Introduction to Differential Topology

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Abstract

This is a preliminary version of introductory lecture notes for Differential Topology. The presentation follows the standard introductory books of Milnor and Guilleman/Pollack. The difference to Milnor's book is that we do not assume prior knowledge of point set topology. All relevant notions in this direction are introduced in Chapter 1. Also the transversality is discussed in a broader and more general framework including basic vector bundle theory. We try to give a deeper account of basic ideas of differential topology than usual in introductory texts. Also many more examples of manifolds like matrix groups and Grassmannians are worked out in detail.

Chapter 1

Continuity, compactness and connectedness.

In this chapter we discuss the relevant topological properties of subsets of Euclidean spaces.

We use the usual symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ $\mathbb{C}$ for the integer numbers, non-negative integer numbers, rational numbers, real numbers and complex numbers.

For k a positive integer we define the *Euclidean spaces*

$$\mathbb{R}^k = \{(x_1, \ldots, x_k) | x_i \in \mathbb{R}, 1 \leq i \leq k\}$$

and $\mathbb{R}^0 := \{0\}$ is a single point.

For $x = (x_1, \ldots, x_k)$ and $x' = (x'_1, \ldots, x'_k)$ let

$$d(x, x') := \sqrt{\sum_{i=1}^{k} (x_i - x'_i)^2} = ||x - x'||$$

be the definition and equation relating *Euclidean metric* and *Euclidean norm.*

In the following we call the subsets of Euclidean spaces just *spaces.*

A function $f : X \to Y$ is *continuous* at $x \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(*) \qquad d(x, x') < \delta \Longrightarrow d(f(x), f(x')) < \varepsilon$$

For $x \in X$ and $\delta > 0$ let $D_X(x, \delta) := \{x' \in X | d(x, x') < \delta\}$ be the ball at x of radius δ.

Then $(*)$ above can be replaced by the condition

$$f(D_X(x, \delta)) \subset D_Y(f(x), \varepsilon)$$

A function $f : X \to Y$ is continuous if it is continuous at each point $x \in X$. The following is a list of properties of continuity. Some of these hold pointwise (Explain!).

1. If $f : X \to Y$ is continuous, and $X' \subset X, Y' \subset Y$ such that $f(X') \subset Y'$ then the *restriction*

   $$f|X' : X' \to Y'$$

   is continuous.

2. If $f : X \to Y$ and $g : Y \to Z$ are continuous then $g \circ f : X \to Z$ is continuous.

3. The inclusion $\iota : X \to Y$ of a subspace, i. e. $\iota(x) = x$ for $x \in X$ is continuous, in particular the identiy functions id_X are continuous.

4. The function $f : X \to \mathbb{R}^k$ is continuous if and only if all the component functions $f_i : X \to \mathbb{R}$ for $i = 1, \dots, k$ are continuous.

5. (i) linear and constant maps are continuous.

   (ii) the multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

   (iii) $x \mapsto \frac{1}{x}, \mathbb{R}^* = \mathbb{R} \setminus 0 \to \mathbb{R}$ is continuous.

Proof of (4.)(Rest is exercise) We discuss continuity at $x \in X$. $\Longrightarrow$: Suppose f is continuous at x, $\varepsilon > 0$ is given. Then there exists $\delta > 0$ such that for all x' with $d(x, x') < \delta$ we have

$$\varepsilon > d(f(x), f(x')) = \sqrt{\sum_{i=1}^{k} (f_i(x) - f_i(x'))^2} \geq |f_i(x) - f_i(x')| = d(f_i(x), f_i(x'))$$

for $i = 1, \dots, k$. Thus f_i is continuous at x for $i = 1, \dots, k$.

$\Longleftarrow$: Let $\varepsilon > 0$ be given. Choose $\delta_i > 0$ such that

$$d(x, x') < \delta_i \Longrightarrow |f_i(x) - f_i(x')| < \frac{\varepsilon}{\sqrt{k}}$$

Then for all x' with $d(x, x') < \delta := \min_i \delta_i$ we have

$$d(f(x), f(x')) = \sqrt{\sum_{i=1}^{k} (f_i(x) - f_i(x'))^2} < \sqrt{\sum_{i=1}^{k} \frac{\varepsilon^2}{k}} = \varepsilon.$$

Thus f is continuous at x. ■

Definition 1.1. (a) For $x \in X$, a subset $V \subset X$ is called a *neighborhood* of x if $x \in V$ and there exists $\delta > 0$ such that $D_X(x, \delta) \subset V$
(b) $U \subset X$ is *open* (in X) if U is neighborhood of each of its points.

Examples. (a) $X = \mathbb{Z} \subset \mathbb{R}$. Then $D_{\mathbb{Z}}(n, \frac{1}{2}) = \{n\}$. In particular all the sets $\{n\}$ are open in $\mathbb{Z}$.
(b) $D_X(x, \delta)$ is open in X. (Exercise 1.1)

Theorem 1.2. *Let X, Y be spaces and $f : X \to Y$. Then the following holds:*
(a) f is continuous at $x \iff$ for each neighborhood N of $f(x)$, $f^{-1}(N)$ is a neighborhood of x.(local characterization)
(b) f is continuous if and only if for all $V \subset Y$ open it follows that $f^{-1}(V) \subset X$ is open.(global characterization)

Proof We write D for both D_X or D_Y as long as this is clear from the context.
(a) $\Longrightarrow$: Let N be a neighborhood of $f(x)$. By definition of neighborhood there is $\varepsilon > 0$ such that $D(f(x), \varepsilon) \subset N$. By definition of continuity at x for this ε we find $\delta > 0$ such that $f(D(x, \delta)) \subset D(f(x), \varepsilon) \subset N$. Thus

$$f^{-1}(N) \supset f^{-1}(f(D(x, \delta)) \supset D(x, \delta)$$

and $f^{-1}(N)$ is neighborhood of x. $\Longleftarrow$: Let $\varepsilon > 0$ be given and $N := D(f(x), \varepsilon)$. This is open by Exercise 1.1 and thus neighborhood of $f(x)$. By assumption $f^{-1}(N)$ is a neighborhood of x. So there exists $\delta > 0$ such that $D(x, \delta) \subset f^{-1}(D(f(x), \varepsilon)$ which implies $f(D(x, \delta)) \subset D(f(x), \varepsilon))$.
(b) Note that by (a) the $\varepsilon - \delta$ condition is equivalent to the neighborhood definition. $\Longrightarrow$: Let $V \subset Y$ be open. In order to show $f^{-1}(V)$ open we show that $f^{-1}(V)$ is neighborhood of each of its points. Let $x \in f^{-1}(V)$ so $f(x) \in V$. Since V is a neighborhood, $f^{-1}(V)$ is a neighborhood of x. $\Longleftarrow$: Let $x \in X$ and $\varepsilon > 0$. Then $f^{-1}(D(f(x), \varepsilon)) \subset X$ is an open subset of X so neighborhood of each of its points. So there is $\delta > 0$ such that $D(x, \delta) \subset f^{-1}(D(f(x), \varepsilon))$, which implies the $\varepsilon\delta$-condition as above. ■

Theorem 1.3. $\emptyset$ and X are open. Intersections of finitely many open sets, and arbitrary unions of open sets are open.

We leave the proof as an exercise.

Remark. Let X be any set and let $\mathcal{O}$ be a system of subsets with $\emptyset, X \in \mathcal{O}$ and satisfying (i) $U, V \in \mathcal{O} \Longrightarrow U \cap V \in \mathcal{O}$, and (ii) $U_i \in \mathcal{O}$ for all $i \in J$ and any set $J \Longrightarrow \cup_{i \in J} U_i \in \mathcal{O}$. Then $(X, \mathcal{O})$ is called a *topological space*. The notion of continuity between topological spaces is easily defined by 1.2 (b). Many results discussed in chapter 1 hold for arbitrary topological spaces but not all. Subsets of Euclidean spaces are examples of so called *metrizable* topological spaces.

Definitions 1.4. (i) For $Y \subset X$ define the *interior* of Y in X by

$$int_X(Y) := \bigcup_{\substack{U \subset Y \\ U \subset X \text{ open}}} U$$

Define the *boundary* of Y in X to be

$$fr_X(Y) := X \setminus (int_X(Y) \cup int_X(X \setminus Y)).$$

(ii) $F \subset X$ is *closed*:$\Longleftrightarrow X \setminus F$ is open.
(iii) Define the *closure* of Y in X by

$$cl_X(Y) := \bigcap_{\substack{F \supset Y \\ F \subset X closed}} F.$$

Remarks. (a) If the space X is clear from the context we omit the subscript and also write $\bar{Y}$ for $cl_X(Y)$.
(b) $cl_X(Y) = int_X(Y) \cup fr_X(Y) \supset Y$ by Exercise 1.2.

In $\mathbb{R}^k$ there is a particularly nice system of open sets: Consider for all $a \in \mathbb{Q}^k$ and $q \in \mathbb{Q}$ the ball

$$D(a, q) = \{x \in \mathbb{R}^k | d(a, x) < q\}.$$

This system is in $1-1$ correspondence with $\mathbb{Q}^k \times \mathbb{Q} = \mathbb{Q}^{k+1}$ thus is countable.

Theorem 1.5. *Each open set $W \subset \mathbb{R}^k$ is a union of subsets* (i) $D(a, q)$, or (ii) $cl(D(a, q))$, for suitable subsets of $\mathbb{Q}^{k+1}$, *thus in particular is countable union of those sets.*

Proof. Let $W \subset \mathbb{R}^k$ be open. For $a \in W \cap \mathbb{Q}^k$ let

$$T(a) := \{(a, q) \in \mathbb{Q}^{k+1} | D(a, q) \subset W\}$$

and let

$$T := \bigcup_{a \in W \cap \mathbb{Q}^k} T(a) \subset \mathbb{Q}^{k+1}.$$

Claim: $W = \cup_T D(a,q) = \cup_T cl(D(a,q))$

Proof of Claim: Let $x \in W$. Since W is open there is $\varepsilon > 0$ such that $D(x,\varepsilon) \subset W$. Let $\lambda \in \mathbb{Q}$ with $0 < \lambda < \frac{\varepsilon}{2}$ and choose $a \in D(x,\lambda) \cap \mathbb{Q}^k$. Then $cl(D(a,\lambda)) \subset W$. In fact $y \in cl(D(a,\lambda)) \implies d(a,y) \leq \lambda \implies d(y,x) \leq d(y,a) + d(a,x) < \lambda + \lambda < \varepsilon \implies y \in D(x,\varepsilon) \subset W$. Moreover $x \in D(a,\lambda)$ because $d(x,a) = d(a,x) < \lambda$, and $(a,\lambda) \in T$ because $D(a,\lambda) \subset cl(D(a,\lambda)) \subset W$. ■

The following technical statement will be helpful in many situations:

Theorem 1.6. *Let $X \subset \mathbb{R}^k$ and $(U_\alpha)_{\alpha \in J}$ be a family of open sets in $\mathbb{R}^k$ with $\cup U_\alpha \supset X$. Then there exists a countable family $(V_i)_{i \in \mathbb{N}}$ with* (i) $\cup V_i \supset X$*, and* (ii) *for each i there exists $\alpha \in J$ such that $V_i \subset U_\alpha$.*

Proof. Just let $U_\alpha = \cup_{(a,q) \in T_\alpha} D(a,q)$ for suitable sets $T_\alpha \subset \mathbb{R}^k$ and let $T := \cup_{\alpha \in J} T_\alpha$. Then after choosing a suitable bijection $\mathbb{N} \leftrightarrow T$ we can identify the collection of $D(a,q)$ for $(a,q) \in T$ with a collection denoted V_i. By construction for each i and thus $(a,q) \in T$ we find $\alpha \in J$ with $D(a,q) \subset U_\alpha$. ■

Theorem 1.7. (pasting) *Let $X = X_1 \cup X_2$ with $X_i \subset X$ closed for $i = 1,2$. Let $f : X \to Y$ be a continuous function with $f|X_i$ continuous for $i = 1,2$. Then f is continuous.*

Proof. For $C \subset Y$ closed note that

$$f^{-1}(C) = (f|X_1)^{-1}(C) \cup (f|X_2)^{-1}(C).$$

By continuity of f_i we conclude $(f|X_i)^{-1}(C) \subset X_i$ is closed and by Exercise 1.4 (c) conclude $(f|X_i)^{-1}(C) \subset X$ is closed for $i = 1,2$. A union of two closed sets is closed, and the result follows. ■

Definition 1.8. A continuous bijection $f : X \to Y$ is called a *homeomorphism* if its inverse $f^{-1} : Y \to X$ is also a continuous function. We write $X \approx Y$ if there exists a homeomorphism between X and Y.

Examples. (a) Let $T := \{(x_1,x_2) \in \mathbb{R}^2 | |x_1| + |x_2| = 1\}$ and let $S^1 := \{(x_1,x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$ be two subsets of $\mathbb{R}^2 = \mathbb{C}$. The space S^1 is called the *unit circle*.

Claim: $T \approx S^1$

Proof. Let $f : S^1 \to T$ and $g : T \to S^1$ be defined by

$$f(x_1, x_2) = (\frac{x_1}{|x_1| + |x_2|}, \frac{x_2}{|x_1| + |x_2|})$$

and

$$g(x_1, x_2) = (\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}).$$

Then it is easy to check that $f \circ g = id_T$ and $g \circ f = id_{S^1}$. Both functions are continuous (extend to $\mathbb{R}^2 \setminus 0$ and conclude from multi-variable calculus), so $g = f^{-1}$ and f is a homeomorphism. ■

(b) Let

$$f : \mathbb{R} \supset [0,1) \to S^1$$

be defined by

$$f(\theta) = e^{2\pi i\theta} = (\cos(2\pi\theta), \sin(2\pi\theta)).$$

The f is a continuous bijection. But $[0,1) \not\approx S^1$ because deletion of a point from $[0,1)$ separates $[0,1)$ while deletion of any point from S^1 does not separate S^1 (compare the following discussion of connecteness).

Products of spaces are introduced in the obvious way: $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^\ell$ then $X \times Y \subset \mathbb{R}^{k+\ell}$. Note that the projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous (check that preimages of open sets are open).

Examples. (a) $S^1 \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ is the *unit cylinder*, $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^{\nvDash} = \mathbb{R}^4$ is the *torus surface*. This space is actually homeomorphic to a doughnut surface in $\mathbb{R}^3$, see exercise 2.2.

(b) $(0,1) \times [0,1) \approx [0,1] \times [0,1)$ (but $(0,1) \not\approx [0,1]$ because the first space is not compact while the second space is). Thus the cancellation rule is **not** true for $\approx$. We construct an explit homeomorphism $f : (0,1) \times [0,1) \to [0,1] \times [0,1)$ as follows:

$$f(x) = \begin{cases} (\frac{x_2}{2}, 1 - 3x_1) & \text{for } 3x_1 \leq 1 - x_2 \\ (x_1 + (1-2x_1)\frac{2x_2-1}{2x_2+1}, x_2) & \text{for } 1 - x_2 \leq 3x_1 \leq 2 + x_2 \\ (1 - \frac{x_2}{3}, 3x_1 - 2) & for\ 3x_1 \geq 2 + x_2 \end{cases}$$

This function is continuous by pasting and easily seen to be a homeomorphism (compare also 1.13).

Definition 1.9. A space $X \subset \mathbb{R}^k$ is compact if for each open covering $(U_\alpha)_{\alpha \in J}$, i. e. $U_\alpha \subset \mathbb{R}^k$ is open and $\cup U_\alpha \supset X$, there exist finitely many $\alpha_1, \ldots, \alpha_r \in J$

such that

$$U_{\alpha_1} \cup \ldots \cup U_{\alpha_r} \supset X.$$

Then $(U_{\alpha_i})_{i=1,\ldots,r}$ is called a *subcovering*.

Remark. Compactness is an intrinsic property, which does not depend on the embedding in $\mathbb{R}^k$. In fact the system of open sets U_α in the definition above can be replaced by the system of sets $U_\alpha \cap X$ of open sets in X (see exercise 1.4 (a)). (Check that this gives rise to an equivalent but intrinsic definition!)

Theorem 1.10. *If $Y \subset X$ is compact then Y is closed in X.*

Proof. By Exercise 1.2 (b) it suffices to show $Y = cl_X(Y)$, i. e. $x \in X \setminus Y \Longrightarrow x \notin cl_X(Y)(\Longleftrightarrow d(x,Y) > 0)$. Now $d(x,y) > 0$ for all $y \in Y$ is obvious. For $y \in Y$ we know that $D_Y(y, \frac{1}{2}d(x,y))$ is open in Y thus the collection of these balls, ranging over all $y \in Y$, defines an open covering of Y. By compactness we get

$$Y = \bigcup_{i=1}^{r} D_Y(y_i, \frac{1}{2}d(x,y_i))$$

for some $y_1, \ldots, y_r \in Y$. So $y \in Y \Longrightarrow y \in D_Y(y_i, \frac{1}{2}d(x,y_i))$ for some i with $1 \le i \le r \Longrightarrow d(y,y_i) < \frac{1}{2}d(y_i,x)$. So

$$d(x,y) \ge d(y_i,x) - d(y_i,y) > \frac{1}{2}d(y_i,x) \ge min_i \frac{1}{2}d(y_i,x) =: \rho > 0.$$

Thus $d(x,Y) \ge \rho > 0$ and the claim follows. ■

Theorem 1.11. *If X is compact and $A \subset X$ is closed then A is compact.*

Proof. Let A be closed and $(U_\alpha)_{\alpha \in J}$ with $U_\alpha \subset X$ open and $\cup_{\alpha \in J} U_\alpha \supset A$. Then $X \setminus A$ is open in X and

$$X = (X \setminus A) \cup \bigcup_{\alpha \in J} U_\alpha.$$

shows that $(X \setminus A, (U_\alpha)_{\alpha \in J}$ is an open covering of X. By compactness of X there is a finite subcovering of X. If we discard $X \setminus A$ from this covering we get an open subcovering of A. ■

Theorem 1.12. *If X is compact and $f : X \to Y$ is continuous then $f(X)$ is compact.*

Proof. Let (U_α) be a covering of $f(X)$ by open subsets of Y. Then $f^{-1}(Y) \subset X$ is open because f is continuous and $\cup_\alpha f^{-1}(U_\alpha) = X$. By compactness

$$X = f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_r})$$

for some $\alpha_1, \ldots, \alpha_r$. It follows that

$$\bigcup_{i=1}^{r} U_{\alpha_i} \supset f(X).$$

■

Theorem 1.13. *If X is compact and $f : X \to Y$ is a continuous bijection then f is a homeomorphism.*

Proof. We have to show that f^{-1} is continuous, or $(f^{-1})^{-1}(A) = f(A)$ is closed for $A \subset X$ closed. But $A \subset X$ closed implies by 1.11 A compact, which imlpies by 1.12 $f(A)$ compact and thus by 1.10 $f(A)$ is closed. ■

Theorem 1.14. (Heine-Borel) *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

For the proof see any book on Analysis, e. g. Rudin's textbook.

Definition 1.15. A pair (U, V) is called a *separation* of a space X if U and V are nonempty and disjoint and $X = U \cup V$.

Remarks. (a) The sets U, V of a separation of X are also closed in X. Moreover, if $\emptyset \neq U \neq X$ is a subset of X and open and closed then $(U, X \setminus U)$ is a separation of X.
(b) U is both open and closed in $X \iff fr_X U = \emptyset$.(*Proof:* $\Longrightarrow$: $fr_X U = X \setminus (int_X U \cup int_X(X \setminus U))$. Since U, $X \setminus U$ are open it follows that $U = int_X U$ and $X \setminus U = int_X(X \setminus U)$ and thus $fr_X(U) = \emptyset$. $\Longleftarrow$: $fr_X U = \emptyset$ implies $X = int_X U \cup int_X(X \setminus U$. Because $int_X U \subset U$ and $int_X(X \setminus U) \subset X \setminus U$ it follws that $int_X U = U$ and $int_X(X \setminus U) = X \setminus U$ which implies that U and $X \setminus U$ are open. ■)

Definition 1.16. A space is connected if it has no separation.

Example. If a space X is discrete and has at least two points then $(\{x\}, X \setminus x)$ is a partition of X and X is not connected. Note that $\{0, 1\} \subset \mathbb{R}$ is a disconnected space.

Theorem 1.17. *A space X is connected if and only if there exists a continuous onto map $f : X \to \{0,1\}$.*

Proof. If (U,V) is a separation define $f : X \to \{0,1\}$ by $f|U$ respectively $f|V$ the constant map onto 0 respectively 1. Conversely define a separation by $U = f^{-1}(0)$ and $V = f^{-1}(1)$. ■

Remark 1.18. If $f : X \to Y$ is continuous and X is connected then Y is connected (*pull back* a partition of Y).

Example. Intervals are connected (proof see Analysis).

Lemma 1.19. (a) *Let (U,V) be a separation of X and $W \subset X$ be connected. Then $W \subset U$ or $W \subset V$.*
(b) *Let $(C_\alpha)_{\alpha \in J}$ be a family of connected spaces in some $\mathbb{R}^k$ with*

$$\bigcap_{\alpha \in J} C_\alpha = \neq \emptyset$$

then

$$C := \bigcup_{\alpha \in J} C_\alpha$$

is connected.

Proof. (a) Otherwise $(W \cap U, W \cap V)$ is a separation of W. (b) Suppose that (U,V) is a separation of C. Since C_α is connected we have for all j: either $C_\alpha \subset U$ or $C_\alpha \subset V$. Since $U \cap V = \emptyset$ but $\cap_{alpha|inJ} C_\alpha \neq \emptyset$ it follows that either $C_\alpha \subset U$ for all j, or $C_\alpha \subset V$ for all j holds. So without restriction $C \subset U$ thus $V = \emptyset$, which contradicts to (U,V) being a separation. ■

Example. In $\mathbb{R}^n$ consider any union of lines through the origin. This is a connected space.

Now for $x \in X$ let

$$C_x := \bigcup_{\substack{C \text{ connected} \\ x \in C}} C,$$

which is a connected space by 1.19 (b). C_x is the maximal connected set containing x or the *component* of X containing x.

Theorem 1.20. *Let $x, y \in X$. Then either $C_x \cap C_y = \emptyset$ or $C_x = C_y$.*

Thus the collection of C_x forms a partition of the space X. So we can decompose X into the corresponding equivalence classes

$$X = \bigcup_{j \in J} C_j$$

where C_j are the distinct components of X, in particular $C_i \cap C_j = \emptyset$ if $i \neq j$.

Proof. If $C_x \cap C_y \neq \emptyset$ then $C_x \cup C_y$ is connected by 1.19, $x \in C_x \cup C_y \implies C_x \cup C_y = C_x \implies C_y \subset C_x$. Similarly $C_x \subset C_y$. ■.

Example. a discrete space (i. e.åll subsets are open $\iff$ points are open sets) has the points as components.
(b) $\mathbb{Q} \subset \mathbb{R}$ has components points. (Let $U \subset \mathbb{Q}$ be any connected set with at least two points, without restriction of the form $q_1 < q_2$. Let $r \in \mathbb{R}$ be a real number with $q_1 < r < q_2$. Then $(-\infty, r) \cap U, (r, \infty) \cap U)$ is a separation of U.

A map $f : X \to Y$ is locally constant if for each $x \in X$ there is some neighborhood U of x sich that $f|U$ is a constant map (onto $f(x)$). Note that locally constant implies continuous.

Theorem 1.21. *Let $f : X \to Y$, X connected and Y discrete. If f is locally constant then f is constant.*

Proof. Let $y = f(x) \in Y$. Let $U := \{x \in X | f(x) = y\}$ and $V := \{x \in X | f(x) \neq y$. Then both U and V are open (because preimages of open sets). Thus $V = \emptyset$ (because otherwise (U, V) is a separation of X. ■
Definition 1.22. (a) A continuous map $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$ is called a *path* in X from x to y.
(b) X is path connected if for all $x, y \in X$ there is a path from x to y in X.

