \( A \cup B \)
  the set of all outcomes in A and B.

- **Intersection:** \( A \cap B \)
  the set of all outcomes in both A and B.

- **Complement:** \( A' \) or \( A^C \)
  the set of all outcomes not in A.

- **Mutually Exclusive:** A and B are mutually exclusive (or disjoint) if \( A \cap B = \emptyset \).

Example: Tossing 2 coins:

Let \( A = \{ \text{at least one H} \} \), \( B = \{ \text{at least one T} \} \), \( C = \{ \text{exactly one H} \} \), and \( D = \{ \text{both are same} \} \). Then, what is \( A \cup B \), \( A \cap B \), \( A' \)? Find all mutually exclusive pairs of sets?

- **Van–Diagram:**
2 Axioms and Properties of Probability

2.1 Axioms of Probability

(A1) \( P(A) \geq 0 \) for any event \( A \).

(A2) \( P(S) = 1 \)

From (A1&2), we know \( 0 \leq P(A) \leq 1 \).

(A3) For any finite collection of mutually exclusive events \( A_1, \ldots, A_k \)

\[
P(A_1 \cup A_2 \cup \cdots \cup A_k) = \sum_{i=1}^{k} P(A_i)
\]

Example: Two coin tosses \( S = \{HH, HT, Th, TT\} \). Let \( A = \{\text{at least one H}\} \), and \( B = \{\text{both are tail}\} \). Then check the above axioms using the set \( A \) and \( B \).

2.2 Basic Properties of Probabilities

- \( P(A) = 1 - P(A^c) \)
- If \( A \) and \( B \) are mutually exclusive, \( P(A \cap B) = 0 \).
- For any two events \( A \) and \( B \),

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

\[
P(A \cap B) = P(A) + P(B) - P(A \cup B)
\]

\[
P(A \cap B^c) = P(A) - P(A \cap B)
\]

- \( P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C)
\]

\[
-P(A \cap B) + P(A \cap B \cap C).
\]
• **Example 1** Five independent components are connected in series. The probability that each component works is 0.9. What is the probability the system does not work?

• **Example 2** The probability for subscribing “Metropolitan” is 0.6, the probability for subscribing “Local” is 0.8, and the probability for subscribing both “Metropolitan” and “Local” is 0.5. What is the probability for the event subscribing at least one of “Metropolitan” and “Local”? What is the probability for subscribing only “Metropolitan”?

• **Example 3** Two groups of students “Visa” or “Master”. 50% of students use “Visa”, 40% of students use “Master”, and 25% of students use both “Visa” and “Master”. What is the probability for using at least one of “Mater” and “Visa”? What is the probability that a selected student has no card? What is the probability that a selected student use “Visa” but not “Master”?

• **Example 4** Three types of defects: type 1, type 2, and type 3. Let $A_1$ = the product has type 1 defect, $A_2$ = the product has type 2 defect, and $A_3$ = the product has type 3 defect. Let

\[
P(A_1) = 0.12 \quad P(A_2) = 0.07 \quad P(A_3) = 0.05.
\]

\[
P(A_1 \cup A_2) = 0.13 \quad P(A_1 \cup A_2) = 0.14 \quad P(A_2 \cup A_3) = 0.1,
\]

and $P(A_1 \cap A_2 \cap A_3) = 0.01$. Then, what is the probability for the event that the product does not have type 1 defect, the probability for the event that the product has both type 1 and type 2 defect, and the probability for the event that the product has at most 2 types of defects?