Note that paths can sometimes be added. If γ_1 is a path from x to y and γ_2 is a path from y to z, both paths in X, then γ defined by

$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a path from x to z (continuous by pasting).
Notation: $\gamma = \gamma_1 \gamma_2$

Examples 1.23. convex or star-shaped spaces are obviously path connected. $\mathbb{R} \setminus 0$ is not path connected (by the intermediate value theorem), but $\mathbb{R}^n \setminus 0$ is path connected for $n > 1$. *Proof:* Let $x, y \in \mathbb{R}^n \setminus 0$ with $x \neq y$ and

$$\ell(t) := x + t(y - x)$$

be the sement from x to y. Then $\ell(t) = 0$ if and only if $x(1-t) = -ty \Longleftrightarrow y = \frac{t-1}{t}x$. Note that $t \neq 0$ or $x = 0$. Let $\lambda := \frac{t-1}{t} < 0$. **Case 1:** If $y \neq \lambda x$ for all $\lambda < 0$ then we are done. **Case 2:** Otherwise segments in $\mathbb{R}^n$ define paths from x to $\hat{x} := \frac{x}{||x||}$ and y to $-\hat{x}$. Then use a rotation $R(t)$ in a plane spanned by $\hat{x}$ and some linearly independent vector z (note $n > 1$ is used precsely here) to define a path from $\hat{x}$ to $-\hat{x}$. In fact if $R(t)$ is rotation about $\frac{t}{\pi}$ for $0 \leq t \leq 1$ then the path $\gamma(t) = R(t)\hat{x}$ is a path from $\hat{x}$ to $-\hat{x}$. Continuity follows from the continuity of $t \mapsto R(t)$ by identifying the set of rotation with a subset of the 2×2-matrices, which can be identified with $\mathbb{R}^4$. ■

Note that the rotation argument from above shows that $n-1$-*spheres*

$$S^{n-1} = \{x \in \mathbb{R}^n | ||x|| = 1\}$$

are path connected for $n > 1$. But $S^0 = \{-1, 1\} \subset \mathbb{R}$ is not path connected.
Remarks. (a) Products of path connected spaces are path connected (proof is clear, take products of paths).
(b) If $f : X \to Y$ is continuous and X is path connected then $f(X)$ is path connected. (*Proof.* Let $y_1, y_2 \in f(X)$ and $y_i = f(x_i)$ for $i = 1, 2$ and let γ be a path in X from x_1 to x_2. Then $f \circ \gamma$ is a path in $f(X)$ from y_1 to y_2.)

Proposition 1.24. *A path connected space X is connected.*
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Proof. Let X be path connected but not connected. Then there is $f : X \to \{0, 1\}$ continuous and onto. Let $x, y \in X$ with $f(x) = 0$ and $f(y) = 1$ and $\gamma : [0, 1] \to X$ be a path from x to y. Then $f \circ \gamma$ is a path in $\{0, 1\}$ from 0 to 1. But each path in $\{0, 1\}$ is constant by 1.21. This is a contradiction. ■

Example (Topologist's sine curve). Let $X = A \cup B$ with

$$A := \{(x, y) \in \mathbb{R}^2 | x = 0, y \in [-1, 1]\}$$

and

$$B := \{(x, y) \in \mathbb{R}^2 | y = \sin \frac{1}{x}, x \in (0, 1]\}$$

One can show that X is connected but not path connected (see e. g. James R. Munkres, Topology)

Theorem 1.25. *Let X be connected such that each $x \in X$ has a path connected neighborhood. Then X is path connected.*

Proof. Suppose X is not path connected. Then let

$$U := \{z \in X | \text{ there is a path from } z \text{ to } x \text{ in} X\}$$

and let $V := X \setminus U$. We claim that (U, V) is a separation. Now $U \cup V = X$ and $U \cap V = \emptyset$ are imediate from the definitions. Let $u \in U$. There is a neighborhood W of u in X which is path connected. It follws that $W \subset U$ because each $w \in W$ can be joined with u by a path in W and then with x. Thus U is open. Similarly, let $v \in V$ and W be a path connected neighborhood of v. If there exists some $u \in W \cap U$ then there is a path from v to x (through u). Thus $W \cap U = \emptyset$ which implies $W \subset V$. Thus V is also open. ■

Chapter 2

Smooth manifolds and maps.

Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^\ell$ be open. A map $f : U \to V$ is *smooth* (or $\mathcal{C}^\infty$) if all partial derivatives:

$$\frac{\partial^j f}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_j}} : U \to \mathbb{R}^\ell,$$

for $1 \leq i_1, \dots, i_j \leq k$ and all $j \geq 1$, exist and are continuous.

Recall that smooth functions in particular are differentiable in the sense of multi-variable calculus, and thus are continuous. The smoothness of f at x can be defined whenever the domain of f is a neighborhood of x (because this neighborhood contains an open disk around x, and that's all we need to define partial derivatives or differentiablity).

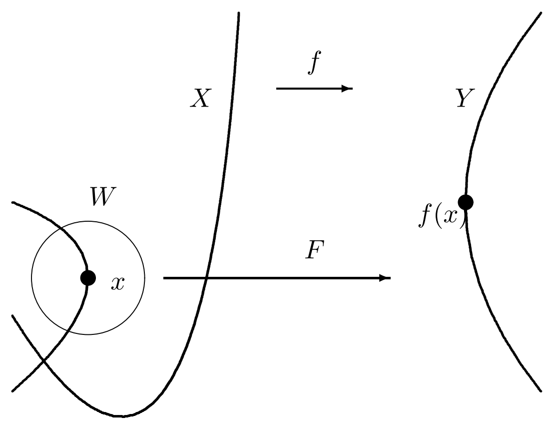
Recall:

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \dots, x_k) = \lim_{h \to 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_k) - f(x_1, \dots, x_k)}{h},$$

We will use some results about differentiable functions without proving them here. Recall that for $f = (f_1, \dots, f_\ell)$ as above, the Jacobi-matrix at some point x is the matrix of first order partial derivatives $(\frac{\partial f_i}{\partial x_j}(x))_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq k}}$. This is an $\ell \times k$-matrix and is the matrix representative of the derivative Df of f at x. Thus $Df(x)$ is a linear map $\mathbb{R}^k \to \mathbb{R}^\ell$. We will give another definition of $Df(x)$ in Step 1 below.

Definition 2.1 Let $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^\ell$ be spaces and $x \in X$. Then a map

$f : X \to Y$ is *smooth at* x if there exists a neighborhood $U \subset \mathbb{R}^k$ of x and a smooth map $F : U \to \mathbb{R}^\ell$ such that $F|(U \cap X) = f|(U \cap X)$. ($F$ is called a *smooth local extension* of f at x). f is *smooth* if f is smooth at x for all $x \in X$.



The smoothness of F is defined using derivatives because U is a neighborhood of x. Note than each smooth local extension of f at x on some neighborhood of x restricts to a smooth local extension on some *open* neighborhood of x.

Remarks. (a) For $X \subset \mathbb{R}^k$, id_X is smooth because it extends to $id_{\mathbb{R}^k}$.
(b) If f is smooth at x then f is continuous at x. This follows because a smooth local extension F is continuous, and restrictions of continuous maps are continuous.
(c) Suppose that f is smooth at x and g is smooth at $f(x)$. Then $g \circ f$ is smooth at x. To see this choose a smooth local extension $F : U \to \mathbb{R}^\ell$ of f at x with U open. Likewise choose a smooth local extension $G : V \to \mathbb{R}^m$ of g at $f(x)$ with $f(x) \in V$ and $V \subset \mathbb{R}^\ell$ open. Since F is continuous and V is open, $F^{-1}(V) \subset U$ is open in U and thus $\mathbb{R}^k$ by Exercise 1.4 (c), with $x \in F^{-1}(V)$. Thus

$$\mathbb{R}^k \supset F^{-1}(V) \xrightarrow{G \circ F} \mathbb{R}^m$$

is smooth, and because

$$(G \circ F)|F^{-1}(V) \cap X = (g \circ f)|F^{-1}(V) \cap X$$

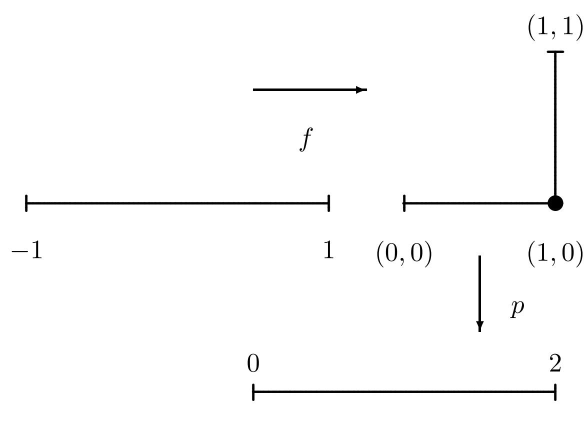
$G \circ F$ defines a smooth local extension of $g \circ f$ at x. ■

Definition 2.2. A map $f : X \to Y$ is is a *diffeomorphism* if f is a differentiable homeomorphism and f^{-1} is smooth. Notation: $X \cong Y$.

Examples. (a) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$ is a smooth homeomorphism but $f^{-1}(x) = \sqrt[3]{x}$ and f^{-1} is not smooth.
(b) Let $X = [-1, 1] \subset \mathbb{R}$ and let $Y = \{(x, y) \in \mathbb{R}^2 | (y = 0 \text{ and } 0 \leq x \leq 1) \text{ or } (x = 1 \text{ and } 0 \leq y \leq 1\}$. It is an easy exercise to prove that X and Y are homeomorphic. Suppose there is a diffeomorphism $f = (f_1, f_2) : X = [-1, 1] \to Y \subset \mathbb{R}^2$. Then (i) $f(x) = (1, 0)$ for some $x \in (-1, 1)$ because for a boundary point x of $[-1, 1]$ it follows $X\backslash x$ is connected. But $Y\backslash(1, 0)$ is not connected. We can assume without restriction that $f(0) = (0, 1)$. (ii) $f'(0) \neq 0$, because: Let G be smooth local extension of f^{-1} at $(1, 0)$. Then $(G \circ f)'(0) = G'(1, 0)f'(0) = 1$ (since $G \circ f = id$ near 0). (This is an application of the chain rule, see 2.6 (a), which actually reads

$$\left(\frac{\partial G}{\partial x_1}(1,0), \frac{\partial G}{\partial x_2}(1,0)\right)\begin{pmatrix} f_1'(0) \\ f_2'(0) \end{pmatrix} = 1.)$$

(iii) Let $p : Y \to [1, 2]$ be defined by $p(x, y) = x$ for $y = 0$ and $0 \leq x \leq 1$ and $p(x, y) = y + 1$ for $x = 1$ and $0 \leq y \leq 1$. Then p is a homeomorphism too. We can assume without restriction that $p \circ f$ is increasing. Then consider $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$. It follows that $\lim_{h\to 0, h<0} \frac{f(h)-f(0)}{h}$ is a positive multiple of e_1 while $\lim_{h\to 0, h>0} \frac{f(h)-f(0)}{h}$ is a negative multiple of e_2, where e_i is the standard basis vector of $\mathbb{R}^2$ for $i = 1, 2$. Thus f is not differentiable. Note that there exist smooth maps $[-1, 1] \to Y$ (find one!) but no diffeomorphisms.

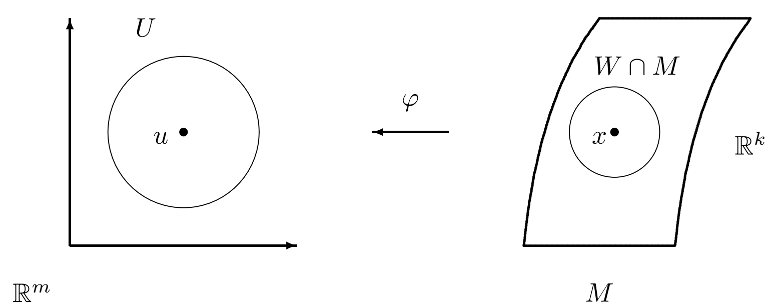


(c) For $a, b > 0$ each ellipse $E_{a,b} = \{(x_1, x_2) | (\frac{x_1}{a})^2 + (\frac{x_2}{b})^2 = 1\}$ is diffeomorphic to $S^1 = E_{1,1}$. In fact $f : S^1 \to E_{a,b}$ defined by $f(x_1, x_2) = (ax_1, bx_2)$ is a diffeomorphism with inverse $(x_1, x_2) \mapsto (\frac{x_1}{a}, \frac{x_2}{b})$.

The goal of *differential topology* could be the classification of arbitrary subsets of Euclidean spaces up to diffeomorphism. But this is a hopeless problem because subsets of Euclidean spaces can be very difficult. What is missing up to this point is some inner structure of the sets X, Y, which allows to transport the usual notions of calculus into this more general setting.

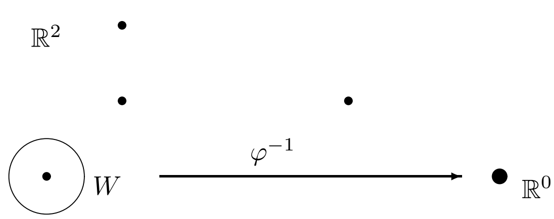
Definition 2.3. A space $M \subset \mathbb{R}^k$ is called a *smooth manifold* (of *dimension* m) if each point $x \in M$ has a neighborhood $W \subset \mathbb{R}^k$ such that $W \cap M$ is diffeomorphic to some open subset of $\mathbb{R}^m$. (Notation: $dim(M) = m$.)

A smooth diffeomorphism $\varphi : \mathbb{R}^m \supset U \to W \cap M$ is called a *parametrization* of $W \cap M \subset M$. The inverse map φ^{-1} then is called a *coordinate system* with *chart* $W \cap M$. Obviously parametrizations and coordinate systems are not unique. Sometimes we also call the composition of a parametrization φ with the inclusion of $\varphi(U)$ into M or $\mathbb{R}^k$ a parametrization.



Actually we should say more precisely *smooth submanifold of* $\mathbb{R}^k$ above. We will come back to the intrinsic nature of smooth manifolds later on. It will follow from 2.8 (b) that $m \leq k$.

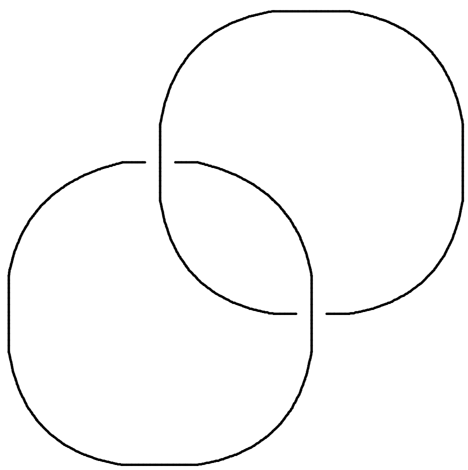
Examples. (a) Let $m = 0$. Then the open neighborhood W of $x \in M$



with $W \cap M \cong \mathbb{R}^0$ shows that each point of M has neighborhood only containing x. Thus 0-dimensional manifolds are just discrete subsets of Euclidean spaces.

(b) An open subset $U \subset \mathbb{R}^m$ is a trivial example of a smooth manifold of dimension m, with parametrization given by id_U.
(c) If M is a smooth manifold of dimension m and $N \subset M$ is an open subset of M then N is a smooth manifold of the same dimension. In fact let $x \in N$ and let $\varphi : U \to M$ be a parametrization at x. Then $\phi := \varphi|(\varphi^{-1}(N))$ is a parametrization for N at x. Note that $\varphi^{-1}(N)$ is open in $\mathbb{R}^m$ because it is open in U.

A 1-dimensional compact smooth manifold $M \subset \mathbb{R}^3$ is called a *link*. If M is connected it is called a *knot*. These names actually refer to M as submanifold. Each such M is diffeomorphic to a disjoint union of circles S^1, see Milnor: Appendix.



Remarks 2.4 (a) Each manifold can be covered by finitely many charts. This follows immediately from 1.6.
(b) (Smooth maps defined on smooth manifolds) Let M be a smooth manifold. Then $f : M \to \mathbb{R}^\ell$ is a smooth map at $x \in M$ in the sense of 2.1. $\Longleftrightarrow$ there is a parametrization $\varphi : U \to M$ such that $f \circ \varphi : \mathbb{R}^m \supset U \to \mathbb{R}^\ell$ is smooth. *Proof.* $\Longrightarrow$: for any parametrization φ on U and smooth local extension $F : W \to \mathbb{R}^\ell$ of f at x with W open, $F \circ \varphi = f \circ \varphi$ is smooth on $\varphi^{-1}(W)$. $\Longleftarrow$: The inverse φ^{-1} defined on a chart of M extends locally around x to a smooth map $\psi : \mathbb{R}^k \supset V \to \mathbb{R}^m$. Then $F := (f \circ \varphi) \circ \psi$ is a smooth local extension of f at x and is defined on a suitable neighborhood of x. Thus f is smooth at x in the sense of 2.1. ■
In particular diffeomorphism of smooth manifolds can be defined using 2.1. or

the alternative description using parametrizations.

Examples 2.5. Recall that

$$S^{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \ldots + x_n^2 = 1\} \subset \mathbb{R}^n$$

Claim: S^{n-1} is a smooth manifold of dimension $(n-1)$.
Proof. Let $y = (y_1, \ldots, y_n) \in S^{n-1}$. Then there is some $1 \le i \le n$ with $y_i \neq 0$.
Case 1: $y_i > 0$. Let $W_i := \{x \in \mathbb{R}^n | x_i > 0\}$. This is open in $\mathbb{R}^n$ and $y \in W_i \cap S^{n-1} = \{x \in S^{n-1} | x_i > 0\}$. Define:

$$\psi_i : W_i \cap S^{n-1} \to D^{n-1} := \{x \in \mathbb{R}^{n-1} | x_1^2 + \ldots + x_{n-1}^2 < 1\}$$

by

$$(*) \qquad \psi_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

Note that $\psi_i(W_i \cap S^{n-1}) \subset D^{n-1}$ is clear because $x_1^2 + \ldots + x_n^2 = 1$ and $x_i > 0$. Also ψ_i is smooth because $(*)$ can be used to extend to a smooth map $\Psi_i : W_i \to \mathbb{R}^{n-1}$.

Now define

$$\varphi_i : D^{n-1} \to W_i \cap S^{n-1}$$

by

$$\varphi_i(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \sqrt{1 - \sum_{j=1}^{n-1} x_j^2}, x_i, \ldots, x_{n-1})$$

Then $\varphi_i = \psi_i^{-1}$ and φ_i is smooth. (Note that we do not have to extend because $D^{n-1} \subset \mathbb{R}^{n-1}$, and in D^{n-1} we stay away from the non-smooth point 0 of the square root.)
Case 2: $y_i < 0$ can be discussed similarly. In this case the parametrization is

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{i-1}, -\sqrt{1 - \sum_{j=1}^{n-1} x_j^2}, x_i, \ldots, x_{n-1})$$

■

Now let $f : M \to N$ be a smooth map between smooth manifolds. We want to **linearize** f at a point $x \in M$ (just like the Jacobi matrix linearizes a map $\mathbb{R}^k \to \mathbb{R}^\ell$).

This linearization will be a **linear map** between vector spaces asscociated to $x \in M$ and $f(x) \in N$ in a natural way.

For $M \subset \mathbb{R}^k$ a smooth manifold of dimension m and $x \in M$ we want to define the *tangent space* $TM_x \subset \mathbb{R}^k$. This will be a vector subspace of dimension m. (Very often the tangent space is visualized as the affine space $x + TM_x$.)

Step 1: Let $U \subset \mathbb{R}^k$ open and $x \in U$. Then let $TU_x := \mathbb{R}^k$. Now if $f : U \to \mathbb{R}^\ell$ is smooth we define $df_x : \mathbb{R}^k \to \mathbb{R}^\ell$ by the following extension of the usual *directional derivative*:

$$df_x(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

In this case df_x actually is just the usual derivative $Df(x)$: In order to see this, let $0 \neq h \in \mathbb{R}^n$. Then for all t sufficiently small we have by definition of the derivative Df:

$$(*) \qquad f(x+th) = f(x) + Df(x)(th) + \rho(th)$$

with $\frac{1}{||h||} \lim_{t \to 0} \frac{\rho(th)}{t} = 0$ (note that $||h||$ is a constant). Also

$$\frac{\rho(th)}{t} = \frac{f(x+th) - f(x)}{t} - Df(x)h$$

For $t \to 0$ it follows that $Df(x)(h) = df_x(h)$ ($df_x(0) = 0$) is immediate from the definition).

The derivative Df of a smooth map $f : \mathbb{R}^k \supset U \to \mathbb{R}^\ell$ with U open is defined in multi-variable calculus by $(*)$ above. It is a map with domain U and taking values in the space of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^\ell$. We let $Df(x)$ be the *linearization* of f at x, even though it is really the *affine approximation* $y \mapsto f(x) + Df(x)(y - x)$, which approximates f near x. For more background see e. g. Michael Spivak: Calculus on Manifolds, Chapter 2. In particular proofs of the rules below can be found there.

Rules 2.6.
(a) If $f : U \to V$ and $g : V \to W$ are smooth and $f(x) = y$ then we have the chain rule

$$d(g \circ f)_x = (dg)_{f(x)} \circ df_x$$

Formally this means that d transforms the commutative "triangle" of smooth maps

$$g \circ f : U \xrightarrow{f} V \xrightarrow{g} W$$

into the commutative "triangle" of linear maps:

$$d(g \circ f)_x : \mathbb{R}^k \xrightarrow{df_x} \mathbb{R}^\ell \xrightarrow{dg_{f(x)}} \mathbb{R}^m$$

(b) $d(id_U) = id_{\mathbb{R}^k}$ for $U \subset \mathbb{R}^k$ open. If $U \subset U'$ is an inclusion of open subsets let $\iota : U \to U'$ be the corresponding inclusion map $\iota(x) = x$ for $x \in U$. Then

$$d\iota_x = id_{\mathbb{R}^k}$$

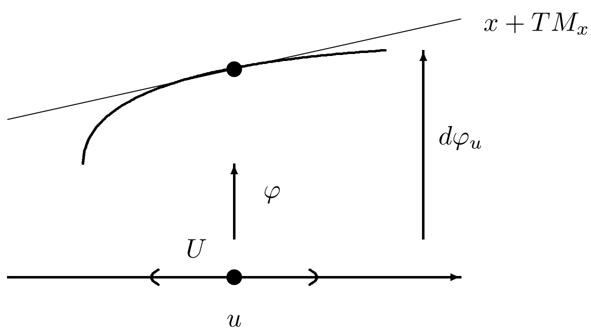
(A general theory generalizing the idea of (a), (b) is the notion of *functor* in *category theory*.)
(c) Let $L : \mathbb{R}^k \to \mathbb{R}^\ell$ be a linear map. Then $dL_x = L$ for all $x \in \mathbb{R}^k$. In fact

$$dL_x(h) = \lim_{t\to 0} \frac{L(x+th) - L(x)}{t} = \lim_{t\to 0} \frac{L(x) + tL(h) - L(x)}{t} = L(h)$$

for all $h \in \mathbb{R}^n$.
(d) If $f : \mathbb{R}^k \supset U \to V \subset \mathbb{R}^\ell$ is a diffeomorphism then $df_x : \mathbb{R}^k \to \mathbb{R}^\ell$ is an isomorphism of vector spaces, in particular $k = \ell$.
(e) (**Inverse function theorem**): Let $f : U \to \mathbb{R}^\ell$ be a smooth map and $x \in U \subset \mathbb{R}^k$ with df_x invertible. Then there is an open neighborhood $W \subset U$ of x such that $f|W : W \to f(W)$ is a diffeomorphism (in particular $f(W)$ is open).

Step 2: Let $M \subset \mathbb{R}^k$ be smooth and $x \in M$. Let $\varphi : U \to M \subset \mathbb{R}^k$ be a parametrization with $\varphi(u) = x \in M$ and $U \subset \mathbb{R}^m$ open. Then $d\varphi_u : \mathbb{R}^m \to \mathbb{R}^k$ is defined.

Definition. $TM_x := d\varphi_u(\mathbb{R}^m) \subset \mathbb{R}^k$ is the *tangent space* to M at x.



We need to show that the definition of TM_x does not depend on the choice of (φ, U).

Let $\chi : V \to M \subset \mathbb{R}^k$ be another parametrization with $\chi(v) = x$. Consider

$$(\chi^{-1} \circ \varphi)|(\varphi^{-1} \circ \chi(V))$$

This maps the open set $U_1 := \varphi^{-1} \circ \chi(V)$ diffeomorphic onto some neighborhood V_1 of v. Now apply 2.6 (a) above to

$$\chi^{-1} \circ \varphi : U_1 \xrightarrow{\varphi} \mathbb{R}^k \xleftarrow{\chi} V_1$$

to get the vector space isomorphism

$$(*) \qquad \mathbb{R}^m \xrightarrow{d\varphi_u} \mathbb{R}^k \xleftarrow{d\chi_v} \mathbb{R}^m$$

Thus the images of $d\varphi_u$ and $d\chi_v$ coincide. ■

Claim: $dim(TM_x) = m$

Proof. This is clear from ()∗) above. We give an alternative proof. Note that

$$\psi = \varphi^{-1} : \varphi(U) \to U$$

is smooth at x. Thus there is $W \subset \mathbb{R}^k$ open with $x \in W$ and an extension

$$\Psi : W \to \mathbb{R}^m$$

with

$$\Psi|(W \cap \varphi(U)) = \varphi^{-1}|(W \cap \varphi(U))$$

Let $U_0 := \varphi^{-1}(W \cap \varphi(U))$. Then by 2.6 (a) the derivative transforms

$$\iota : U_0 \xrightarrow{\varphi|U_0} W \xrightarrow{\Psi} \mathbb{R}^m,$$

where ι is the inclusion map, into

$$id : \mathbb{R}^m \xrightarrow{d\varphi_u} \mathbb{R}^k \xrightarrow{d\Psi_x} \mathbb{R}^m$$

It follows that the rank of $d\varphi_u$ is $\geq m$ and by dimension reasons thus $rank(d\varphi_u) = m$. Thus

$$dim(im(d\varphi_u)) = dim(TM_x) = m$$

■

Let $f : \mathbb{R}^k \supset M \to N \subset \mathbb{R}^\ell$ be a smooth map between smooth manifolds, and $f(x) = y$.

Definition 2.7. For $x \in M$ the *derivative*

$$df_x : TM_x \to TN_{f(x)}$$

is defined as follows: Extend f locally at x to the smooth map $F : W \to \mathbb{R}^\ell$ and let

$$df_x(h) := dF_x(h)$$

(compare step 1 above for the definition of dF_x).

Claims:
(a) df_x does not depend on the choice of $F : W \to \mathbb{R}^\ell$.
(b) $dF_x(TM_x) \subset TN_{f(x)}$.

Proof. Let

$$\varphi' : U' \to M \subset \mathbb{R}^k$$

and

$$\chi : V \to N \subset \mathbb{R}^\ell$$

be parametrizations with $x \in \varphi'(U')$ and $f(x) \in \chi(V)$. Then let

$$U := (\varphi')^{-1}(f^{-1}(\chi(V)) \cap W)$$

and let

$$\varphi := \varphi'|U$$

Note that $f^{-1}(\chi(V))$ is an open neighborhood of x in M. Then $\varphi(U) \subset W$ and $f(\varphi(U)) \subset \chi(V)$.

Now consider

$$\chi^{-1} \circ f \circ \varphi : U \to V,$$

which is best visualized in the commutative square:

$$\begin{array}{ccc} W & \xrightarrow{F} & \mathbb{R}^\ell \\ \varphi \big\uparrow & & \big\uparrow \chi \\ U & \xrightarrow{\chi^{-1} \circ f \circ \varphi} & V \end{array}$$

This transforms (by two applications of 2.6 (a)) into the commutative square

$$\begin{array}{ccc} \mathbb{R}^k & \xrightarrow{dF_x} & \mathbb{R}^\ell \\ \big\uparrow d\varphi_u & & \big\uparrow d\chi_v \\ \mathbb{R}^m & \xrightarrow{d(\chi^{-1} \circ f \circ \varphi)_u} & \mathbb{R}^n \end{array}$$

It follows that

$$dF_x(TM_x) = dF_x \circ d\varphi_u(\mathbb{R}^m) = d\chi_v \circ d(\chi^{-1} \circ f \circ \varphi)_u(\mathbb{R}^m) \subset d\chi_v(\mathbb{R}^n) =: TN_{f(x)}$$

Thus (b) holds. Since $dF_x|TM_x = d\chi_v \circ d(\chi^{-1} \circ f \circ \varphi)_u \circ (d\varphi_u)^{-1}$ and the right hand side does not depend on F so does df_x. This proves (a). Note that on the other hand the left hand side does not depend on any choices of parametrizations. ■

Here is an alternative description of TM_x using smooth paths in M:

$TM_x = \{\gamma'(0) \in \mathbb{R}^k | \gamma : (-\delta, \delta) \to M$ is a smooth path with $\gamma(0) = x$ and $\delta > 0\}$

Proof. Recall that

$$\gamma'(s) = \lim_{t \to 0} \frac{\gamma(s+t) - \gamma(s)}{t} = d\gamma_s(1)$$

for $s \in (-\delta, \delta)$. Now given a smooth path $\gamma : (-\delta, \delta) \to M$ with $\gamma(0) = x$ there is a parametrization φ of M near x such that

$$\gamma = \varphi \circ \varphi^{-1} \circ \gamma : (-\varepsilon, \varepsilon) \to M$$

holds and is smooth, for some $\varepsilon < \delta$. Thus by the chain rule

$$\gamma'(0) = (\varphi \circ \varphi^{-1} \circ \gamma)'(0) = d\varphi_u(d(\varphi^{-1} \circ \gamma)_0(1)) \in TM_x$$

by the definition. Conversely let $h \in \mathbb{R}^m$, and consider the path

$$\gamma_h(t) := u + th \in \mathbb{R}^m.$$

Let (φ, U) be as above with $\varphi(u) = x$. Then $\gamma_h(t) \in U$ for t sufficiently small, $t \in (-\varepsilon, \varepsilon)$, and $\gamma_h(0) = u$. Consider $\gamma := \varphi \circ \gamma_h$. Then

$$\gamma'(0) = d\gamma_0(1) = d\varphi_u(\gamma_h'(0)) = d\varphi_u(h)$$

■

Note that for $v = \gamma'(0)$ and $f : M \to N$ a smooth map, we have

$$df_x(v) = df_x(\gamma'(0)) = (f \circ \gamma)'(0).$$

This calculation actually involves the chain rule 2.8 (a) below.

The rules 2.6 now *globalize* to rules for the derivatives between smooth manifolds.

Rules 2.8.
(a) (**Chain Rule**) Let $f : M \to N$ and $g : N \to P$ be smooth maps between smooth manifolds. Then

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

(b) $d(id_M)_x = id_{TM_x}$. Moreover if $\iota : M \to N$ is an inclusion of manifolds, $M, N \subset \mathbb{R}^k$ smooth, then

$$d\iota_x : TM_x \to TN_x$$

is an inclusion of a subspace of a vector space.
(c) Let $f : M \to N$ be a diffeomorphism. Then

$$df_x : TM_x \to TN_{f(x)}$$

is a vector space isomorphism.

Proof. (a): The idea is to use the chain rule for suitable extensions. For the smooth map $f : \mathbb{R}^k \supset M \to N \subset \mathbb{R}^\ell$ we can find a neighborhood W of x in $\mathbb{R}^k$ and a smooth local extension $F : W \to \mathbb{R}^\ell$ of f. Similarly for $g : \mathbb{R}^\ell \supset N \to P \subset \mathbb{R}^t$ we can find V neighborhood of $f(x)$ in $\mathbb{R}^\ell$ and a smooth local extension $G : V \to \mathbb{R}^t$ of g. Then

$$(G \circ F)|F^{-1}(V) : F^{-1}(V) \to \mathbb{R}^t$$

is a smooth local extension of $g \circ f$ at $x \in M$, and $F^{-1}(V) \subset W$ is an open subset of $\mathbb{R}^k$ (compare exercise 1.4 (c)). It follows that

$$d(g \circ f)_x = d(G \circ F)_x = dG_{f(x)} \circ dF_x = dg_{f(x)} \circ df_x$$

from the chain rule in 2.6 (a).
For the proof of (b) we can use that each path in M is also a path in N and the alternative definition of tangent spaces.
(c): Just note that the chain rule implies the following:

$$(df^{-1})_{f(x)} \circ df_x = id_{TM_x}$$

and

$$df_x \circ (df^{-1})_{f(x)} = id_{TN_{f(x)}}.$$

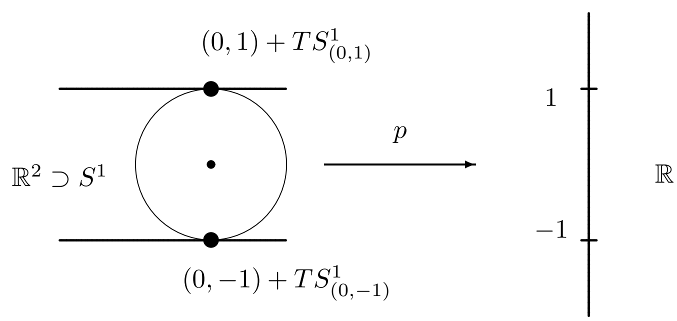
Chapter 3

Regular values and Sard's theorem.

Let $f : M \to N$ be a smooth map between smooth manifolds.

Definition 3.1. A point $x \in M$ is called a *regular point* of f if $df_x : TM_x \to TN_{f(x)}$ is onto. Let $C = C(f) \subset M$ denote the set of points at which f is not regular. Then $f(C) \subset N$ is the set of *critical values* of f and $N \setminus f(C)$ is the set of *regular values* of f.

Examples. (a) If $y \in N$ with $f^{-1}(y) = \emptyset$ then y is a regular value of f. This is abuse of notation because in this case y is not a value of f.
(b) Let $p : S^1 \to \mathbb{R}$ be the restriction of the projection $P : \mathbb{R}^2 \to \mathbb{R}$, $(x_1, x_2) \mapsto x_2$, to S^1.



Then the kernel of $dP_x = P$ is given by $\mathbb{R} \times \{0\}$ at each point $x \in \mathbb{R}^2$. But

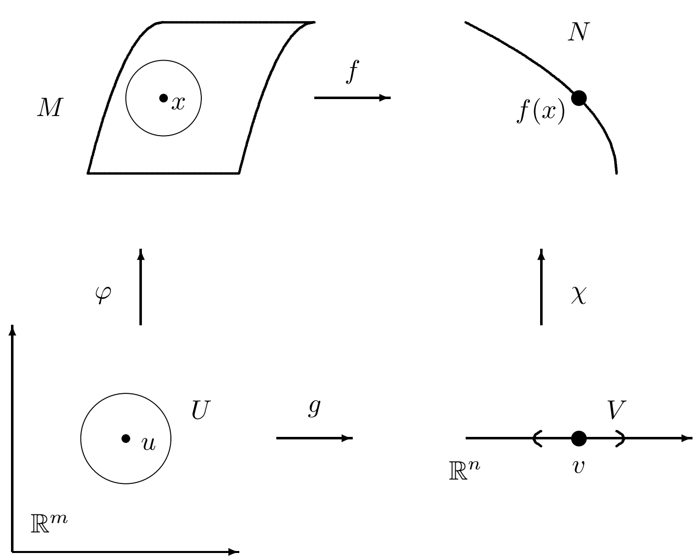
$TS^1_x = \mathbb{R} \times \{0\}$ precisely at $x = (0, -1)$ and $x = (0, 1)$. Thus dp_x is onto for $x \neq (0, \pm 1)$ and is the zero-map at $(0, \pm 1)$. Thus $C(f) = \{-1, 1\}$ and $(0, \pm 1)$ are the two critical points of p. Note that the two critical values are the maximum and minimum values of the function.
(c) If $m < n$ then each point $x \in M$ is a critical point, and the set of critical values is the image $f(M)$.

Theorem 3.2. *Let $f : M \to N$ be smooth with dim$(M) = m$ and dim$(N) = n$. Let x be a regular point of f. Then there exists parametrizations $\varphi : U \to M$ at x and $\chi : V \to N$ at y such that*

$$(\chi^{-1} \circ f \circ \varphi)(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$$

Proof. For chosen parametrizations consider the commutative diagram, i. e. $g = \chi^{-1} \circ f \circ \varphi$:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \varphi \uparrow & & \uparrow \chi \\ U & \xrightarrow{g} & V \end{array}$$



We can assume without restriction that $f(\varphi(U)) \subset \chi(V)$ (otherwise *shrink*

U). Since $dg_u : \mathbb{R}^m \to \mathbb{R}^n$ is onto (which requires $n \leq m$, and $\varphi(u) = x$) there are invertible matrices A, B such that

$$A(dg_u)B = (I_n, 0_{n\times(m-n)}),$$

where we identify the linear map dg_u with a matrix representation and use block matrix notation with I_n denoting the $n \times n$ identity matrix and $0_{n\times(m-n)}$ is the $n \times (m-n)$ matrix with only zero entries. (This follows from the fact that the *matrix* dg_u can be brought into the form $(I_n, 0_{n\times(m-n)})$ by suitable row and column operations.) Now replace φ by $\varphi \circ B$ and χ by $\chi \circ A^{-1}$, where now we identify the matrices A, B with the linear maps induced by them. Similarly g is replaced by a new map but we will keep the old notation. Let $G : U \to \mathbb{R}^m$ be defined by

$$G(x) := (g(x), x_{n+1}, \ldots, x_m)$$

for $x = (x_1, \ldots, x_m)$. Then $dG_u = id_{\mathbb{R}^m} = I_m$ is invertible thus G is locally invertible at u by the inverse mapping theorem. Let $G^{-1} : U' \to U$ be a local inverse (Note that we actually might have to shrink U in order to have G invertible). Then we have the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \varphi\circ G^{-1}}\uparrow & & \uparrow{\scriptstyle \chi} \\ U' & \longrightarrow & V \\ {\scriptstyle \subset}\downarrow & & \downarrow{\scriptstyle \subset} \\ \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \end{array}$$

and

$$(\chi^{-1} \circ f \circ \varphi \circ G^{-1})(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$$

because

$$(x_1, \ldots, x_m) = G(y) = (g(y), x_{n+1}, \ldots, x_m)$$

for some $y \in U$, implies

$$(\chi^{-1} \circ f \circ \varphi \circ G^{-1})(G(y)) = g(y) = (x_1, \ldots, x_n)$$

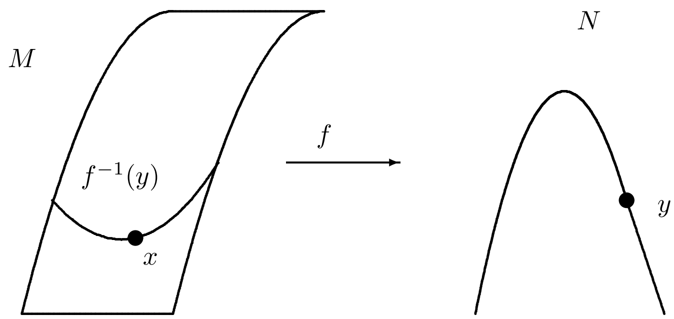
So replacing φ by $\varphi \circ G^{-1}$ will lead to the conclusion. ■

3.2. gives a description of a smooth map f near a regular point and shows that locally it can be given in charts by the standard projection. The general local description of smooth maps is an interesting subject on its own. The corresponding theory is called *singularity theory*.

Corollary 3.3. *Let $f : M \to N$ be smooth and $y \in N$ a regular value. Then*

$$f^{-1}(y) \subset M \subset \mathbb{R}^k$$

is a smooth manifold of dimension $m-n$, where $m = dim(M)$ and $n = dim(N)$.



Proof. Every $x \in f^{-1}(y)$ is a regular point of f. Using the final parametrization φ from 3.2 with $\varphi(u) = x$ and $u = (u_1, \dots, u_m)$ it follows that $\varphi|((\{(u_1, \dots u_n)\} \times \mathbb{R}^{m-n}) \cap U)$ is a parametrization of $f^{-1}(y)$ at x with inverse $\varphi^{-1}|(f^{-1}(y) \cap \varphi(U))$.■

Let $M' \subset M$ so that $TM'_x \subset TM_x$ is an inclusion of a subspace of a vector space (see 2.8 (b)). Then the *normal space* to M' in M at x is

$$\nu(M', M)_x := \{v \in TM_x| < v, TM'_x >= 0\},$$

where $< x, y >= \sum_{i=1}^k x_i y_i$ is the usual inner product of vectors. Note that the normal space at x is a vector space of dimension $dim(M) - dim(M')$.

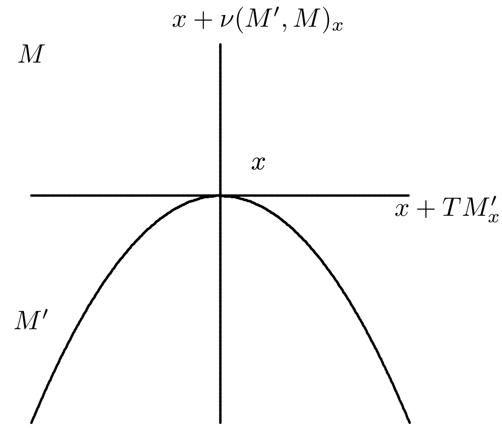
Corollary 3.4. *For f, y as in 3.3.we have:*

$$T(f^{-1}(y))_x = ker(df_x)$$

and

$$df_x|\nu(f^{-1}(y), M)_x : \nu(f^{-1}(y), M)_x \to TN_y$$

is an isomorphism of vector spaces for all $x \in f^{-1}(y)$.



Proof. Consider the commutative diagram:

$$\begin{array}{ccc} f^{-1}(y) & \xrightarrow{\text{inclusion}} & M \\ {\scriptstyle f|(f^{-1}(y))}\downarrow & & \downarrow{\scriptstyle f} \\ \{y\} & \xrightarrow{\text{inclusion}} & N \end{array}$$

it follows that $df_x(T(f^{-1}(y))_x) = 0$ because $T(\{y\}_y) = \{0\}$. But

$$dim(T(f^{-1}(y))_x) = m - n = dim(f^{-1}(y))$$

and

$$dim(ker(df_x) = m - n$$

from the usual dimension formula for linear maps. Since $T(f^{-1}(y))_x \subset ker(df_x)$ we conclude that the spaces are equal, and df_x induces an isomorphism also because of dimensions. ■

Examples 3.5.
(a) Let $f : \mathbb{R}^m \to \mathbb{R}$ be defined by

$$f(x) = x_1^2 + \ldots + x_m^2.$$

Then

$$df_x = grad(f)(x) = (2x_1, \ldots, 2x_m).$$

Thus each point $r \neq 0$ is a regular value of f and for $r > 0$:

$$f^{-1}(r) = \{x \in \mathbb{R}^m| \; ||x||^2 = r\}$$

is a smooth manifold of dimension $m-1$. Note that $f^{-1}(r) = \emptyset$ for $r < 0$, and $f^{-1}(0) = \{0\}$ is a manifold but of dimension different than the one given in 3.3.
(b) Let $M(n)$ be the set of all $n \times n$-matrices with real components, which can be identified with $\mathbb{R}^{n^2}$ and thus is a smooth manifold. Then

$$Sym(n) := \{B \in M(n)|B^t = B\} \subset M(n)$$

is the linear subspace of *symmetric* $n \times n$-matrices. This is also a smooth manifold of dimension $\frac{n(n+1)}{2}$. (Use that a symmetric matrix is determined by the diagonal and all components above the diagonal.) Then let

$$O(n) := \{A \in M(n)|AA^t = I\} \subset M(n)$$

be the set of *orthogonal* matrices, where I is the identity matrix.

Claim: $O(n)$ is a smooth manifold (in $\mathbb{R}^{n^2}$) of dimension $\frac{n(n-1)}{2}$.

Proof. Let

$$f : M(n) \to Sym(n)$$

be defined by

$$f(A) = AA^t.$$

This is in each component a polynomial map in the entries thus is a smooth map. Note that $f^{-1}(I) = O(n)$. We will show that I is a regular value. Let $A \in f^{-1}(I)$, $B \in TM(n)_A = M(n)$. Then

$$\begin{aligned} df_A(B) &= \lim_{t \to 0} \frac{f(A+tB) - f(A)}{t} = \lim_{t \to 0} \frac{(A+tB)(A+tB)^t - AA^t}{t} \\ &= \lim_{t \to 0} (BA^t + AB^t + tBB^t) = BA^t + AB^t \end{aligned}$$

Now $T(Sym(n))_C = Sym(n)$ for each symmetric matrix C. Let $C \in Sym(n)$ and $B := \frac{1}{2}CA$. Then

$$df_A(B) = \frac{1}{2}CAA^t + A(\frac{1}{2}A^tC^t) = \frac{1}{2}C + \frac{1}{2}C^t = C$$

Thus df_A is onto. It follows that

$$dim(O(n)) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

■

By Exercise 2.4, $O(n)$ is compact. Because

$$det : O(n) \to \{-1, 1\}$$

is continuous (consider the explicit formula for calculating the determinant from the entries of the matrix), $O(n)$ is not connected. It can be shown that

$$SO(n) = det^{-1}(1)$$

is connected. Note that obviously $SO(n)$ is also a smooth manifold of the same dimension $\frac{n(n-1)}{2}$.

Theorem 3.6. *Let $f : M \to N$ be a smooth map with M compact and $dim(M) = dim(N)$. Let $y \in N$ be a regular value of f. Then $f^{-1}(y)$ is a finite set. Moreover, there is an open neighborhood V of y such that $|f^{-1}(y)| = |f^{-1}(y')|$ for all $y' \in V$, where $|\ |$ denotes the number of elements of a set.*

Proof. Note that $f^{-1}(y)$ is a closed subset of the compact space M and thus is compact by 1.11. By 3.2., for each $x \in f^{-1}(y)$ there is a neighborhood U_x such that $f|U_x : U_x \to f(U_x)$ is a diffeomorphism (because in suitable parametrizations giving the identity map). In particular $U_x \cap f^{-1}(y) = \{x\}$. Then $(U_x \cap f^{-1}(y))_{x \in f^{-1}(y)}$ is an open covering of the compact space $f^{-1}(y)$. A finite subcovering can only contain finitely many points. So let $f^{-1}(y) = \{x_1, \dots, x_r\}$ and $U_{x_i} = U_i$, $f(U_i) =: V_i$ as above. Then

$$V := (V_1 \cap \dots \cap V_r) \setminus f(M \setminus (U_1 \cup \dots \cup U_r))$$

is an open set (In fact: $U_1 \cup \dots \cup U_r$ is open by 1.3. Thus $M \setminus (U_1 \cup \dots \cup U_r)$ is compact by 1.11 thus $f(M \setminus (U_1 \cup \dots \cup U_r))$ is compact by 1.12., thus closed in N by 1.10. Its complement in N is open and thus the intersection of this complement with the open set $V_1 \cap \dots \cap V_r$ (use 1.3. again) is open.) Now let $y' \in V$ then $y' \in V_i$ for $i = 1, \dots, r$ and there are uniquely determined $x_i' \in U_i$ for $1 \le i \le r$ with $f(x_i') = y'$. This shows $|f^{-1}(y')| \ge r$. Suppose $|f^{-1}(y')| > r$. Then there is $x \in M \setminus (U_1 \cup \dots \cup U_r)$ with $f(x) = y' \in V$, which is a contradiction. ■

Remark. Let f, y be as above and let C be the set of critical points of f. Then

$$N \setminus f(C) \ni y \mapsto |f^{-1}(y)| \in \mathbb{N} \cup \{0\}$$

is a locally constant function. Thus this function is constant on the components of $N \setminus f(C)$. The set $N \setminus f(C)$ is open, as we will see now.

Theorem 3.7. *Let $f : M \to N$ be a smooth map(with arbitrary dimensions $dim(M)$ and $dim(N)$). Then the map*

$$M \ni x \mapsto rank(df_x) \in \mathbb{N} \cup \{0\}$$

has the following property: for all $x \in M$ there exists a neighborhood U such that

$$rank(df_a) \geq rank(df_x)$$

for all $a \in U$.(The rank cannot drop locally!)

Proof. It suffices to prove this for $f : V \to \mathbb{R}^\ell$ and $V \subset \mathbb{R}^m$ open (use parametrizations). Then df_x is the linear map defined by the Jacobi-matrix. Let $rank(df_x) = d$. Then there is a $d \times d$ submatrix of the Jacobi-matrix $(\frac{\partial f_i}{\partial x_j}(x))_{\substack{1\leq i\leq m\\1\leq j\leq \ell}}$ of rank d. Since f is smooth the coefficients of the corresponding matrix $(\frac{\partial f_{i_r}}{\partial x_{j_s}}(x))_{\substack{1\leq r\leq d\\1\leq s\leq d}}$ depend continuously on x, meaning: if $||x - x'|| < \delta$ then for all $1 \leq r, s \leq d$ we have

$$\left|\frac{\partial f_{i_r}}{\partial x_{j_s}}(x) - \frac{\partial f_{i_r}}{\partial x_{j_s}}(x')\right| < \varepsilon$$

Now consider the map

$$\lambda : V \ni a \mapsto det\left((\frac{\partial f_{i_r}}{\partial x_{j_s}}(a))_{\substack{1\leq r\leq d\\1\leq s\leq d}}\right) \in \mathbb{R}$$

This is a continuous map because

$$det : M(d) = \mathbb{R}^d \to \mathbb{R}$$

is continuous, and compositions of continuous functions are continuous. Since $\mathbb{R}\backslash 0$ is open we can define $U := \lambda^{-1}(\mathbb{R}\backslash 0)$. Then for all $a \in U$ the Jacobi-matrix has a submatrix of rank d, thus has rank $\geq d$. ■

The following is an immediate consequence of 3.7. using that for $f : M \to N$ and $dim(N) = n$ we have that C is the complement of $\{x \in M | rank(df_x) = n\}$. See also 3.20 for a more general statement.

Corollary 3.8. *Let $f : M \to N$ be smooth and Let $C \subset M$ be the set of critical points. Then*
(a) *C is closed*
(b) *If M is compact then the set of critical values $f(C) \subset N$ is compact and thus closed in N by* 1.13.

Theorem 3.9.(Fundamental theorem of algebra) *Let*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$$

be a complex polynomial with $n \geq 1$ and $a_0 \neq 0$. Then P has a root.

Proof. (i) Let $S^2 \subset \mathbb{R}^3$ be the *Riemann sphere* with north pole $N = (0,0,1)$ and south pole $S = (0,0,-1)$. For coordinate systems we will use the stereographic projections

$$h_+ : S^2 \setminus N \to \mathbb{R}^2 = \mathbb{C}$$

respectively

$$h_- : S^2 \setminus S \to \mathbb{R}^2 = \mathbb{C}$$

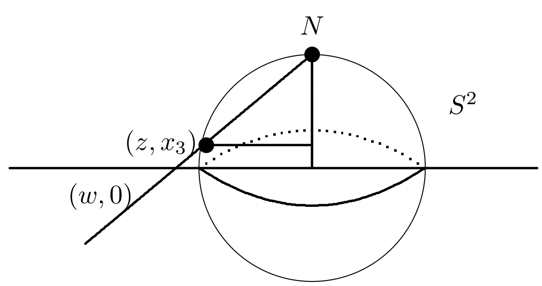
defined by

$$h_+(z, x_3) = \frac{z}{1 - x_3}$$

respectively

$$h_-(z, x_3) = \frac{z}{1 + x_3}$$

where we identify $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. Geometrically h_+ assigns to each point in $S^2 \setminus N$ the intersection point of the line through this point and N with the plane $x_3 = 0$ identified with $\mathbb{C}$.



Similarly a calculation shows:

$$h_+^{-1}(z) = \left(\frac{2z}{|z|^2 + 1}, 1 - \frac{2}{|z|^2 + 1}\right)$$

and

$$h_-^{-1}(z) = \left(\frac{2z}{|z|^2 + 1}, \frac{2}{|z|^2 + 1} - 1\right)$$

Note that $h_+ h_-^{-1}(z) = h_- h_+^{-1}(z) = \frac{1}{(\bar{z})}$. (Check by computation that all the formulas are correct!)

(ii) **Claim:** Let P be the given polynomial and let $f : S^2 \to S^2$ be defined by $f(x) = h_+^{-1} P h_+(x)$ for $x \neq N$ and $f(N) = N$. Then f is smooth.

Proof. This is clear for $x \neq N$. Let $x = N$. Then $S^2 \setminus S$ is a neighborhood of N with coordinate system h_-. Now f is smooth at N if and only if $Q(z) := h_- f h_-^{-1}(z)$ is smooth at 0 by 2.4 (b). But for $z \neq 0$,

$$\begin{aligned} Q(z) =& (h_- h_+^{-1}) P (h_+ h_-^{-1})(z) = \frac{1}{\overline{P(\frac{1}{\overline{z}})}} = \frac{1}{\overline{a_0 \overline{z}^{\,-n} + \ldots + a_n}} \\ =& \frac{1}{\overline{a_0} z^{-n} + \ldots + \overline{a_n}} = \frac{z^n}{\overline{a_0} + \ldots + \overline{a_n} z^n} \end{aligned}$$

and $Q(0) = h_- f(N) = 0$. Note that the first expression is also defined at $z = 0$ because $a_0 \neq 0$. This proves the claim because the first expression is smooth. ■

(iii) Now a critical point of f is N, or $h_+^{-1}(z)$ if

$$P'(z) = \sum a_{n-j} j z^{j-1} = 0$$

because for $x \neq N$

$$df_x = (dh_+^{-1})_{P h_+(x)} \circ dP_{h_+(x)} \circ (dh_+)_x.$$

With $h_+(x) = z$, the Jacobi-matrix is

$$dP_z = P'(z) = \lim_{h \to 0, h \in \mathbb{C}} \frac{P(z+h) - P(z)}{h},$$

where we identify complex numbers with certain real 2×2-matrices using the correspondence

$$a + ib \longleftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

This follows from the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

for analytic functions $f = u + iv$. It is easy to prove that products and sums of analytic functions are analytic. Since $f(z) = z$ is obviously analytic all polynomial functions are analytic. Note that the determinant of the matrix corresponding to $a + ib$ is $a^2 + b^2 = |a + ib|^2$. Now $f(C) \subset S^2$ is compact and consists of only finitely many points (the number of zeroes of P' is finite). Since

$S^2 \setminus f(C)$ is connected. (Exercise: it is path connected, proceed similarly to the proof in 1.23. to avoid a point.) Thus

$$S^2 \setminus f(C) \ni y \mapsto |f^{-1}(y)|$$

is constant. But $f^{-1}(y) \neq \emptyset$ for some $y \in S^2 \setminus f(C)$ because otherwise all all the values actually taken by the function would be critical values and there are only finitely many. Since $n \geq 1$ this is not possible. Thus $|f^{-1}(y)| > 0$ for *all* regular values. But $|f^{-1}(y)| > 0$ for all critical values by definition. Thus f is onto. Thus there exists $x \neq N$ such that $f(x) = S$ (we know $f(N) = N$), which implies the existence of $z \in \mathbb{C}$ such that $P(z) = 0$. ■

We now address the *most important* question:

How often or rare are regular values?

The answer to this question is given by Sard's theorem. In order to state the result we need to introduce some new concepts.

Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two vectors with $a_i < b_i$ for $i = 1, \ldots, n$. Then

$$W(a,b) := \prod_{i=1}^{n} [a_i, b_i] \subset \mathbb{R}^n$$

is an n-dimensional *rectangle*. An n-dimensional rectangle is a *cube* with side $d = b_i - a_i$ for $1 \leq i \leq n$. Let

$$vol(W(a,b)) = \prod_{i=1}^{n} (b_i - a_i)$$

be the volume of $W(a,b)$.

Definition 3.10. $A \subset \mathbb{R}^n$ is called a *set of measure zero* if for each $\varepsilon > 0$ the set A can be covered by countably many rectangles (or cubes), i. e. there are $(W_i)_{i \in \mathbb{N}}$ with

$$\sum_{i=1}^{\infty} vol(W_i) < \varepsilon$$

We introduce the following notation. For $c \in \mathbb{R}^k$ let

$$\mathbb{R}^{n-k}_c := \{c\} \times \mathbb{R}^{n-k} \subset \mathbb{R}^n.$$

Note that in 3.10. the rectangles or cubes can easily be replaced by open rectangles or cubes. For the proof of Sard's theorem we will actually only need the case $k = 1$ from the following 3.11 (d). But the proof of 3.21 will require the more general version for all positive integers k.

Lemma 3.11. (a) *Countable unions of sets of measure zero, or subsets of sets of measure zero, are sets of measure zero.*
(b) $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ *is a set of measure zero. An open subset of* $\mathbb{R}^n$ *is not a set of measure zero.*
(c) *If* $U \subset \mathbb{R}^n$ *is open and* $A \subset U$ *is a set of measure zero, and if* $f : U \to \mathbb{R}^n$ *is smooth, then* $f(A) \subset \mathbb{R}^n$ *is a set of measure zero.*
(d) (Measure zero set Fubini) *Let* $A \subset \mathbb{R}^n$ *be closed such that* $A \cap \mathbb{R}^{n-k}_c$ *is a set of measure zero for all* $c \in \mathbb{R}^k$*. Then* A *is a set of measure zero.*

Proof. (a) Let $N_1, N_2, \dots, N_i, \dots$ be a sequence of sets of measure 0. Let W_i^j, $j = 1, 2, \dots$ be a covering of N_i with $\sum_{j=1}^{\infty} vol(W_i^j) < \frac{\varepsilon}{2^i}$. Then $(W_i^j, i = 1, 2, \dots; j = 1, 2, \dots)$ is a covering of $N_1 \cup N_2 \cup \dots$ and

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} vol(W_i^j) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon \left(\frac{1}{1-\frac{1}{2}} - 1 \right) = \varepsilon$$

The assertion about subsets is obvious.
(b) We begin by proving the first claim for all compact subsets $K \subset \mathbb{R}^{n-1}$. Then K is closed and bounded thus $K \subset S$ for a sufficiently large rectangle in $\mathbb{R}^{n-1}$. Then for each $\delta > 0$ the set $S' := S \times [-\frac{\delta}{2}, \frac{\delta}{2}]$ is a rectangle in $\mathbb{R}^n$ containing $K \subset \mathbb{R}^n$. Let $\delta < \frac{\varepsilon}{vol(S)}$. Then $vol(S') = vol(S) \times \delta < \varepsilon$ for given ε. To prove the claim for $\mathbb{R}^{n-1}$ write $\mathbb{R}^{n-1} = \cup_{i=1}^{\infty} C_i$ with C_i compact, and apply (a). If $U \subset \mathbb{R}^n$ is open then U contains an open rectangle of volume $\delta > 0$. Then the volume of any covering of U by rectangles will be $\geq \delta$ (Think about this!) Thus U is not a set of measure zero.
(c) We can assume that $U = \cup_{i=1}^{\infty} C_i$ with C_i compact rectangles (this is proved similarly to theorem 1.5.) Then it suffices to prove that all the sets $f(C_i \cap A)$ have measure zero, given that A has measure zero. So we will show that $f(A \cap K)$ has measure zero for each compact rectangle $K \subset U$. Since the partial derivatives of f on K are bounded there is a positive constant $C \in \mathbb{R}$ such that

$$||f(x) - f(y)|| \leq C||x - y||$$

for all $x, y \in K$. It follws from this that if W is a cube of length a then $f(W \cap K)$ is contained in a cube $\tilde{W}$ of length $\sqrt{n}Ca$. Thus $f(W \cap K) \subset \tilde{W}$ with

$$vol(\tilde{W}) \leq n^{n/2} C^n vol(W).$$

(Recall that $|x_i| \leq \frac{a}{2}$ for $i = 1, \dots, n$ implies $\sum_{i=1}^{n} x_i^2 \leq n(\frac{a}{2})^2$), and a ball of radius r is contained in a cube of length $2r$.) Now let $A \subset \cup_i W_i$ with W_i a sequence of cubes with

$$\sum vol(W_i) < \frac{\varepsilon}{C^n n^{n/2}}.$$

Then $f(A \cap K) \subset \cup_i f(K \cap W_i) \subset \cup_i \tilde{W}_i$ for a sequence of cubes $\tilde{W}_i$ with

$$\sum_i vol(\tilde{W}_i) \leq C^n n^{n/2} \sum_i vol(W_i) < \varepsilon$$

This proves the claim. (Why does the argument not work for smooth maps $f : \mathbb{R}^m \supset U \to \mathbb{R}^n$ and $m > n$?)

(d) Without restriction we can assume that A is compact (cover $\mathbb{R}^n$ by countably many compact sets and intersect with A). It is clear that the proof can be reduced to $k = 1$ by using induction. (To see this use that for $\mathbb{R}^k \ni c = (c', c'') \in \mathbb{R}^{k-1} \times \mathbb{R}$ we have $A \cap \mathbb{R}^{n-k}_c = (A \cap \mathbb{R}^{n-(k-1)}_{c'}) \cap \mathbb{R}^{n-k}_{c'\ \ c''}$.) Thus we assume $k = 1$. We first prove the following:

Claim: Every covering of $[a, b] \subset \mathbb{R}$ by open finite intervals of length $< b - a$ has a subcovering by intervals with total length $\leq 6|b - a|$. *Proof.* Choose a finite minimal subcovering, thus find intervals $I_1, \dots, I_p$ of the covering such that $[a, b] \subset \cup_{j=1}^p I_j$ but $[a, b] \not\subset \cup_{i=1, i\neq q}^p I_i$ for all $1 \leq q \leq p$ (throw away redundant intervals from a *finite* subcovering). We can enumerate the intervals according to their left endpoint, i. e. if $I_j = (a_j, b_j)$ then $i < j$ if $a_i < a_j$. Note that $a_i = a_j$ does not occur because of minimality. Then $a_i < a_{i+1} < b_i < b_{i+1}$. It follows that the total overlap is at most the length of $[a, b]$. Thus the total length of the intervals is at most $3|a - b|$. (If $a_{i+2} < b_i$ then we could discard (a_{i+1}, b_{i+1}).) ■

Now assume $A \subset [a, b] \times \mathbb{R}^{n-1}$ and $A \cap \mathbb{R}^{n-1}_c$ is a set of measure zero for all $c \in [a, b]$. For every $\varepsilon > 0$ we can find a covering of $A \cap \mathbb{R}^{n-1}_c$ by open rectangles R^i_c in $\mathbb{R}^{n-1}_c$ of total volume $< \varepsilon$. Now for sufficiently small $\delta > 0$ the open sets $I^\delta_c \times R^i_c$ cover

$$A \cap \bigcup_{x \in I^\delta_c} \mathbb{R}^{n-1}_x$$

where $I^\delta_c := (c - \delta, c + \delta)$ for $c \in \mathbb{R}$. (This follows from the compactness of A: In fact if no $\delta > 0$ with the above property exists then you can construct the sequence of *nonempty* compact sets:

$$A_n := (A \cap \bigcup_{x \in [c - \frac{1}{n}, c + \frac{1}{n}]} \mathbb{R}^{n-1}_x) \setminus \bigcup_i [c - \frac{1}{n}, c + \frac{1}{n}] \times R^i_c.$$

Then $A_n \supset A_{n+1}$, and it follows (see e. g. Rudin, Principles of Mathematical Analysis 2.36 Corollary, page 38) that $\cap_n A_n \neq \emptyset$. But this is a contradiction because the intersection of the A_n is $A \setminus \cup_i R^i_c$ is empty by construction of the

R_c^i.) The sets I_c^δ form a covering of $[a,b]$, which has a finite subcovering by intervals I_j, $j = 1, \ldots N$, of total length $\leq 6|b-a|$ by the claim above. Let R_j^i denote the set R_c^i if $I_j = I_c^\delta$. Then the $I_j \times R_j^i$ form a covering of A with total volume $\leq 6|b-a|\varepsilon$ (Just calculate:

$$\sum_{i,j} vol(I_j \times R_j^i) = \sum_j vol(I_j) \sum_i vol(R_j^i) < \varepsilon \sum_j vol(I_j) \leq \varepsilon 6|b-a|.)$$

Since we can make this arbitrarily small, A is a set of measure zero. This proves (d). ■.

Definition 3.12. Let $N \subset \mathbb{R}^\ell$ be a smooth manifold of dimension n. Then $A \subset N$ is called a set of measure zero if for *some* covering of N by coordinate systems $\psi_j : W_j \to \mathbb{R}^n$ with $W_j \subset N$ open, the sets $\psi_j(W_j \cap A)$ are sets of measure zero in $\mathbb{R}^n$ for all j.

Remark. It follows from 3.11 (c) that in definition 3.12 *any* covering can be used.

Theorem 3.13. *Let $A \subset N$ be a set of measure zero in a smooth manifold. Then $N \setminus A$ is dense, i. e. $cl_N(N \setminus A) = \overline{N \setminus A} = N$.*

Proof. Suppose the claim is not true. Then there is a point $y \in N$ with $y \notin \overline{N \setminus A}$. So there is an open neighborhood V of $y \in N$ in $N \setminus (\overline{N \setminus A}) \subset A$ and a corresponding coordinate system (ψ, W) at y, which maps $W \cap V$ onto an open subset of $\mathbb{R}^n$. But $\psi(W \cap V \cap A) = \psi(W \cap V)$ is open in $\mathbb{R}^n$ and has zero measure by 3.11 (a) and (c). This contradicts 3.11 (b). ■

Theorem 3.14. (Sard's theorem) *Let $f : M \to N$ be smooth. Then the set of critical values of f is a set of measure zero. In particular the regular values are dense.*

Remark. For $dim(M) < dim(N)$ this proves that $f(M) \subset N$ is a set of measure zero.

Proof of Sard's theorem. Let $C \subset M$ be the set of critical points of f. Consider a countable covering of M by parametrizations $\varphi_i : U_i \to M$, $i \in \mathbb{N}$. Then

$$f(C) = \bigcup_{i \in \mathbb{N}} f \circ \varphi_i(C_i)$$

where $C_i \subset U_i$ is the set of critical points of $f \circ \varphi_i$. Moreover $x \in M$ is a critical point for f if and only if it is critical for $\chi \circ f$ and any coordinate system χ for N at $f(x)$ (this is because χ is a diffeomorphism of open sets.) Thus

Sard's theorem follows, using the very definition of a set of measure 0, from the following case:

Theorem 3.15. (Euclidean version of Sard's theorem) *Let $U \subset \mathbb{R}^m$ be open and $f : U \to \mathbb{R}^n$ be smooth. Then $f(C) \subset \mathbb{R}^n$ is a set of measure zero, where C is the set of critical points of f.*

Proof. The proof will be done by induction on m. The case $m = 0$ is obvious. We will do the induction step from $m-1$ to m. Without restriction we can assume that $n \geq 1$.

It follows from 3.8. and continuity of derivatives that we have the following descending sequence of *closed* sets:

$$C \supset C_1 \supset C_2 \supset \ldots \supset C_i \supset C_{i+1} \supset \ldots$$

where

$$C_i := \{x \in U | \text{all partial derivatives of } f \text{ of order } \leq i \text{ vanish at } x\}$$

(For example $C_1 = \{x \in U | df_x = 0\}$.) In order to prove the claim we will prove three lemmas. The result then is immediate from 3.11 (a).

Lemma 1. *$f(C \setminus C_1)$ is a set of measure zero.*

Proof. For each $x \in C \setminus C_1$ we construct an open set $V = V_x$ with the property that $f(V_x \cap (C \setminus C_1))$ has measure zero. Then because of 1.6., $C \setminus C_1$ is covered by countably many sets V_i, $i \in \mathbb{N}$ with $V_i \subset V_x$ for some $x \in C \setminus C_1$. Since also $f(V_i \cap (C \setminus C_1))$ has measure zero the claim follows. Let $x \in C \setminus C_1$. It follows that there is a nonvanishing partial derivative $\frac{\partial f_1}{\partial x_1}(x) \neq 0$. Define

$$h : U \to \mathbb{R}^m$$

by

$$h(x) = (f_1(x), x_2, \ldots, x_m).$$

Then

$$dh_x = (grad(f_1)(x)^t, e_2, \ldots e_m)^t$$

is invertible. (Note that we consider $grad(f_1)(x)$ as a row vector here while e_i is the i-th canonical basis column vector of $\mathbb{R}^m$ for $i = 1, \ldots, m$.) Thus h maps an open neighborhood V of x diffeomorphically onto $V' \subset \mathbb{R}^m$. The map $g := f \circ h^{-1} : V' \to \mathbb{R}^n$ has the very same regular *values* as $f|V$, and

$$g(t, x_2, \ldots, x_m) = (t, y_2, \ldots, y_n)$$

for suitable $y_2, \dots y_n \in \mathbb{R}$. (*Proof.* Let $(t, x_2, \dots, x_m) \in V'$ be arbitrary. We know that $(t, x_2, \dots, x_m) = h(x) = (f_1(x), x_2, \dots, x_m)$ for some $x = (x_1, \dots, x_m) \in V$. Therefore $g(t, x_2, \dots, x_m) = f(h^{-1}(h(x))) = f(x) = (f_1(x), y_2, \dots, y_n)$ and $t = f_1(x)$.) Now consider for each t:

$$g^t := g|(t \times \mathbb{R}^{m-1}) \cap V' : (t \times \mathbb{R}^{m-1}) \cap V' \to t \times \mathbb{R}^{n-1}$$

We know that the image of the linear map described by the matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ * & & \left(\frac{\partial (g^t)_i}{\partial x_j}\right) & & \end{pmatrix}$$

is spanned by the columns of the matrix. Now $(t, x_2, \dots, x_m)$ is critical for g^t if and only if it is critical for g and not in C_1. By induction hypothesis we know that the set of critical values of g^t in $t \times \mathbb{R}^{n-1}$ has measure zero. It follows from Fubini's result 3.11 (d) (see Remark below and also 3.20.) that the claim holds for g. ■

Remark. For the application of the measure zero set-Fubini, note that the set of critical values of g is *not* necessarily closed. But we can argue as follows: $C \subset V$ is a closed set. Thus there is $\tilde{C} \subset \mathbb{R}^m$ closed such that $\tilde{C} \cap V = C$, e. g. $\tilde{C} = cl_{\mathbb{R}^m} C$. Now $V = \cup_{j \in \mathbb{N}} K_j$ for compact sets $K_j \subset V$ by 1.6. Then

$$C = \bigcup_{j \in \mathbb{N}} \tilde{C} \cap K_j = \bigcup_{j \in \mathbb{N}} C \cap K_j.$$

Now $\tilde{C} \cap K_j$ is closed in K_j thus compact. Thus C is countable union of compact sets, and thus $g(C)$ is a countable union of compact sets.

Lemma 2. *For $k \geq 1$ the set $f(C_k \setminus C_{k+1})$ has measure zero.*

Proof. Let $x \in C_k \setminus C_{k+1}$. Then there is a $(k+1)$-st partial derivative at x, which is not zero. Let $\rho(x)$ be a k-th partial derivative such that $\frac{\partial \rho}{\partial x_1}(x) \neq 0$. Define $h : U \to \mathbb{R}^m$ by

$$h(x) = (\rho(x), x_2, \dots, x_m).$$

Then h maps an open neighborhood V of x diffeomorphically onto an open set $V' \subset \mathbb{R}^m$. We have

$$h(C_k \cap V) \subset \{0\} \times \mathbb{R}^{m-1}.$$

It follows that all critical points of $g := f \circ h^{-1}$ of type C_k are contained in the hyperspace $\{0\} \times \mathbb{R}^{m-1}$. Let

$$\overline{g} : (\{0\} \times \mathbb{R}^{m-1}) \cap V' \to \mathbb{R}^n$$

be the restriction of g. By induction hypothesis it follows that the set of critical values of $\overline{g}$ is a set of measure zero. But all critical points of g (of type C_k) are critical points of $\overline{g}$. It follows that $f((C_k \setminus C_{k+1}) \cap V)$ is a set of measure zero. Now as before find a covering of $C_k \setminus C_{k+1}$ by countably many sets, each contained a set V as above. ■

Lemma 3. *For $k > \frac{m}{n} - 1$ we have that $f(C_k)$ is a set of measure zero.*

Proof. Let $S \subset U$ be a cube of length δ. We will show that for $k > \frac{m}{n} - 1$ the set $f(C_k \cap S)$ has measure zero. Then we can cover U by countanbly many cubes to conclude that $f(C_k)$ has measure zero. For $x \in C_k \cap S$ with $x + h \in S$ we have

$$f(x+h) = f(x) + R(x,h)$$

with

$$(*) \qquad ||R(x,h)|| < a||h||^{k+1}.$$

(Here we use the Taylor remainder term for smooth functions of several variables. You can find this e. g. in Gerald B. Folland: Advanced Calculus, or any other book discussing Multivariable Calculus.) Since S is compact we can assume that a only depends on f and S (and not on x). In fact, we have

$$R(x,h) = \sum_{|\alpha|=k+1} \frac{D^\alpha f(x+\vartheta h)}{\alpha!} h^\alpha$$

by the mean value theorem for some $0 \leq \vartheta \leq 1$, where we use obvious multi-index notation, and $D^\alpha f$ attains a maximimum on S. Now subdivide S into r^m cubes of length $\frac{\delta}{r}$ and let S_1 be that one of the resulting cubes with $x \in S_1 \cap C_k$. Then each point in S_1 is of the form $x + h$ with

$$||h|| < \sqrt{m}\frac{\delta}{r},$$

because

$$||(\frac{\delta}{r}, \ldots, \frac{\delta}{r})|| = \sqrt{\sum_{k=1}^{m} (\frac{\delta}{r})^2} = \sqrt{m}\frac{\delta}{r}$$

It follows from $(*)$ above that $f(S_1)$ is contained in a cube of length $\frac{b}{r^{k+1}}$ with center $f(x)$, and $b = 2a(\sqrt{m}\delta)^{k+1}$. Thus $f(C_k \cap S)$ is contained in a union of at most r^m cubes of total volume at most

$$r^m (\frac{b}{r^{k+1}})^n = b^n r^{m-(k+1)n} \to 0$$

for $r \to \infty$ if $k+1 > \frac{m}{n}$. Thus $f(C_k \cap S)$ is a set of measure zero. (Note that the natural number r determines the covering by cubes and can be chosen arbitrarily large.) ■

For most of the remaining results in this chapter we will consider the special case of smooth maps

$$f : M \to \mathbb{R}.$$

The study of non-degenerate maps and their relation with topology in this case is called *Morse theory*.

In this case $x \in M$ is a critical point of $f \iff grad(f)(x) = 0$.

Example. Let M be compact. Then each function $f : M \to \mathbb{R}$ has at least two critical points, namely where the function attains maximum respectively minimum value.

For $U \subset \mathbb{R}^m$ open and $f : U \to \mathbb{R}$ smooth let

$$H_x(f) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right) \in M(n)$$

be the *Hesse-matrix* of f at x.

Definition 3.15. A critical point $x \in U$ is called *non-degenerate* if $H_x(f)$ is an invertible matrix. The function f is called a *Morse function* if all the critical points of f are non-degenerate.

Observation. *Non-degenerate critical points are isolated.*

Proof. Given f let $g := grad(f) : U \to \mathbb{R}^m$ be the map

$$U \ni x \mapsto (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m})$$

The point x is critical for f means $g(x) = 0$. Now $dg_x = H_x(f)$ non-degenerate means that g maps a neighborhood U of x diffeomorphically onto a neighborhood of $0 \in \mathbb{R}^m$. It follows that there exists no $x' \neq x$ in U with $g(x') = 0$. Thus there is no further critical point of f in U. ■.

Definition 3.16. Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional manifold and $f : M \to \mathbb{R}$ be a smooth map, $x \in M$ a critical point. Then x is *non-degenerate* for f if for some (and because of 3.17. every) parametrization $\varphi : U \to M$ at x we have: x is nondegenerate for $f \circ \varphi$.

In fact let x be a critical point, and without restriction let $\varphi_1, \varphi_2 : U \to M$ be two parametrizations with $\varphi_1(0) = \varphi_2(0) = x$ (use translations and shrink).

In order to prove that 0 is nondegenerate for $f \circ \varphi_1 \iff 0$ is nondegenerate for $f \circ \varphi_2$ it suffices to prove:

Lemma 3.17. *Let $U \subset \mathbb{R}^m$ be open and let $0 \in U$ be a non-degenerate critical point of $f : U \to \mathbb{R}$, $\psi : U \to U$ a diffeomorphism with $\psi(0) = 0$. Then 0 is also non-degenerate critical point of $f' = f \circ \psi$.*

Proof. By the chain rule we get for $x \in U$

$$\frac{\partial f'}{\partial x_i}(x) = \sum_{k=1}^{m} \frac{\partial f}{\partial x_k}(\psi(x))\frac{\partial \psi_k}{\partial x_i}(x)$$

It follows that

$$\frac{\partial^2 f'}{\partial x_i \partial x_j}(0) = \sum_{k,l} \frac{\partial^2 f}{\partial x_k \partial x_l}(0)\frac{\partial \psi_l}{\partial x_j}(0)\frac{\partial \psi_k}{\partial x_i}(0) + \sum_{k} \frac{\partial f}{\partial x_k}(0)\frac{\partial^2 \psi_k}{\partial x_i \partial x_j}(0).$$

Thus because $\frac{\partial f}{\partial x_k}(0) = 0$ it follows that

$$H_0(f') = (d\psi_0)^t H_0(f) d\psi_0.$$

Therefore $H_0(f)$ is invertible if and only if $H_0(f')$ is invertible
(because $det(d\psi_0) \neq 0$).

Theorem 3.18.(Morse lemma) *Let $f : M \to \mathbb{R}$ be smooth and $a \in M$ be a non-degenerate critical point of f. Then there exists a parametrization $\varphi : U \to M$ with $\varphi(0) = a$, and a natural number $p := ind(f, a)$, the index of f at a, with $0 \leq p \leq m := dim(M)$, such that for all $x = (x_1, \ldots, x_m) \in U$ the following holds:*

$$f \circ \varphi(x_1, \ldots, x_m) = f(a) - x_1^2 - x_2^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_m^2$$

Remark. At an index m point the function has a local maximum. Thus going away from a in any direction the values of the function are descending. At an index 0 point the function has a local minimum and the values of the function are decreasing in any direction close to the point. At an index p critical point there are p descending *directions* and $(m - p)$ ascending directions (generalized monkey saddle).

Example. Let $T \subset \mathbb{R}^3$ be a doughnut surface like in Exercises 2.2. Let f be the restriction of the projection $p(x_1, x_2, x_3) = x_1$ to T. This function has two critical points of index 1, a criticial point of index 2 and a critical point of index 0.

The Morse lemma says that locally a smooth function at a nondegenerate critical point is determined by a *quadratic form.*

Proof of Morse lemma. We first prove that the index at a nondegenerate critical point is well-defined. Let $f \circ \varphi_1$ and $f \circ \varphi_2$ be as above. Then $H_0(f \circ \varphi_i)$ is a diagonal matrix with p_i terms -2 and $m - p_i$ terms $+2$ on the diagonal. But by 3.17 using suitable ψ we get

$$H_0(f \circ \varphi_0) = (d\psi_0)^t H_0(f \circ \varphi_2)(d\psi_0),$$

and in block matrix form

$$\begin{pmatrix} a_{11}^2(\pm 2) & 0 \\ 0 & \star \end{pmatrix} = \begin{pmatrix} a_{11} & \star \\ \star & \star \end{pmatrix} \begin{pmatrix} \pm 2 & 0 \\ 0 & \star \end{pmatrix} \begin{pmatrix} a_{11} & \star \\ \star & \star \end{pmatrix}$$

The general result follows from this by noting that we can permute the coordinates.

To prove the extistence of φ as above we need the following:

Lemma 3.19. *Let V be a convex neighborhood of $0 \in \mathbb{R}^m$ and $f : V \to \mathbb{R}$ be a smooth map with $f(0) = 0$. Then*

$$f(x) = \sum_{i=1}^{m} x_i g_i(x)$$

for smooth functions $g_i : V \to \mathbb{R}$ with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof. By the fundamental theorem of calculus

$$f(x) = \int_0^1 \frac{d}{dt}(f(tx))dt = \int_0^1 \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(tx) x_i dt = \sum_{i=1}^{m} \left(\int_0^1 \frac{\partial f}{\partial x_i}(tx)dt \right) x_i,$$

so we define

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$$

■

Note that for proving 3.18 we can assume that $f : U \to \mathbb{R}$ with $f(0) = 0$, 0 is a non-degenerate critical point, and $U \subset \mathbb{R}^m$ convex. Then for $x \in U$ we know that $f(x) = \sum_{j=1}^m g_j(x)x_j$ and since 0 is critical we have $g_j(0) = \frac{\partial f}{\partial x_j}(0) = 0$. If we apply 3.19 to the maps g_j we get $g_j(x) = \sum_{i=1}^m x_i h_{ij}(x)$ for smooth functions $h_{ij} : U \to \mathbb{R}$. Thus

$$f(x) = \sum_{i,j=1}^{m} x_i x_j h_{ij}(x) = \frac{1}{2} \sum_{i,j=1}^{m} x_i x_j (h_{ij}(x) + h_{ji}(x)) = \sum_{i,j=1}^{m} x_i x_j \overline{h}_{ij}(x),$$

where

$$\overline{h}_{ij} = \frac{1}{2}(h_{ij} + h_{ji}).$$

It is an exercise to show that

$$\overline{h}_{ij}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0) = \frac{1}{2}H_0(f)_{ij}$$

and thus is symmetric and nonsingular. The rest of the proof is an induction. Suppose by induction hypothesis that for $r \geq 0$ we have constructed a neighborhood U_1 of 0, and a parametrization $\varphi_1 : U_1 \to U$ with $\varphi_1(0) = 0$ such that

$$f \circ \varphi_1(u) = \pm u_1^2 \pm \ldots \pm u_{r-1}^2 + \sum_{r \leq i,j \leq m} H_{ij}(u) u_i u_j$$

with $H(u) = (H_{ij}(u))_{r \leq i,j \leq m}$ a symmetric and nonsingular matrix for all $u \in U_1$. The case $r = 0$ has been discussed above. It is known from linear algebra (*diagonalization of symmetric matrices*) that there exists an orthogonal matrix A such that

$$AH(0)A^t = diag(\lambda_r, \ldots, \lambda_m),$$

where *diag* means diagonal matrix, and all $\lambda_i \neq 0$. (see e. g. P. M. Cohn, Elements of linear algebra, theorem 8.8.) Let $\tilde{u} = (u_r, \ldots, u_m)$ and

$$\tilde{u}^t H(u)\tilde{u} = (A\tilde{u})^t AH(u)A^t(A\tilde{u}) = (\tilde{u}')^t H'(u')\tilde{u}',$$

where

$$u' = \begin{pmatrix} I_{r-1} & 0_{r-1,m-r+1} \\ 0_{m-r+1,r-1} & A \end{pmatrix} u$$

and

$$H'(0) = AH(0)A^t = diag(\lambda_r, \ldots, \lambda_m).$$

To simplify notation we write u for u' and H for H'. We know that $H_{rr}(0) = \lambda_r \neq 0$. Let $g(u) := \sqrt{|H_{rr}(u)|}$. This is a smooth nonzero function in a small neighborhood $U_2 \subset U_1$ of 0 with $g(u)^2 = \mp H_{rr}(u)$. Now we calculate for $u \in U_2$:

$$
\begin{aligned}
f \circ \varphi_1(u) =& \pm u_1^2 \pm \ldots \pm u_{r-1}^2 \pm (u_r g(u))^2 \\
&+ u_r g(u) \sum_{i>r} u_i \frac{H_{ir}(u)}{H_{rr}(u)} (\pm g(u)) + u_r g(u) \sum_{j>r} u_j \frac{H_{rj}(u)}{H_{rr}(u)} (\pm g(u)) \\
&+ \sum_{i,j>r} u_i u_j H_{ij}(u) \\
&= \pm u_1^2 \pm \ldots \pm u_{r-1}^2 \pm \left(u_r g(u) + \sum_{i>r} u_i \frac{H_{ir}(u)}{H_{rr}(u)} g(u) \right)^2 \\
&\mp \sum_{i,j>r} u_i u_j \frac{H_{ir}(u) H_{jr}(u)}{H_{rr}(u)^2} g(u)^2 + \sum_{i,j>r} u_i u_j H_{ij}(u) \\
&= \pm u_1^2 \pm \ldots \pm u_{r-1}^2 \pm \tilde{u}_r^2 + \sum_{i,j>r} u_i u_j (H_{ij}(u) - H_{ir}(u) H_{jr}(u)) \\
&=: f \circ \varphi_2^{-1}(\tilde{u})
\end{aligned}
$$

with $\varphi_2^{-1}(\tilde{u}) = \varphi_1(u)$ which holds by definition if and only if

$$
\tilde{u} = \varphi_2(\varphi_1(u)) := (u_1, \ldots, u_{r-1}, \tilde{u}_r, u_{r+1}, \ldots, u_m).
$$

Thus we define

$$
\rho : U_2 \to U
$$

by $\rho(u) = \tilde{u}$. Then ρ is a diffeomorphism in a neighborhood U_3 of 0 and we use ρ to define

$$
\varphi_2 := \rho \circ \varphi_1^{-1}.
$$

It follows that

$$
f \circ \varphi_2^{-1}(\tilde{u}) = \pm \tilde{u}_1^2 \pm \ldots \pm \tilde{u}_r^2 + \sum_{(r+1) \leq i,j \leq m} \tilde{u}_i \tilde{u}_j \tilde{H}_{ij}(\tilde{u})
$$

with

$$
\tilde{H}_{ij}(\tilde{u}) := H_{ij}(u) - H_{ir}(u) H_{jr}(u)
$$

for $r + 1 \leq i, j \leq m$. Note that this is again symmetric and nonsingular in a neighborhood of 0. This shows the inductive step and proves the result. ■

Remark. For M compact and $f : M \to \mathbb{R}$ a Morse function with critical points x_i, $1 \leq i \leq N$ let

$$
\chi(M, f) := \sum_{i=1}^{N} (-1)^{ind(f, x_i)} = \sum_{p} (-1)^p n(p),
$$

where $n(p)$ is the number of critical points of index p. We will prove later on that the number $\chi(M, f)$ is actually independent of f. It is called the *Euler characteristic* of M and is one of the most important concepts of geometric topology.

The following lemma actually has been used in the proof of Lemma 1 for the proof of Sard's theorem. (Note that the measure zero set Fubini easily extends to countable unions of closed sets.)

Lemma 3.20. *Let $f : M \to N$ be smooth. Then the set of critical values of f is a countable union of closed sets.*

Proof. Let $M = \cup_{j\in\mathbb{N}}\varphi_j(U_j)$ for parametrizations φ_j, and thus

$$f(C) = \cup_{j\in\mathbb{N}} f \circ \varphi_j(C_j)$$

where C is the set of critical points of f and C_j is the set of critical points of $f \circ \varphi_j$. Thus it suffices to prove 3.20 for a function $g : U \to N$ with $U \subset \mathbb{R}^m$ an open set. Now cover U by countably many cubes W_i, $i \in \mathbb{N}$. Then

$$g(C) = \cup_{i\in\mathbb{N}} g(C \cap W_i),$$

with each $W_i \cap C$ is compact, and thus $g(W_i \cap C)$ is compact and thus closed. ■

Convention. If M is a smooth manifold then *for almost all $x \in M$* means for all x in the complement of a set of measure zero.

Theorem 3.21. *Let $M \subset \mathbb{R}^k$ and $f : M \to \mathbb{R}$ be smooth. For $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ let*

$$f^a(x) = f(x) + < a, x > .$$

Then f^a is a Morse function for almost all $a \in \mathbb{R}^k$.

Remark. For $||a||$ sufficiently small the function f^a approximates f pointwise, *uniformly* on compact submanifolds.

Corollary 3.22. *For each smooth manifold M there exists a Morse function.*

Proof. Approximate the projection

$$M \subset \mathbb{R}^k \xrightarrow{\ p\ } \mathbb{R}$$

where p is the projection onto the last coordinate. ■

Proof of 3.21.
Case 1: Let $M = U \subset \mathbb{R}^k$ be open. For $g := grad(f) : U \to \mathbb{R}^k$ it follows that

$$grad(f^a)(x) = g(x) + a$$

and

$$H_x(f^a) = dg_x = H_x(f).$$

Thus we have that x is a critical point of f^a if and only if $g(x) = -a$. Then $-a$ is regular for g if and only if $H_x(f^a)$ is invertible for all x with $g(x) = -a$. Thus it follows from Sard's theorem, applied to the function g, that f^a is Morse for almost all $a \in \mathbb{R}^k$.

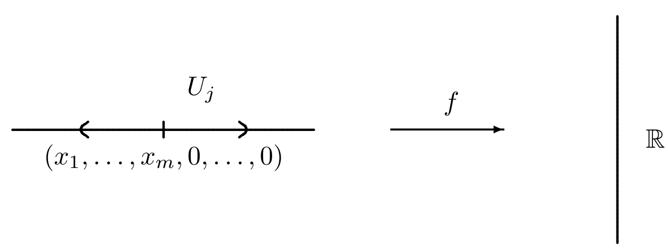
Case 2: Let $(U_j)_{j\in\mathbb{N}}$ be an open covering of M. Then f^a is not Morse if and only if $f^a|U_j$ is not Morse for some $j \in \mathbb{N}$. Moreover

$$\{a \in \mathbb{R}^k | f^a \text{ is not Morse}\} = \bigcup_{j\in\mathbb{N}} S_j,$$

where

$$S_j := \{a \in \mathbb{R}^k |\ f^a|U_j \text{ is not Morse}\}$$

We will show that S_j is a set of measure zero for all $j \in \mathbb{N}$.
(i) Suppose that $U_j \subset \mathbb{R}^m \times \{0\} \subset \mathbb{R}^k$.



Then for almost all $b \in \mathbb{R}^m$ we have $x \mapsto f(x) + < b, x >$ is Morse, but $b_{m+1}x_{m+1} + \ldots + b_k x_k = 0$ for arbitrary $(b_{m+1}, \ldots b_k) \in \mathbb{R}^{k-m}$ and $x \in U_j$. Thus the set of all $a \in \mathbb{R}^k$ such that $f^a|U_j$ is not Morse has the form $N \times \mathbb{R}^{k-m}$ where N is a set of measure zero in $\mathbb{R}^k$. It follows from the zero measure Fubini that $N \times \mathbb{R}^{k-m}$ is a set of measure zero in $\mathbb{R}^k$.
(ii) Next suppose that $(p_1, \ldots, p_m)$ is a coordinate system on U_j where
$p_i(x_1, \ldots, x_k) = x_i$ for $1 \le i \le k$ are the usual coordinate projections. Then define for $c \in \mathbb{R}^{k-m}$ and $x = (x_1, \ldots x_k) \in U_j$:

$$f^{(0,c)}(x) = f(x) + c_{m+1}x_{m+1} + \ldots + c_k x_k$$

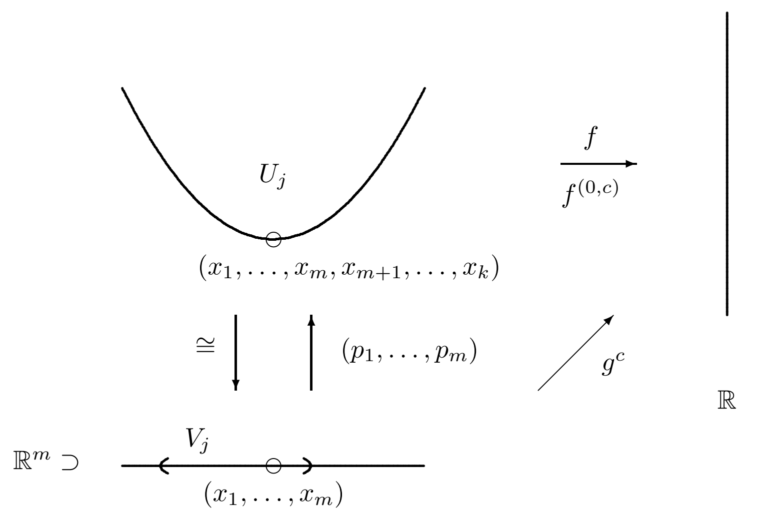
If we apply *Case 1* to the function g^c (see picture below) it follows that for fixed c for almost all $b \in \mathbb{R}^m$ the function

$$f^{(b,c)}(x) = f^{(0,c)}(x) + b_1 x_1 + \ldots + b_m x_m$$

is a Morse function. Therefore

$$S_j \cap (\mathbb{R}^m \times \{c\})$$

is a set of measure zero for all $c \in \mathbb{R}^m$. It follows that S_j is a set of measure zero.



(iii) Because of (ii) it suffices to find a covering $(U_j)_{j \in \mathbb{N}}$ such that for each j there exists a sequence of projections $(p_{i_1}, p_{i_2}, \ldots, p_{i_m})$ mapping U_j diffeomorphically into $\mathbb{R}^m$. Then the restriction to M has the differential

$$(p_{i_1}, \ldots, p_{i_m})|TM_x,$$

thus is a local diffeomorphism at x. This gives for each $x \in M$ a neighborhood V_x with properties as in (ii). Then cover M by countably many open V_i such that for each i, $V_i \subset V_x$ for some $x \in M$, and thus the condition of Case (ii) is satisfied on V_i. ■

Similarly to the study of regular points is the study of points at which df_x has rank equal to $dim(M)$. At such a point the function is called *immersive*. For those functions one can prove a result analogous to 3.2.

Theorem 3.23. *Let $f : M \to N$ be a smooth map and $x \in M$ with rank$(df_x) =$ dim$(M) = m \leq$ dim$(N) = n$. Then there exist parametrizations $\varphi : U \to M$ and $\psi : V \to N$ such that*

$$\psi^{-1} \circ f \circ \varphi(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0) \in \mathbb{R}^n$$

The proof of this is left as Exercise 4.2.

We conclude this chapter by proving a *local* converse of 3.3. For an inclusion of smooth manifolds $Z \subset M$ let

$$codim_M(Z) := dim(M) - dim(Z) = dim(\nu(Z, M)_x)$$

for all $x \in Z$.

Theorem 3.24. *Let $Z, M \subset \mathbb{R}^k$ be smooth manifolds, $Z \subset M$ and $z \in Z$. Then there is a neighborhood W of z in M and a smooth map $f : W \to \mathbb{R}^\ell$ with regular value 0 such that*

$$f^{-1}(0) = W \cap Z,$$

where $\ell := codim_M(Z)$.

Proof. By 3.23 applied to the inclusion $Z \subset M$ we find parametrizations $\varphi : U \to M$ and $\psi : V \to \mathbb{R}^n$ with $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ open sets such that

$$\varphi^{-1} \circ \psi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$$

for all $(x_1, \ldots, x_n) \in V$. Note that $\ell = m - n$. The set $\varphi^{-1} \circ \psi(V) \subset \mathbb{R}^n \times \{0\}$ is open in $\mathbb{R}^n \times \{0\}$. Therefore we can find open $U_1 \subset U$ with $\varphi^{-1} \circ \psi(V) = (\mathbb{R}^n \times \{0\}) \cap U_1$. Now $\psi(V)$ is open in Z. So there is $W_1 \subset \mathbb{R}^k$ open with $W_1 \cap Z = \psi(V)$. Let $\varphi_1 := \varphi | U_1$ and $U_2 := \varphi_1^{-1}(W_1)$. Let $\varphi_2 := \varphi | U_2$. Then

$$\varphi_2^{-1}(Z) = \varphi_2^{-1}(\psi(V)) = U_2 \cap (\mathbb{R}^n \times \{0\})$$

Let $W := \varphi_2(U_2)$ and $f := p \circ \varphi_2^{-1}$ where $p : \mathbb{R}^m \to \mathbb{R}^\ell$ is the projection

$$p(x_1, \ldots, x_m) = (x_{n+1}, \ldots, x_m).$$

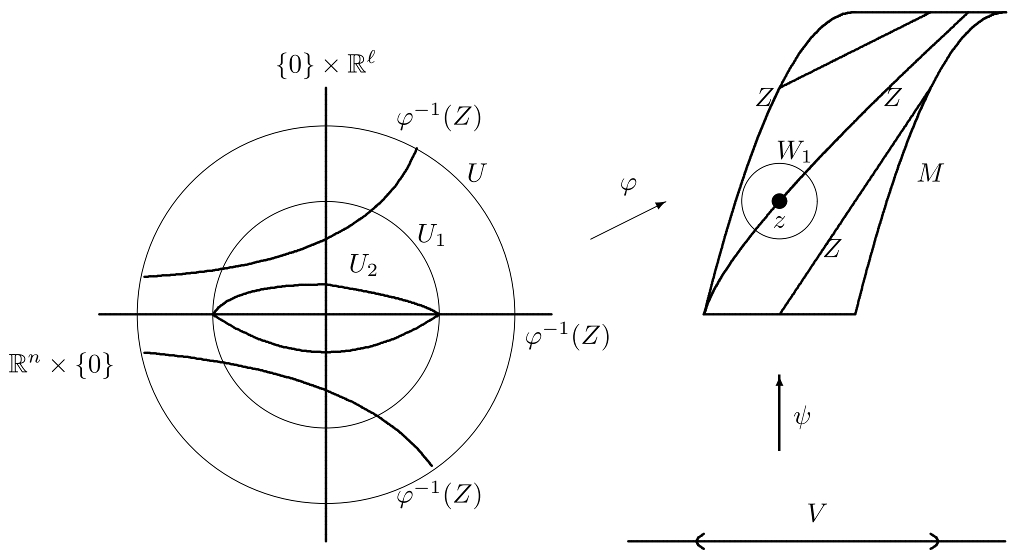
Now 0 is a regular value for f because for $y \in f^{-1}(0)$

$$df_y = dp_x \circ d(\varphi_2^{-1})_y,$$

which is the composition of p with an isomorphism, and $\varphi_2(x) = y$. Now

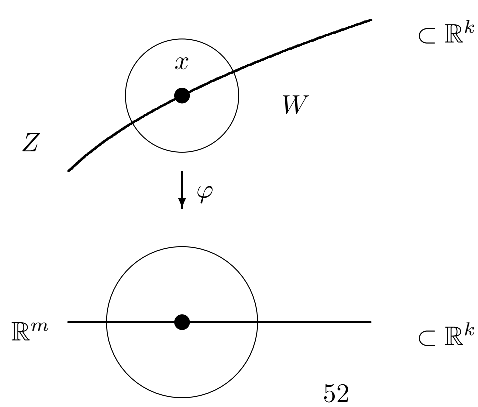
$$p \circ \varphi_2^{-1}(y) = 0 \Longleftrightarrow \varphi_2^{-1}(y) \in \mathbb{R}^n \times \{0\} \Longleftrightarrow y \in \psi(V) = W \cap Z,$$

thus $f^{-1}(0) = W \cap Z$.



■

Remark. This is true in particular for $M = \mathbb{R}^k$ and $Z \subset \mathbb{R}^k$ a manifold of dimension m, and thus can be used as equivalent way to define smooth manifolds. Note that in the proof we have shown that *locally* a smooth manifold $Z \subset \mathbb{R}^k$ looks like $\mathbb{R}^m \subset \mathbb{R}^k$.



We have

$$\varphi(W \cap Z) = \mathbb{R}^m \times \{0\}$$

and without restriction (why?)

$$\varphi(W) = \mathbb{R}^k.$$

Chapter 4

Manifolds with boundary and orientations.

Let $B^m := \{x \in \mathbb{R}^m | \; ||x|| \leq 1\}$ or $S^1 \times I \subset \mathbb{R}^3$ where $I = [0,1]$. These subsets of Euclidean spaces are not smooth manifolds. Note that $D^m \subset B^m$ is a smooth manifold of dimension m. But for points in $fr_{B^m}(D^m) = S^{m-1}$ the condition of an m-dimensional manifold is not satisfied (even though S^{m-1} is of course an $(m-1)$-dimensional manifold). Similarly $S^1 \times (0,1)$ is a smooth surface. The points in $fr_{S^1\times[0,1]}S^1 \times (0,1)$ form the disjoint union of two circles $S^1 \times \{0\}$ and $S^1 \times \{1\}$, and are 1-dimensional manifolds. The model space for a neighborhood of a point in S^{m-1} in B^m is not $\mathbb{R}^m$ but the half-space

$$H^m := \{x \in \mathbb{R}^m | x_m \geq 0\}.$$

Note that

$$fr_{\mathbb{R}^m} H^m = fr_{H^m}(int_{\mathbb{R}^m} H^m) = \{x \in \mathbb{R}^m | x_m = 0\}.$$

Definition 4.1. A space $M \subset \mathbb{R}^k$ is called *m-dimensional manifold with boundary* if for each $x \in M$ there is an open neighborhood in M, which is diffeomorphic to an open subset of H^m.

Remarks. (a) A diffeomorphism

$$\varphi : H^m \supset U \to \varphi(U) \subset M$$

with U open in H^m (and thus $\varphi(U)$ open in M) is called a *parametrization* at x for all $x \in \varphi(U)$.

(b) Recall from Exercise 1.4. that $U \subset H^m$ is open if $U = U' \cap H^m$ for some open subset U' in $\mathbb{R}^m$. Let $x \in U$. If $x \in int(H^m)$ then there exists $\varepsilon > 0$ such that $D(x, \varepsilon) \subset U' \cap int(H^m) = U \cap int(H^m)$, which is open in $\mathbb{R}^m$. If $x \in fr(H^m)$ then for all $\varepsilon > 0$ we have $D(x, \varepsilon) \cap (\mathbb{R}^m \setminus H^m) \neq \emptyset$. Thus in a manifold with boundary we can distinguish between the points with a parametrization defined on an open subset of $\mathbb{R}^m$, and those *not* defined on an open subset of $\mathbb{R}^m$.
(c) The *(manifold) boundary of M* is the set

$$\partial M := \{x \in M | \exists\ \varphi \text{ parametrization at } x \text{ with } x \in \varphi(U \cap fr(H^m))\} \subset M.$$

It follows that at each $x \in M \setminus \partial M$ there is a parametrization $\varphi : U \to M$ with $\varphi(u) = x$ and $x \in U \cap int(H^m)$. The *interior of M* is defined by

$$Int(M) := M \setminus \partial M,$$

and thus is a usual smooth manifold (without boundary) in the sense of chapter 2. Note that in general $\partial M \neq fr_{\mathbb{R}^k}(M)$.

Examples. (a) Let $M = \{x \in \mathbb{R}^2 | 1 \leq ||x|| < 2\}$ is a smooth 2-dimensional manifold with boundary and $\partial M = S^1$ while $fr_{\mathbb{R}^2}(M) = S^1 \cup 2S^1$.
(b) Let $M = S^1 \times I \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ with $I = [0, 1]$. Then $\partial M = S^1 \times \{0\} \cup S^1 \times \{1\}$ and $fr_{\mathbb{R}^3}(M) = M$, $int_{\mathbb{R}^3}(M) = \emptyset$ but $Int(M) = S^1 \times (0, 1)$.

Question. Is it always true that $\partial M \subset fr(M)$?

Convention. By definition each manifold in the sense of 2.3 is also a manifold with boundary. But in general one uses the word *manifold with boundary* only if $\partial M \neq \emptyset$.

Example. $M = I = [0, 1] \subset \mathbb{R}$ is a manifold with boundary but not $I \times I \subset \mathbb{R}^2$. In fact a diffeomorphism from a neighborhood U of $(1, 1) \in I \times I$ in $I \times I$ to an open neighborhood of 0 in H^2 would restrict to a diffeomorphism from $U \cap fr(I \times I)$ to a neighborhood of 0 in $\mathbb{R}^{m-1} \times \{0\} \subset H^m$ (compare 4.4. to check that this is true). But such a diffeomorphism does not exist by 2.2.

Theorem 4.2. *Let N be a manifold with boundary and M a manifold without boundary. Then $M \times N$ is a manifold with boundary and*

$$\partial(M \times N) = M \times \partial N$$

and

$$dim(M \times N) = dim(M) + dim(N).$$

Proof. Let $U \subset \mathbb{R}^m$ be open and $\varphi : U \to M$, and $V \subset H^n$ be open and $\psi : V \to N$ be parametrizations. Then $\varphi \times \psi : U \times V \to M \times N$. Note that $U \times V \subset \mathbb{R}^m \times H^n = H^{m+n}$ is open (compare Exercise 2.1.) ■

Example. For M a smooth manifold, $\partial(M \times I) = M \times \partial I = M \times \{0\} \cup M \times \{1\}$.

Remark 4.3.(Definition of df_x, TM_x etc.) Let

$$g : H^m \supset U \to \mathbb{R}^\ell$$

be smooth. Then for all $u \in U$ we define $TU_u = \mathbb{R}^m$. For the definition of derivatives we distinguish two cases.
Case 1: Let $u \in int(H^m)$. Then g is defined on an open neighborhood of u in $\mathbb{R}^m$ thus dg_u is defined as before.
Case 2: Let $u \in fr(H^m)$. Then extend g to a smooth mapping

$$\tilde{g} : U' \to \mathbb{R}^\ell.$$

This is possible by definition of smoothness. Then define

$$dg_u := d\tilde{g}_u : \mathbb{R}^m \to \mathbb{R}^\ell$$

Let $\tilde{\tilde{g}}$ be a second extension defined on U'.
Claim: $d\tilde{\tilde{g}}_u = d\tilde{g}_u$
Proof of Claim. There exists a sequence (u_i) in $U \cap int(H^m)$, which converges to u for $i \to \infty$. It follows that $d\tilde{g}_{u_i} = d\tilde{\tilde{g}}_{u_i}$ for all $i \in \mathbb{N}$. But the mapping $U' \ni x \mapsto d\tilde{g}_x$ is a smooth mapping from U' into the space of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^\ell$ (which we identify with $\mathbb{R}^{k\times\ell}$). This is because each component of the Jacobi-matrix is smooth and thus continuous. The same applies to $\tilde{\tilde{g}}$. Thus we have $d\tilde{g}_{u_i} \to d\tilde{g}_u$ and $d\tilde{\tilde{g}}_{u_i} \to d\tilde{\tilde{g}}_u$ for $i \to \infty$. The claim follows because limits are unique. ■
Thus dg_u is well-defined for all $u \in U$.

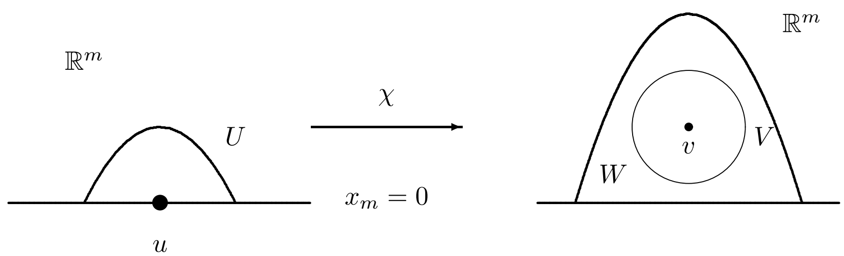
Now if $M \subset \mathbb{R}^k$ is a mooth manifold with boundary, $x \in M$ and $\varphi : U \to M$ is a parametrization with $\varphi(u) = x$ then we define

$$TM_x := d\varphi_u(\mathbb{R}^m) \subset \mathbb{R}^k,$$

which is an m-dimensional vector space. As in chapter 2 it can be shown that this does not depend on the choice of (φ, U). Now if $f : M \to N$ is smooth then $df_x : TM_x \to TN_{f(x)}$ is defined as in chapter 2 by local extension of f at x. The proof of the chain rule in this general setting is left as an exercise.

Theorem 4.4. *Let $U \subset H^m$ and $V \subset H^m$ be open sets, and $\chi : U \to V \subset H^m$ be a diffeomorphism. Then*

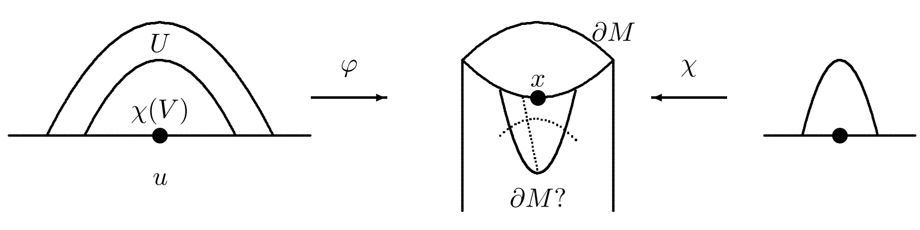
$$\chi(U \cap fr(H^m)) = V \cap fr(H^m).$$



Proof. It suffices to show $\chi(U \cap fr(H^m)) \subset V \cap fr(H^m)$. (Then $\chi^{-1}(V \cap fr(H^m)) \subset U \cap fr(H^m)$ implies that $V \cap fr(H^m)) \subset \chi(U \cap fr(H^m))$.) Let $u \in U \cap fr(H^m))$. For the sake of contradiction suppose $\chi(u) = v \in int(H^m) \cap V$ (which is an open set in $\mathbb{R}^m$ because $int(H^m)$ is open in $\mathbb{R}^m$). Since $d\chi_v^{-1}$ is bijective, χ^{-1} is defined on an open neighborhood of v in $\mathbb{R}^m$. By the inverse function theorem χ^{-1} maps an open neighborhood W onto an open neighborhood of u in $\mathbb{R}^m$. Since $\chi^{-1}(W) \subset U$ it follows $u \in int(H^m)$, which is a contradiction. ■.

Corollary 4.5. *Let M be an m-dimensional manifold with boundary. Then ∂M is an $(m-1)$-dimensional manifold without boundary ($\partial\partial M = \emptyset$.)*

Proof. Let $x \in \partial M$ and $\varphi : H^m \supset U \to M$ be a parametrization with $\varphi(u) = x$ and $u \in fr(H^m) \cap U$.
Claim: $\varphi(U \cap fr(H^m)) = \varphi(U) \cap \partial M$ (It suffices to prove this claim because then $\varphi|U \cap fr(H^m)$ is a parametrization for ∂M at x, note that $\varphi(U) \cap \partial M$ is open in ∂M.)



First note that $\varphi(U \cap fr(H^m) \subset \partial M \cap \varphi(U)$ is clear by definition of ∂M.

We have to prove that $\partial M \cap \varphi(U) \subset \varphi(U \cap fr(H^m))$. Suppose for the sake of contradiction that this is false. Then there is some $\psi : H^m \supset V \to M$ with $\psi(V) \subset \varphi(U)$ (after shrinking let's say), $v \in V \cap fr(H^m)$ and $y = \psi(v) \notin \varphi(U \cap fr(H^m))$. Consider $\varphi^{-1} \circ \psi =: \chi : V \to \chi(V)$ is a diffeomorphism and $\chi(V) \subset U$. We have $v \in fr(H^m) \cap V$ but $\chi(v) \in int(H^m) \cap U$. This contradicts 4.4. ■.

Obviously if $f : M \to N$ is smooth then $\partial f := f|\partial M : \partial M \to N$ is a smooth map.

Corollary 4.6. *Let $f : M \to N$ be a diffeomorphism between manifolds with boundary. Then*

$$\partial f := f|\partial M : \partial M \to \partial N$$

is also a diffeomorphism.

Proof. It suffices to show that $f(\partial M) \subset \partial N$. Suppose $x \in \partial M$, $f(x) \notin \partial N$. Consider parametrizations $\varphi : U \to M$ and $\psi : V \to N$ with U, V open in H^m, and $\varphi(u) = x$ and $\psi(v) = f(x)$. We can assume $\varphi(U) \subset f^{-1}\psi(V)$. Then $\chi := \psi^{-1} \circ f \circ \varphi : U \to V$ is a diffeomorphism onto its image. Because $u \in fr(H^m) \cap U$ it follows that $\chi(u) \in \chi(U) \cap H^m$. This contradicts 4.4. ■

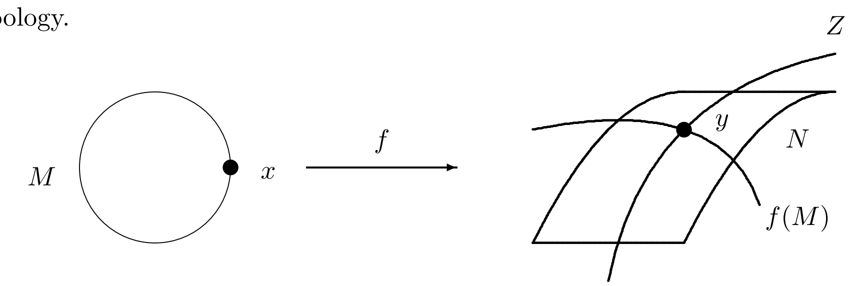
Note that for $f : M \to N$ it follows immediately that $d(\partial f)_x = df_x|T)(\partial M)_x$.

Recall that for two linear subspaces V_1, V_2 of a vector space V there is the subspace

$$V_1 + V_2 := \{v = v_1 + v_2 | v_1 \in V_1, v_2 \in V_2\} \subset V,$$

which is the smallest subspace of V containing both V_1 and V_2.

The following definition is *probably* the most important one in differential topology.



Definition 4.7. Let $f : M \to N$ be a smooth map between manifolds with boundary (but empty boundaries possible) and let $Z \subset N$ be a manifold. Then

f is called *transversal to Z at y* if for all $x \in f^{-1}(y)$ we have

$$df_x(TM_x) + TZ_y = TN_y.$$

If f is transversal to Z at all points $y \in Z$ then f is called *transversal to Z* with notation $f \pitchfork Z$. For two manfiolds $M_1, M_2 \subset N$ the notation is $M_1 \pitchfork M_2$ if the inclusion $\iota : M_1 \subset N$ is transversal to M_2 (or equivalently the inlcusion M_2 into N is transversal to M_1).

Theorem 4.8. *Let $f : M \to N$ be a smooth mapping, $Z \subset int(N)$ a smooth submanifold with $\partial Z = \emptyset$ (but ∂M or ∂N could be nonempty). Suppose that $f \pitchfork Z$, and if $\partial M \neq \emptyset$ also $(\partial f) \pitchfork Z$. Then $f^{-1}(Z) \subset M$ is a smooth manifold with*

$$\partial(f^{-1}(Z)) = (\partial f)^{-1}(Z) = \partial M \cap f^{-1}(Z)$$

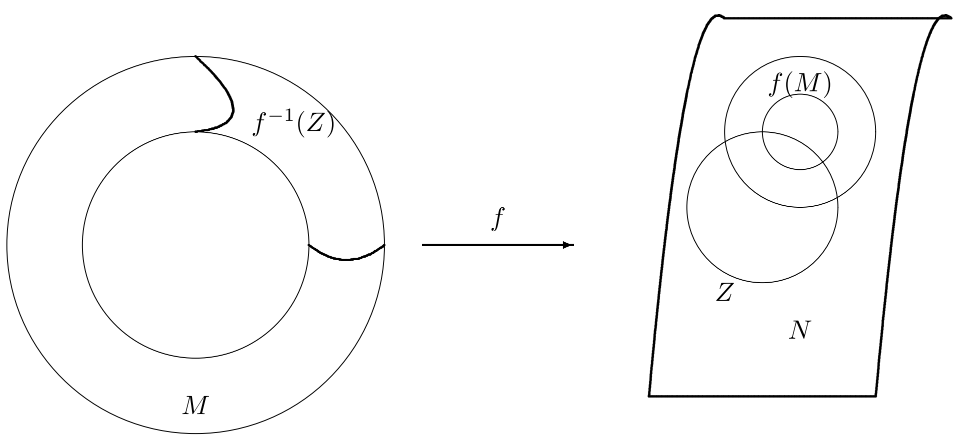
and

$$codim_M f^{-1}(Z) = codim_N(Z) = codim_{\partial M}(\partial f)^{-1}(Z).$$

Moreover

$$f^{-1}(Z) \pitchfork \partial M.$$

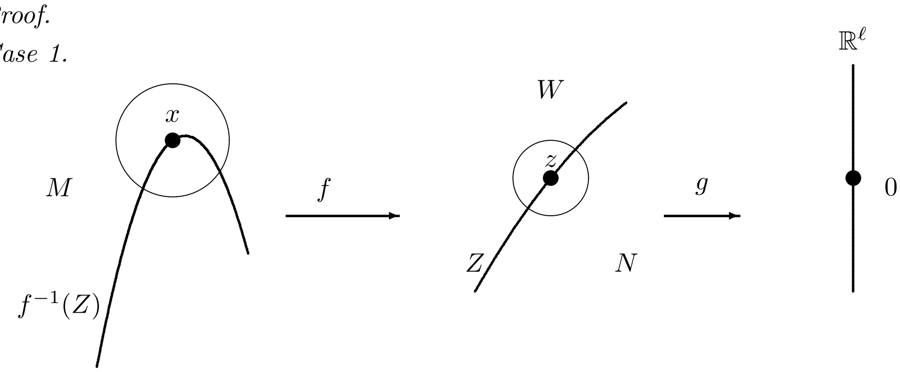
Here is a picture of a typical example:



Examples. (a) The notion of transversality generalizes that of a regular value. In fact if $Z = \{z\}$ is a point in $Int(N)$ then $f : M \to N$ is transversal to Z if and only if z is a regular value of f.
(b) If $f : M \to N$ is a *submersion*, i. e. df_x is onto for all $x \in M$ then f is transversal to any submanifold of N.
(c) If $f(M) \cap Z = \emptyset$ then $f \pitchfork Z$.

(d) If $Z \subset N$ is an open subset of N then $f \pitchfork Z$ for all $f : M \to N$. (e) If $f = \gamma : \mathbb{R} \to \mathbb{R}^2$ is a curve in $\mathbb{R}^2$ and $Z = \mathbb{R} \times \{0\}$ then $f \pitchfork Z$ if at all intersection points the derivative vector of f has a component in the y-direction.

Proof.
Case 1.



Let $\partial M = \emptyset$. Let $x \in f^{-1}(Z)$ thus $f(x) = z \in Z$. By 3.21. there is an open neighborhood W of z in $Int(N)$ and a smooth map $g : W \to \mathbb{R}^\ell$ with $Z \cap W = g^{-1}(0)$. W is also an open neighborhood in N because $Int(N) \subset N$ is open (and $\partial N \subset N$ is closed (Exercise!)). Consider

$$g \circ f : f^{-1}(W) \to \mathbb{R}^\ell$$

with $f^{-1}(W) \subset M$ open. We have $d(g \circ f)_x = dg_z \circ df_x$ by the chain rule. Consider

$$TM_x \xrightarrow{df_x} TN_z \xrightarrow{p} \nu(Z,N)_z \xrightarrow{dg_z} \mathbb{R}^\ell,$$

where p is the projection from $TN_z = TZ_z \oplus \nu(Z,N)_z$ onto $\nu(Z,N)_z$. Then $p \circ df_x$ is onto because

$$df_x(TM_x) + TZ_z = TN_z$$

and

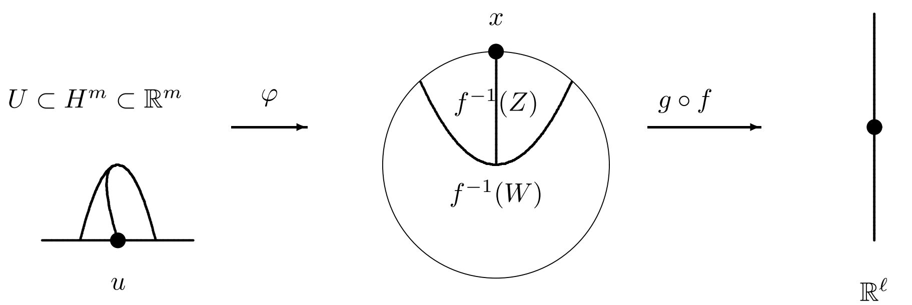
$$\nu(Z,N)_z \oplus TZ_z = TN_z.$$

(The argument from linear algebra is as follows: Each $v \in \nu(Z,N)_z$ can be written as $v'+w$ with $v' \in TZ_z$ and $w \in df_x(TM_x)$. Then $p(w) = p(v')+p(w) = p(v'+w) = p(v) = v$.) Moreover, dg_z is an isomorphism by 3.4. Thus 0 is a regular value and

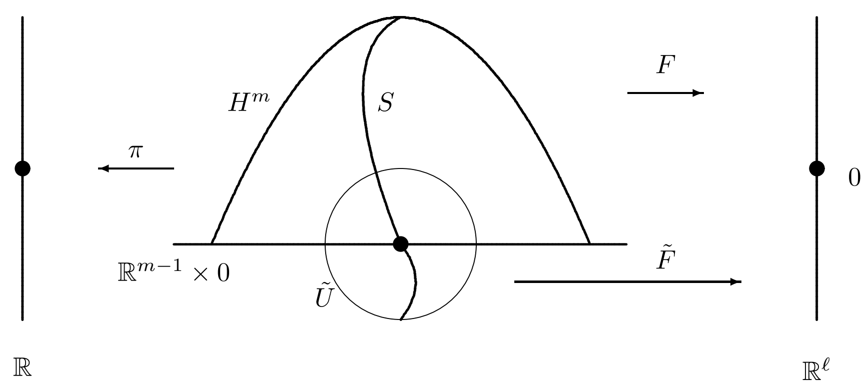
$$(g \circ f)^{-1}(0) = f^{-1}(g^{-1}(0)) = f^{-1}(W \cap Z) = f^{-1}(Z) \cap f^{-1}(W).$$

Note that $f^{-1}(W) \subset M$ is open, and we see that $f^{-1}(Z)$ is a smooth manifold by 3.3.

Case 2. By Case 1 we know that $f^{-1}(Z) \cap Int(M)$ is a manifold. Let $x \in f^{-1}(Z) \cap \partial M$. Choose g as above. Consider $g \circ f | f^{-1}(W)$ and replace Z by 0:



We have $(g \circ f)^{-1}(0) = f^{-1}(Z) \cap f^{-1}(W)$. Choose parametrization $\varphi : U \to M$ at x, $U \subset H^m$ open and $\varphi(U) \subset f^{-1}(W)$. Then $f^{-1}(Z)$ is a smooth manifold (close to x) if $\varphi^{-1} f^{-1}(Z) = F^{-1}(0)$ is a manifold with boundary in H^m. Thus consider $F := g \circ f \circ \varphi : U \to \mathbb{R}^\ell$. Then $u = \varphi^{-1}(x)$ is a regular point of F and ∂F. Extend F to a smooth map $\tilde{F}$ defined on a neighborhood $\tilde{U}$ of u in $\mathbb{R}^m$. By definition $dF_u = d\tilde{F}_u$ thus u is also regular point of $\tilde{F}$. Thus $\tilde{F}^{-1}(0) =: S \subset \mathbb{R}^m$ is a smooth manifold (This is actually subtle: Use that by 3.7. the rank of $d\tilde{F}$ can locally only increase and restrict $\tilde{F}$ to an open subset with this property. Then all points in this neighborhood are regular and thus 0 is a regular value for the restriction of $\tilde{F}$ to this subset.) It suffices to prove that $S \cap H^m$ is a smooth manifold with boundary and $S \pitchfork \partial H^m$. Let $\pi : S \to \mathbb{R}$ be the restriction of the projection $\mathbb{R}^m \ni (x_1, \dots, x_m) \mapsto x_m \in \mathbb{R}$ to S, also denoted by π. Then $S \cap H^m = \{s \in S | \pi(s) \geq 0\} = F^{-1}(0)$.



Claim: 0 is a regular value of $\pi | S$.

Proof of Claim. Suppose, for the sake of contradiction, that $s \in S$, $\pi(s) = 0$ ($\iff s \in S \cap \partial H^m$), and $d\pi_s = 0 = \pi|TS_s$. It follows that

$$TS_s \subset \mathbb{R}^{m-1} \times 0 = TH_s^m = ker(\pi).$$

Now $TS_s = ker(d\tilde{F}_s)$ by 3.4., and $d(\partial F)_s = dF_s|\mathbb{R}^{m-1} \times 0$. Because $ker(dF_s) = TS_s \subset \mathbb{R}^{m-1} \times 0$ it follows

$$ker(dF_s) = ker(d(\partial F)_s)$$

Since $dF_s : \mathbb{R}^m \to \mathbb{R}$ and $d(\partial F)_s) : \mathbb{R}^{m-1} \to \mathbb{R}$ are both onto it follows that

$$dim(ker(dF_s)) = m - 1$$

and

$$dim(ker(d(\partial F)_s) = m - 2$$

This is a contradiction. ■

The rest of the proof of 4.8. follows from the following lemma. Note that $f^{-1}(Z) \pitchfork \partial M$ then follows from $(S \cap H^m) \pitchfork \partial H^m$, which follows from $TS_s = T(S \cap \partial H^m)_s \not\subset \mathbb{R}^{m-1} \times \{0\}$ shown in the proof above. The claims about codimension are easy consequences of 3.3. and 4.5.

Lemma 4.9. *Let S be a manifold without boundary and $\pi : S \to \mathbb{R}$ be a smooth map with regular value 0. Then*

$$\pi^{-1}[0, \infty) = \{s \in S|\pi(s) \geq 0\}$$

is a manifold with boundary $\pi^{-1}(0)$.

Proof. The set $\{s \in S|\pi(s) > 0\}$ is open in S and thus a manifold of dimension $dim(S)$. Let $\pi(s) = 0$. By 3.2. there are parametrizations φ, χ at s, with φ defined on $V \subset \mathbb{R}^{dim(S)}$ open, such that $\chi^{-1} \circ \pi \circ \varphi$ is the natural projection and $\chi[0, \varepsilon) \subset [0, \varepsilon)$ for some $\varepsilon > 0$ (Use that χ is a diffeomorphism thus has non-zero derivative at 0.) Note that

$$\varphi(V) \cap \{s|\pi(s) \geq 0\} \cong U \cap H^{dim(S)}.$$

Thus after restricting the domain of φ suitably we have a parametrization of a neighborhood of s. ■.

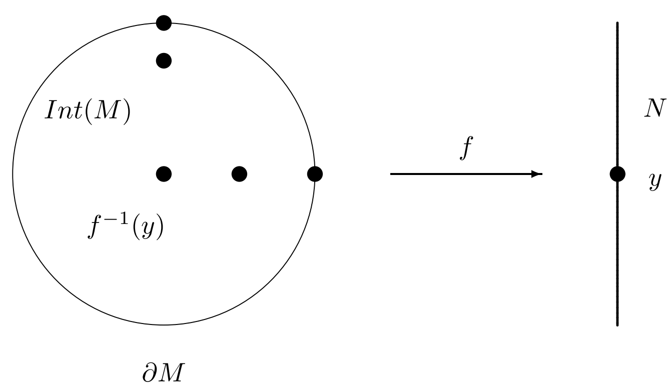
Example. Apply 4.9. to

$$\pi : B^m = \{x \in \mathbb{R}^m|||x|| \leq 1\} \to \mathbb{R},$$

where π is the restriction of the projection $\mathbb{R}^m \to \mathbb{R}$, $(x_1, \ldots, x_m) \mapsto x_m$. This shows that $B^{m-1} \subset B^m$ is a smooth manifold with $\partial B^{m-1} = (\partial B^m) \cap B^{m-1}$.

Theorem 4.10 (Sard's theorem with boundary). *Let $f : M \to N$ be a smooth map with $\partial N = \emptyset$. Then almost all points in N are regular values of both f and ∂f.*

Recall that for *almost all points* means for all points except on a set of measure zero.



Proof. By the usual theorem of Sard we know that both the sets $\{y \in N | y \text{ is critical for } f|Int(M)\}$ and $\{y \in N | y \text{ is critical for } \partial f\}$ have measure zero thus also their union. But if $y \in N$ is critical for f but not for $f|Int(M)$ then there exists some $x \in \partial M$ such that $f(x) = y$ and x is a critical point of f. But then x is also a critical point for ∂f and thus y is a critical value of ∂f. ■

Theorem 4.11. *Each smooth 1-dimensional connected manifold is diffeomorphic to one of $[0,1], [0,1), (0,1)$ or S^1.*

Let J be an interval, i. e. a subset of $\mathbb{R}$ diffeomorphic to one of $[0,1], [0,1)$ or $(0,1)$. We will need the following:

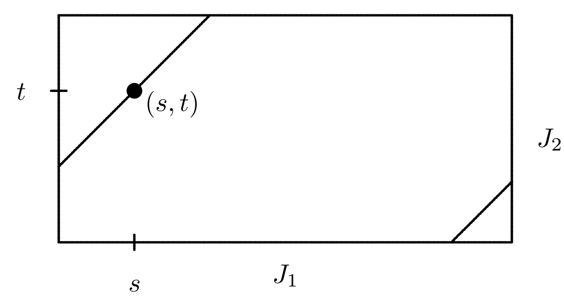
Definition. Let M be a manifold of dimension 1, $M \subset \mathbb{R}^k$. Then $\gamma : J \to M$ is called a *parametrization by arc length in M* if $\gamma(J) \subset M$ is open, and $\gamma : J \to \gamma(J)$ is a diffeomorphism, and $||\gamma'(s)|| = 1$ for all $s \in J$.

Lemma. *Let $\gamma_1 : J_1 \to M$ and $\gamma_2 : J_2 \to M$ be parametrizations by arc length. Then $\gamma_1(J_1) \cap \gamma_2(J_2)$ has at most two components. If the intersection has only one component then γ_1 extends to a parametrization by arc length of $\gamma_1(J_1) \cup \gamma_2(J_2)$. If the intersection has two components then $M \cong S^1$.*

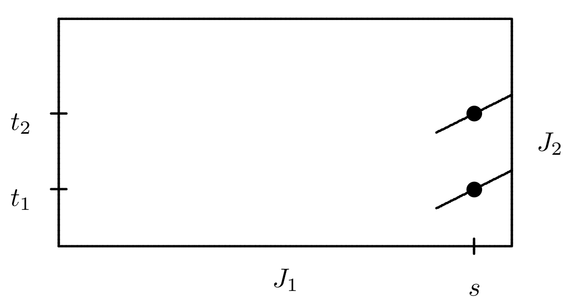
Let $U \subset J_1$ be open with $\gamma_1(U) \subset \gamma_2(J_2)$. Then $\gamma_2^{-1} \circ \gamma_1$ maps U diffeomorphically onto a subset of J_2, which is open. Now for all $s \in U$ we have $(\gamma_2^{-1} \circ \gamma_1)'(s) = \pm 1$. (This follows from $TM_{\gamma_1(s)} \subset \mathbb{R}^k$ is a 1-dimensional subspace, and thus there are precisely two vectors of length 1. Also both $d(\gamma_1)_s$ and $d(\gamma_2)_t$ are isomorphisms between 1-dimensional vector spaces, $d(\gamma_2^{-1})_{\gamma_2(t)} = (d(\gamma_2)_t)^{-1}$ by chain rule, and $d(\gamma_1)_s(1) = \gamma_1'(s)$ respectively $d(\gamma_2)_t(1) = \gamma_2'(t)$, see e. g. page 24.) Let $\Gamma \subset J_1 \times J_2$ be defined by

$$\Gamma := \{(s,t) | \gamma_1(s) = \gamma_2(t)\}.$$

This is of course the graph of $\gamma_2^{-1} \circ \gamma_1$ defined on $\gamma_1^{-1}(\gamma_2(J_2))$. Note that for each $s \in J_1$ there is at most one $t \in J_2$ such that $(s,t) \in \Gamma$ (and vice versa). Note that Γ is a union of segments of slope ± 1 (recall that a function $\mathbb{R} \supset J_1 \to J_2 \subset \mathbb{R}$ with derivative ± 1 at all points is an affine map with slope ± 1.

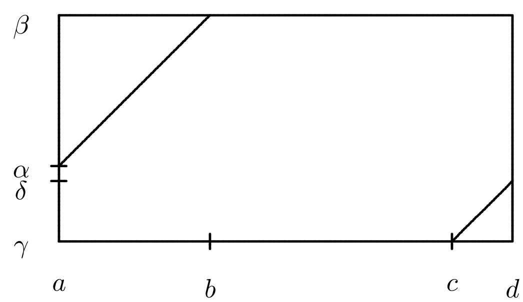


Note that none of the segments of Γ can end in the interior of $J_1 \times J_2$. This follows from Γ closed (consider a sequence in Γ, $(s_i, t_i) \to (s,t) \in J_1 \times J_2$, and note by continuity of γ_1, γ_2 it follows that $\gamma_1(s) = \gamma_2(t)$, which implies $(s,t) \in \Gamma$), and $\gamma_2^{-1} \circ \gamma_1$ is a local diffeomorphism. In fact suppose $\gamma_1(t) = \gamma_2(s)$ is such an endpoint. Now $\gamma_1(J_1)$ and $\gamma_2(J_2)$ are both open thus $\gamma_1(J_1) \cap \gamma_2(J_2)$ is open. It follows that $\gamma_2^{-1} \circ \gamma_1$ is defined on a neighborhood of s.

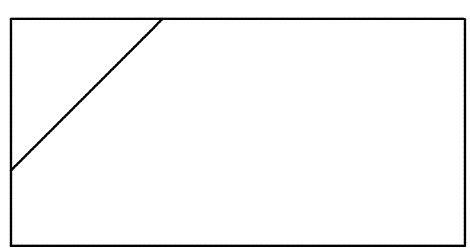


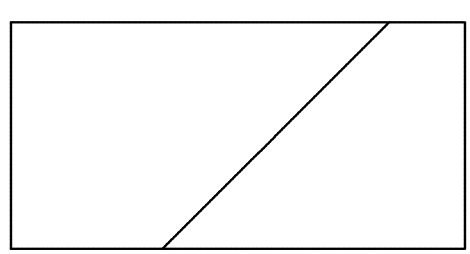
If two segments would end in $fr(J_1 \times J_2)$ then we could e. g. find $t_1 \neq t_2$

such that $\gamma_2(t_1) = \gamma_1(s)$ and $\gamma_2(t_2) = \gamma_1(s)$. But $\gamma_2(t_1) \neq \gamma_2(t_2)$ because γ_2 is one-to-one. This is a contradiction. An analogous argument works if two segments end on any of the other intervals in $fr(J_1 \times J_2)$. It follows that Γ has at most two components, which the same slope in the case of two components (because any segment begins on one side and ends on another one). The three typical situations are as follows:



or with Γ connected:





Case 1. Let Γ be connected. Then $\gamma_2^{-1} \circ \gamma_1$ extends to some affine map $L : \mathbb{R} \to \mathbb{R}$ and

$$\gamma : J_1 \cup L^{-1}(J_2) \to \gamma_1(J_1) \cup \gamma_2(J_2)$$

with

$$\gamma | J_1 = \gamma_1 \quad \text{and} \quad \gamma | L^{-1}(J_2) = \gamma_2 \circ L$$

Note that $J_1 \cap L^{-1}(J_2) \neq \emptyset \iff J_1 \cap \gamma_1^{-1}\gamma_2(J_2) \neq \emptyset \iff \gamma_1(J_1) \cap \gamma_2(J_2) \neq \emptyset$. Therefore $J_1 \cup L^{-1}(J_2)$ is an interval. Moreover if $\gamma_2(J_2) \not\subset \gamma_1(J_1)$ then γ properly extends γ_1.

Case 2. The set Γ has two components, which are segments of slope 1 (without restriction). By precomposing γ_2 with some translation we can arrange that $\gamma = c$ and $\delta = d$. Thus we have

$$a < b \leq c < d \leq \alpha < \beta.$$

Now map $[a, \alpha]$ onto S^1 using the map $t \mapsto e^{i\theta}$ and

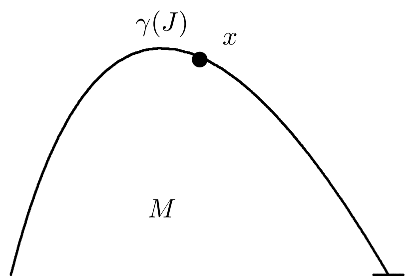
$$\theta := \frac{2\pi t}{a - \alpha}.$$

Thus if we replace t by $t + (\alpha - a)$ we map to the same point on S^1. Now define $h : S^1 \to M$ by

$$h(e^{i\theta}) := \begin{cases} \gamma_1(t) & \text{for } a < t < d = \delta \\ \gamma_2(t) & \text{for } c = \gamma < t < \beta \end{cases}$$

This is well-defined because $\gamma_1(a) = \gamma_2(\alpha)$ and $\gamma_2^{-1} \circ \gamma_1 = id$ on (c, d). Moreover, because γ_2 extends beyond α and takes the very same values there that γ_1 takes on $[a, b]$ it follows that h is smooth. Then $h(S^1)$ is compact and open in M (see 4.12. below), and thus because of M connected it follows that $h(S^1) = M$. But h is bijective and smooth. Because h has a smooth local inverse at all points, the inverse map is smooth too (see 4.12. below). ■

Proof of 4.11. Let $\gamma : J \to M$ be a *maximal* parametrization by arc length. The existence of a maximal parametrization follows from Zorns Lemma: Define a partial order on the set of all parametrizations in M by arc length using $(\gamma_1 : J_1 \to M) \prec (\gamma_2 : J_2 \to M)$ if $J_1 \subset J_2$ and $\gamma_2 | J_1 = \gamma_1$. Then every chain has an upper bound, which is defined on the union of all domains of parametrizations in the chain. Thus we can find a maximal parametrization by arc length in M. Now suppose $M \neq S^1$. Suppose that $\gamma(J) \neq M$. Note that $\gamma(J) \subset M$ is open. $\gamma(J)$ cannot be both open and closed in M because otherwise M is not connected. So there is a sequence of points in $\gamma(J)$ with a limit $x \in M \setminus \gamma(J)$ (see picture on next page). Parametrize a neighborhood of x by arc length and apply 4.12 to define a larger parametrization. This is a contradiction. ■.



Example. Recall from Chapter 2 that a compact smooth 1-dimensional manifold $L \subset \mathbb{R}^3$ with $\partial L = \emptyset$ is called a link. If L is also connected it is called a knot.

Remarks 4.12. (a) Let $f : M \to N$ be smooth ($\partial M = \partial N = \emptyset$), injective and df_x an isomorphism for all $x \in M$. Then $f : M \to f(M)$ is a diffeomorphism and $f(M) \subset N$ is open. *Proof:* Both statements are immediate from the inverse function theorem, compare 2.6 (e).
(b) The components of a manifold are open. *Proof:* Let $C \subset M$ be a component, and $x \in C$. Then there is an open neighborhood U of x with $U \cong D^m \subset \mathbb{R}^m$, thus U connected. By 1.19 (a) $U \subset C$. Thus x has an open neighborhood in C.
(c) It follows from (b) that a compact manifold has at most finitely many components M_j, i. e. $M = \cup_{1 \le j \le N} M_j$ and $M_i \cap M_j = \emptyset$ for $i \neq j$.
(d) It follows from (b) that each connected manifold is path connected because each $x \in M$ has a path connected neighborhood, see. 1.25.
(e) It follows from (a) that components of manifolds are manifolds. Thus 4.11. gives a complete *classification* of 1-dimensional manifolds.

Remark 4.13. Note that 4.11 (e) implies that each link with k components in $\mathbb{R}^3$ is as a smooth manifold diffeomorphic to a disjoint union of circles S^1. This disjoint union can be realized by placing all circles in $\mathbb{R}^2 \times 0 \subset \mathbb{R}^3$, all with radius $\frac{1}{2}$ and centers in the natural numbers. Then the problem of *knot theory* is to find out whether the above diffeomorphism of subsets of $\mathbb{R}^3$ *extends* to a global diffeomorphism of $\mathbb{R}^3$.

Theorem 4.14. *Let M be a compact manifold with boundary $\partial M \neq \emptyset$. Then* *there exists no smooth map $f : M \to \partial M$ with $f|\partial M = id_{\partial M}$.*

Proof. Let f be as above and $y \in \partial M$ a regular value of f (regular values are dense by 4.10). By 4.8. we know that $f^{-1}(y)$ is a smooth 1-dimensional manifold with $\partial f^{-1}(y) = (\partial f)^{-1}(y) = \{y\}$. But $f^{-1}(y) \subset M$ is compact thus a finite union of closed segments and circles. Thus $(\partial f)^{-1}(y)$ is an even number of points. This is a contradiction. ■

Example. The map $id_{S^{m-1}}$ does not extend to a smooth map $B^m \to S^{m-1}$.

Theorem 4.15. *Each smooth map $f : B^m \to B^m$ has a fixed point, i. e. there exists $x \in M^m$ such that $f(x) = x$.*

Proof. Suppose the claim is not true. Then define $g : B^m \to S^{m-1}$ by assigning to $x \in B^m$ the point on S^{m-1} on the line through x and $f(x)$, which is closer to x than to $f(x)$. The computation is as follows: $g(x) = x + tu$ with $t \geq 0$ and $u := \frac{x-f(x)}{||x-f(x)||}$. We determine t such that $||g(x)|| = 1$. Therefore

$$1 = ||g(x)||^2 =< x + tu, x + tu >= ||x||^2 + 2t < x, u > +t^2.$$

This implies

$$t^2 + 2t < x, u > +(||x||^2 - 1) = 0$$

and thus by the quadratic formula

$$t = - < x, u > + \sqrt{< x, u >^2 + 1 - ||x||^2} \geq 0.$$

Note that $1 - ||x||^2 \geq 0$, and we have chosen t correspondingly such that $g(x)$ is closer to x than to $f(x)$. Thus g is smooth and $g|S^{m-1} = id_{S^{m-1}}$. This is a contradiction to 4.14. ■

Theorem 4.16 (Brower Fixed Point Theorem). *Each continuous map $G : B^m \to B^m$ has a fixed point.*

Proof. Let G be continuous without any fixed point. For each $\varepsilon > 0$ there exists a smooth function $P_1 : \mathbb{R}^m \to \mathbb{R}^m$ with

$$||P_1(x) - G(x)|| < \varepsilon$$

for all $x \in B^m$. (P_1 exists by the Weierstrass approximation theorem for functions $\mathbb{R}^m \to \mathbb{R}$, which can be easily deduced from the Stone-Weierstrass theorem, see e. g. Rudin, Principles of Mathematical Analysis, 7.32.) Let $P(x) := \frac{P_1(x)}{1+\varepsilon}$. Then $P(B^m) \subset B^m$ because

$$||P(x)|| = \frac{1}{1+\varepsilon}||P_1(x)|| = \frac{1}{1+\varepsilon}||P_1(x) - G(x) + G(x)|| \leq \frac{1}{1+\varepsilon}(\varepsilon + 1) = 1$$

Because

$$||P(x) - P_1(x)|| = ||P(x) - (1+\varepsilon)P(x)|| \le \varepsilon ||P(x)|| \le \varepsilon$$

it follows that

$$||P(x) - G(x)|| \le ||P(x) - P_1(x)|| + ||P_1(x) - G(x)|| < 2\varepsilon.$$

Let $G(x) \ne x$ for all $x \in B^m$. Then $x \mapsto ||G(x) - x||$ attains a minimum $\mu > 0$ on B^m. Choose P as above with $||P(x) - G(x)|| < \mu$. Then

$$||P(x)-x|| = ||P(x)-G(x)+G(x)-x|| \ge ||G(x)-x|| - ||P(x)-G(x)|| > \mu - \mu = 0.$$

Thus P is smooth and has no fixed point. This is a contradiction to 4.15. ■.

Remark. Note that there exist smooth maps of open n-balls without any fixed point. In fact let $g : D^n \to \mathbb{R}^n$ be a diffeomorphism and let $t_a : \mathbb{R}^n \to \mathbb{R}^n$ be a translation. Then $g^{-1} \circ t_a \circ g : D^n \to D^n$ has no fixed points. In fact, if $x \in D^n$ is a fixed point of $g^{-1} \circ t_a \circ g$ then $g^{-1}(t_a(g(x)) = x$ thus $t_a(g(x)) = g(x)$ is a fixed point of t_a in $\mathbb{R}^n$. But a translation has no fixed points.

We now introduce the important concept of orientation.

Let V be a real vector space of dimension m. Two ordered bases $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_m)$ are called *oriented in the same way* if the matrix of the change of bases has positive determinant. Thus if we write $w_i = \sum_{1 \le j \le m} \lambda_{ij} v_j$ for $1 \le i \le m$ we have $det((\lambda_{ij})_{i,j}) > 0$. *Oriented in the same way* is an equivalence relation on the set of ordered bases (Exercise).

Definition 4.17. An *orientation* of a vector space V is an equivalence class of ordered bases.

Each ordered basis $(v_1, \ldots, v_m)$ determines the orientation $[v_1, \ldots, v_m] = \sigma$. Let $-\sigma$ denote the opposite orientation, i. e. $-\sigma = [w_1, \ldots, w_m)$, where the matrix of basis change $(v_1, \ldots, v_m) \longleftrightarrow (w_1, \ldots w_m)$ has negative determinant. Note that each vector space has precisely two orientations.

Example. For $m \ge 1$, $\mathbb{R}^m$ has the standard orientation $[e_1, \ldots, e_m]$. where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ for $1 \le i \le m$ is the i-th canonical basis vector with 1 in the i-th place. $\mathbb{R}^0 = \{0\}$ is *formally* oriented by ± 1.

If V is a vector space with orientation σ and $(v_1, \ldots, v_m)$ is an ordered basis then let $sign(w_1, \ldots, w_m) = +1$ if $[w_1, \ldots, w_m] = \sigma$ and $sign(w_1, \ldots, w_m) = -1$ if $[w_1, \ldots, w_m] = -\sigma$.

Let $L : V \to W$ be an isomorphism of vector spaces and let $\sigma = [v_1, \dots, v_m]$ be an orientation for V. Then $L(\sigma) = [L(v_1), \dots, L(v_m)]$ is an orientation for W.

Example. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, and $\mathbb{R}^2$ be oriented by $\sigma = [e_1, e_2]$. Then $L(\sigma) = \sigma$ respectively $L(\sigma) = -\sigma$ iff $det(L) > 0$ respectively $det(L) < 0$.

Definition 4.18. An *orientation of a manifold* M of dimension m is a family $\sigma = (\sigma_x)_{x \in M}$ of orientations σ_x of TM_x with the following compatibility property: For $m \geq 1$ we require that each $x \in M$ has a neighborhood U with coordinate system $h : U \to H^m$ such that for all $y \in U$, $dh_y(\sigma_y)$ is the same orientation of $\mathbb{R}^m$.

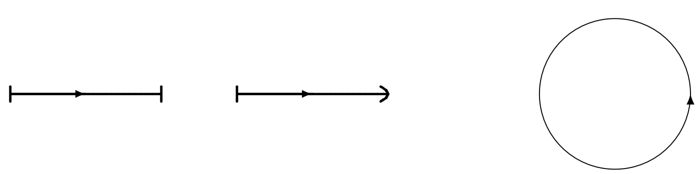
M is called *orientable* if there exists an orientation of M.

Theorem 4.19. *If M is connected and orientable then there exist precisely two orientations of M.*

Proof. Let $\sigma = (\sigma_x)$ be an orientation of M. Then $-\sigma := (-\sigma_x)$ is also an orientation of M. This follows, because for coordinate systems h, we have $dh_y(-\sigma) = -dh_y(\sigma)$. Let σ' be an arbitrary further orientation. Define $\varepsilon : M \to \{1, -1\}$ by $\varepsilon(x) = +1$ if $\sigma_x = \sigma'_x$, and $\varepsilon(x) = -1$ if $\sigma'_x = -\sigma_x$. Since ε is a locally constant function by definition 4.18 and M connected it follows by 1.21 that ε is constant. This implies $\sigma = \sigma'$ or $-\sigma = \sigma'$. ■

Convention. An *oriented manifold* is a pair (M, σ), where σ is an orientation of M. We often write just M instead of (M, σ) and let $-M$ denote $(M, -\sigma)$.

Example 4.20. Each 1-dimensional manifold is orientable:



Notice: If $f : M \to N$ is a diffeomorphism with M oriented then N is oriented by $\sigma'_y := df_x(\sigma_x)$ for $y = f(x)$ and σ_x the orientation of M at x. Thus orientability is an *invariant of diffeomorphism*. We will say that two oriented manifolds (M, σ) and (N, σ') are *(oriented) diffeomorphic* if there exists a diffeomorphism $f : M \to N$ with $df(\sigma) = \sigma'$, i. e. $df_x(\sigma_x) = \sigma'_{f(x)}$ for all $x \in M$. Note that M

is always diffeomorphic to $-M$ by the identity map. But it is not always true that M is *oriented* diffeomorphic to $-M$ as the following examples show.

Example. Note that the oriented 1-manifold S^1 (use the counter clockwise orientation of the circle) is oriented diffeomorphic to $-S^1$. An oriented diffeomorphism can be defined by restricting the reflection $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$ to S^1. Also the oriented closed interval $[0,1]$ (use the orientation of $\mathbb{R}$ defined by the order restricted to $[0,1]$) is diffeomorphic to $-[0,1]$ using the diffeomorphism of $[0,1]$ defined by restricting $\mathbb{R} \ni t \mapsto 1-t \in \mathbb{R}$. But the oriented smooth 1-manifold $[0,1)$ (oriented in the same way as $[0,1]$) is *not* oriented diffeomorphic to $-[0,1)$ because this would imply the existence of a smooth bijective map $f : [0,1) \to [0,1)$ with $f'(t) < 0$ for all t. Thus there is a lack of symmetry of $[0,1)$ with respect to orientations.

We now define a new important family of smooth manifolds.

For $r, \ell \in \mathbb{N}$ let $V_{\ell,r}$ be the set of all r-tuples $(v_1, \ldots, v_r)$ of linearly independent vectors $v_i \in \mathbb{R}^\ell$. We will consider

$$V_{\ell,r} \subset M(\ell,r) \cong \mathbb{R}^{\ell \cdot r},$$

where $M(\ell,r)$ is the space of all real $\ell \times r$-matrices. This is an open subset of $\mathbb{R}^{\ell \cdot r}$. This is of course a smooth manifold of dimension $\ell \cdot r$. $V_{\ell,r}$ is the (non-compact) *Stiefel manifold*.

Lemma 4.21. *The natural map*

$$\kappa : V_{\ell,\ell-1} \to S^{\ell-1},$$

which maps $(v_1, \ldots, v_{\ell-1})$ to the unique vector $v \in S^{\ell-1}$ such that

$$det(v, v_1, \ldots, v_{\ell-1}) > 0$$

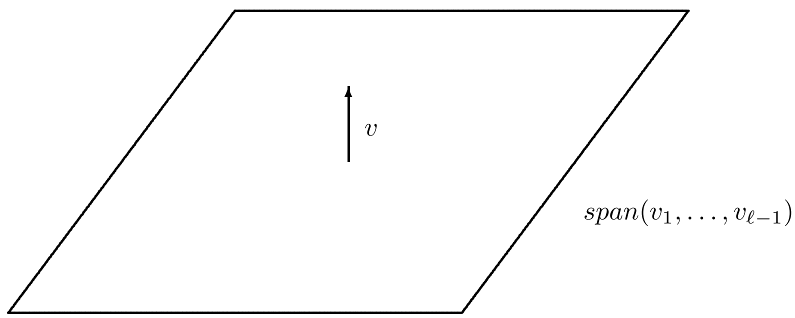
and v is perpendicular to the hyperspace $span(v_1, \ldots, v_{\ell-1}) \subset \mathbb{R}^\ell$ spanned by $v_1, \ldots, v_{\ell-1}$ is smooth.
(*Notation: $v \perp span(v_1, \ldots, v_{\ell-1})$ or $v \in span(v_1, \ldots, v_{\ell-1})^\perp$.*)

Proof. Let $S : (v_1, \ldots, v_{\ell-1}) \to (\tilde{v}_1, \ldots, \tilde{v}_{\ell-1})$ be the *Schmidt orthonormalization* map (see e. g. Curtis: Linear Algebra). It is easy to check that this is a smooth map. Note that S preserves the orientation of $span(v_1, \ldots, v_{\ell-1})$ defined by $(v_1, \ldots, v_{\ell-1})$. Then $v \in \mathbb{R}^\ell$ is the unique solution of the system of ℓ linear equations:

$$det(v, \tilde{v}_1, \ldots, \tilde{v}_{\ell-1}) = 1$$

$$< v, \tilde{v}_i >= 0 \text{ for } 1 \leq i \leq \ell - 1,$$

which depends smoothly on $(v_1, \ldots, v_{\ell-1})$. (The existence of a unique solution is already clear from the original definition. But the system of equations above shows the smooth dependence of v on $(v_1, \ldots, v_{\ell-1})$). ■



In the following the notation M^m means a manifold of dimension m.

Theorem 4.22. *Let $f : M^m \to \mathbb{R}^{m+1}$ be an immersion (i. e. df_x is injective for all $x \in M$). Suppose that M is orientable with orientation σ. Then there is a smooth map*

$$\eta : M^m \to S^m$$

with $\eta(x) \perp df_x(TM_x)$ for all $x \in M$.

Proof. $df_x(TM_x) \subset \mathbb{R}^{m+1}$ is a hyperspace in $\mathbb{R}^{m+1}$. Thus there are precisely two vectors of length 1 in $df_x(TM_x)^\perp$ and precisely one vector $\eta(x)$ with

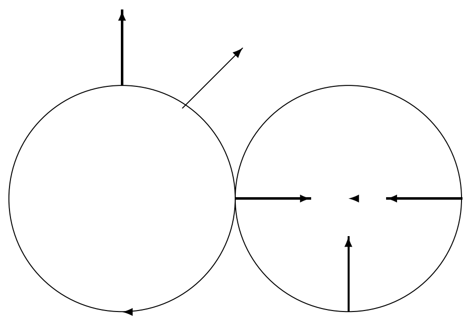
$$det(\eta(x), df_x(v_1), \ldots, df_x(v_m)) > 0,$$

where $(v_1, \ldots, v_m)$ is an oriented basis of TM_x. The map $\eta : M \to S^m$ is smooth. In fact, let $x \in M$ and let U be an open subset of H^m and $\varphi : U \to M$ be a parametriation at x with $[d\varphi_u(e_1, \ldots, e_m)] = \sigma_{\varphi(u)}$ for all $u \in U$ (respectively $-\sigma_{\varphi(u)}$ for all $u \in U$), i. e. $d\varphi_u(e_1, \ldots, e_m)$ (respectively $d\varphi_u(-e_1, \ldots, e_m)$) is an oriented basis of TM_x. Then the map

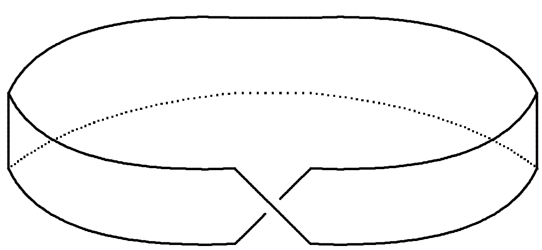
$$\rho : U \ni u \mapsto df_{\varphi(u)} \circ d\varphi_u(e_1, \ldots, e_m) \in V_{m+1,m}$$

is smooth. Then $\eta \circ \varphi = \kappa \circ \rho$ is smooth by 4.21. ■

$\eta = \eta_f$ is the *normal vectorfield* corresponding to the immersion f. The following picture shows the image of some immersion $f : S^1 \to \mathbb{R}^2$ (the *figure eight*) and some normal vectors at x but drawn at $f(x)$. The orientation of f is also indicated.



Example 4.23 The *Möbius band* $M^2 \subset \mathbb{R}^3$ is a nonorientable surface with boundary. If it would be orientable it would have a smooth normal vector field. But it is easy to see that no such normal vector field exists. In fact, imagine the surface is colored red when you see the tip of the normal vectors, and is colored green if you look at the tail of the vector. Then a smooth normal vector field on a surface defines a coloring by red on one *side* and by green on the other *side*. The same argument works for each manifold $M^m \subset \mathbb{R}^{m+1}$. Thus orientability coincides with being two-sided for a manifold $M^m \subset \mathbb{R}^{m+1}$. If $\partial M = \emptyset$ and M is connected then two-sided-ness implies that $\mathbb{R}^{m+1} \setminus M^m$ has precisely two components (Prove this!)



4.24. Product orientation. Let M, N be oriented manifolds with $\partial M = \emptyset$

or $\partial N = \emptyset$. Then an orientation for $M \times N$ is defined as follows. First identify

$$T(M \times N)_{(x,y)} = TM_x \times TN_y,$$

where the product of vector spaces is defined as usual. (Recall from page 24 that the tangent space to $M \times N$ at (x, y) is the space of derivatives at 0 of paths in $M \times N$ through (x, y) at time 0, and note that paths in $M \times N$ through (x, y) have the form $\gamma = (\gamma_1, \gamma_2)$ with γ_1 a path in M through x and γ_2 a path in N through y.) Then let $\alpha = (v_1, \ldots, v_m)$ and $\beta = (w_1, \ldots, w_n)$ be ordered bases of TM_x and TN_y. Let $(\alpha \times 0, 0 \times \beta)$ be the ordered basis:

$$((v_1, 0), \ldots, (v_m, 0), (0, w_1), \ldots (0, w_n))$$

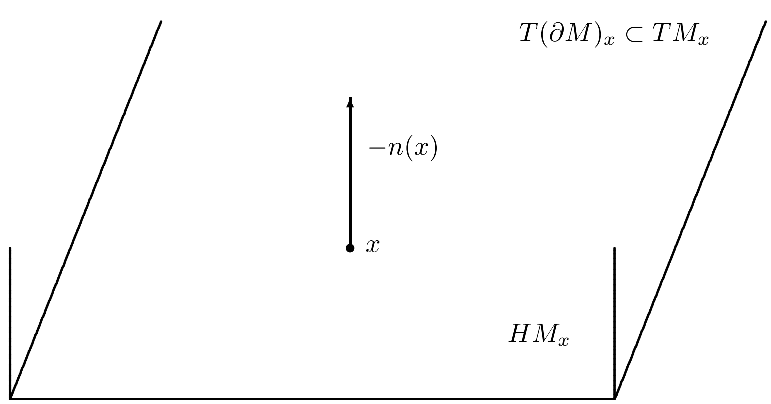
of $TM_x \times TN_y$. Then define the orientation of $M \times N$ at (x, y) such that the following rule holds:

$$sign(\alpha \times 0, 0 \times \beta) := sign(\alpha) sign(\beta).$$

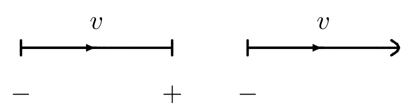
This is easilty seen to be well-defined.

Exercise. Show that $(m, n) \mapsto (n, m)$ is an oriented diffeomorphism $M \times N \xrightarrow{\cong} N \times N$.

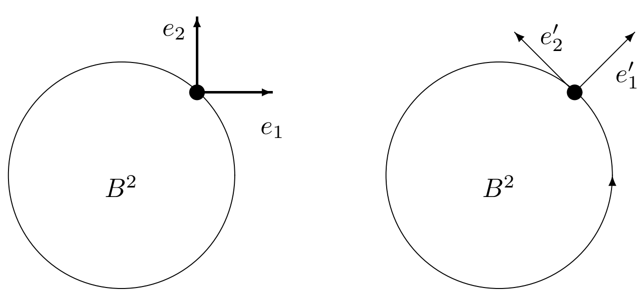
4.25. Boundary orientation. Each orientation of a manifold M induces an orientation of ∂M in the following way. Let $x \in \partial M$. Then $T(\partial M)_x \subset TM_x$ is a hyperspace. Let $HM_x \subset TM_x$ denote the uniquely determined half space of TM_x defined by the derivatives of paths $\gamma'(0)$ for $\gamma : [0, \delta) \to M$ with $\gamma(0) = x$ (Exercise!). There is a unique vector $n(x) \perp T(\partial M)_x$ with $-n(x) \in HM_x$ and $||n(x)|| = 1$.



The vector $-n(x)$ is called the *interior* normal vector, and the vector $n(x)$ is called the *exterior* normal vector. Let $(v_1, \ldots, v_m)$ be an oriented basis for TM_x with $v_1 = n(x)$ and for $m \geq 2$, $(v_2, \ldots, v_m) \in T(\partial M)_x$. Then let $T(\partial M)_x$ be oriented by $[v_2, \ldots, v_m]$. Thus in general for arbitrary bases let $sign(v_2, \ldots, v_m) = sign(n(x), v_2, \ldots, v_m)$. When $m = 1$ and the length 1 vector v in TM_x determines the orientation of M at $x \in \partial M$ then let $\sigma(T(\partial M)_x) = +1$ respectively -1 if $v = n(x)$ respectively $v = -n(x)$. Here $\sigma(V)$ denotes the formal orientation of the 0-dimensional vector space V as described following 4.17.



Example 4.26. (i) If $M^m \subset N^m$ then M is oriented by the orientation of N.
(ii) Consider e. g. $B^2 \subset \mathbb{R}^2$ and S^1 oriented as the boundary of B^2:



(ii) For a 1-dimensional manifold and $P \in \partial M$ we call $\sigma(P) \in \{\pm 1\}$ the *orientation number* at P. It is an important observation that for a *compact* 1-dimensional manifold M,

$$\sum_{P \in \partial M} \sigma(P) = 0.$$

(This obviously holds for each component of M.)

Important Example 4.27. Let M be oriented. For $t \in I = [0, 1]$ let $M_t := \{t\} \times M \subset I \times M$. Then M_t is oriented by the diffeomorphism $x \mapsto (t, x)$, $M \to M_t$ (*). Now consider $\partial(I \times M) = M_1 \cup M_0$ (first without orientations).

Each ordered basis of $T(M_1)_{(1,x)}$ has the form $0 \times \beta$ where β is an ordered basis of TM_x. The boundary orientation in $(1,x)$ is defined by

$$sign(0 \times \beta) = sign(n(1,x), 0 \times \beta)$$

and the product orientation ($1 = e_1 \in TI_1 = \mathbb{R}$):

$$sign(1 \times 0, 0 \times \beta) = sign(1)sign(\beta) = sign(\beta).$$

It follows that the boundary orientation of M_1 is equal to the (*)-orientation. For M_0 we have

$$sign(-1 \times 0, 0 \times \beta) = sign(-1)sign(\beta) = -sign(\beta).$$

Thus M_0, oriented by product and boundary orientation of $I \times M$, has the opposite orientation as that defined by (*). Thus we write for the *oriented* boundary:

$$\partial(I \times M) = M_1 - M_0.$$

4.28. Preimage orientation. We first prove an algebraic lemma.

Lemma. *Let $V = V_1 \oplus V_2$ be a direct sum of vector spaces. Then each orientation of two of the vector spaces induces an orientation of the third vector space.*

Proof. Let β_i be an oriented basis of V_i for $i = 1, 2$, and let $\beta = (\beta_1, \beta_2)$. Define an orientation of $V_1 \oplus V_2$ by

$$(*) \qquad sign(\beta) = sign(\beta_1)sign(\beta_2)$$

If V and some V_i is oriented then (*) can be used to orient V_j for $i \neq j$, $i, j \in \{1, 2\}$. ■

Let $f : M \to N$ be a smooth map, $Z \subset Int(N)$ a submanifold and $f \pitchfork Z$, $\partial f \pitchfork Z$. We assume that $\partial N = \partial Z = \emptyset$ and that all manifolds are oriented. Let $S := f^{-1}(Z) \subset M$ and $y = f(x) \in Z$. Because of transversality we have

$$df_x(TM_x) + TZ_y = TN_y.$$

We know by definition that

$$\nu(S, M)_x \oplus TS_x = TM_x,$$

and $df_x(TS_x) \subset TZ_y$. Thus we have

$$df_x(\nu(S, M)_x) \oplus TZ_y = TN_y.$$

This sum is direct but not necessarily orthogonal. It is direct because $codim_N Z = codim_M S = dim(\nu(S, M))_x$ for all $x \in S$, see 4.8.

Thus the orientations of TN_y and TZ_y induce an orientation of $df_x(\nu(S, M)_x)$, which is ismorphic under $((df_x)|)^{-1}$ to $\nu(S, M)_x$. Thus we can orient $\nu(S, M)_x$ using $((df_x)|)^{-1}$. Then use

$$\nu(S, M)_x \oplus TS_x = TM_x$$

to orient TS_x. (Note that instead of $\nu(S, M)_x$ an arbitrary oriented subspace H with $H \oplus TS_x = TM_x$ can be used.)

Chapter 5

Smooth homotopy and vector bundles.

The motivation is the following: Given $f : M \to N$, $Z \subset N$ a closed submanifold, and $dim(M) + dim(Z) = dim(N)$. Suppose that $f \pitchfork Z$ and all manifolds are oriented and have no boundary. Then $f^{-1}(Z) \subset M$ is a 0-dimensional manifold, oriented by 4.28. Thus, if M is compact,

$$\sum_{x \in f^{-1}(Z)} \sigma(x) \in \mathbb{Z}$$

is defined.

We want to study how this sum is changing under deformations.

Definition 5.1. A *smooth homotopy* between smooth maps $f, g : X \to Y$ of spaces is a smooth map

$$F : I \times X \to Y$$

such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$. The notation is $f \simeq g$.

Remark. $\simeq$ is an equivalence relation. (Exercise)

Examples. Smooth homotopy of smooth paths $I \to Y$ (relative to endpoints) or smooth homotopy of smooth loops $S^1 \to Y$ is related with the notion of *fundamental group.* If $U \subset \mathbb{R}^m$ is convex, or star shaped with respect to some $y \in U$. Then given any smooth map $f : U \to Y$ the homotopy

$$F : I \times U \to Y$$

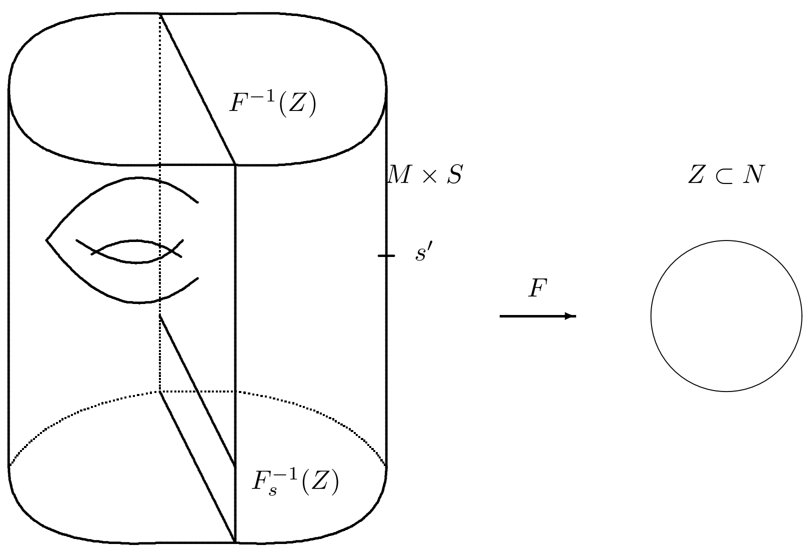
defined by

$$(t, u) \mapsto f(ty + (1-t)y)$$

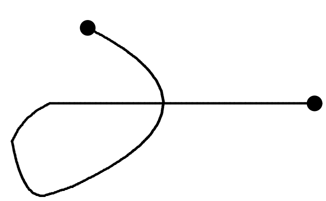
proves that f is homotopic to the constant map at y.

Theorem 5.2 (Transversality). *Let $F : M \times S \to N$ be a smooth map between manifolds, $Z \subset N$, at most $\partial M \neq \emptyset$. If $F \pitchfork Z$ and $\partial F \pitchfork Z$ then for almost all $s \in S$ also $F_s \pitchfork Z$ and $\partial F_s \pitchfork Z$, where $F_s : M \to N$ is defined by $F_s(x) = F(x, s)$ for all $s \in S$ and $x \in M$.*

Example. The following picture illustrates a case $M = B^2$, $S = \mathbb{R}$, $Z = S^1$ and $N = \mathbb{R}^2$:



Note that there are $s \in \mathbb{R}$ for which $F_s^{-1}(Z)$ is *not* a smooth manifold. For instance $F_{s'}^{-1}(Z)$ could be a curve of the form



with boundary in ∂M, or it could be the disjoint union of a line and a point e. g. for some s'' just below or above s' which also is not a 1-*dimensional* manifold.

Proof of 5.2. By 4.8. we know that $W = F^{-1}(Z) \subset M \times S$ is a smooth manifold with $\partial W = (\partial M \times S) \cap W$. Let $\pi : M \times S \to S$ be the projection.
Claim: If $s \in S$ is a regular value of $\pi|W$ (respectively $\partial(\pi|W)$) then $F_s \pitchfork Z$ (respectively $\partial F_s \pitchfork Z$). (Then 5.2. follows from Sard's theorem.)
Proof of Claim. Let $F_s(x) = z \in Z$. Since $F \pitchfork Z$ we know that

$$dF_{(x,s)}(T(M \times S)_{(x,s)} + TZ_z = TN_z.$$

Thus for all $a \in TN_z$ there exists $b \in T(M \times S)_{(x,s)}$ such that

$$dF_{(x,s)}(b) - a \in TZ_z.$$

Let $b = (w, e) \in TM_x \times TS_s$. (If $e = 0$ then $d(F_s)_x(w) - a \in TZ_z$ and $F_s \pitchfork Z$.) By assumption we know that $TW_{(x,s)}$ maps onto TS_s under the restriction of $d\pi_{(x,s)} : TM_x \times TS_s \to TS_s$. Moreover $d\pi_{(x,s)}(w', e) = e$ for $w' \in TM_x$ and $e \in TS_s$. Since $F(W) \subset Z$ it follows that $dF_{(x,s)}(w', e) \in TZ_z$ for $(w', e) \in TW_{(x,s)}$. Let $v := w - w'$. It follows using $(w, e) - (w', e) = (w - w', 0)$ that

$$\begin{aligned} c := d(F_s)_x(v) - a &= dF_{(x,s)}((w, e) - (w', e)) - a \\ &= (dF_{(x,s)}(w, e) - a) - (dF_{(x,s)}(w', e) \in TZ_z. \end{aligned}$$

Thus for each $a \in TN_z$ there exist vectors $v \in TM_x$ and $c \in TZ_z$ with

$$d(F_s)_x(v) + c = a,$$

which implies $F_s \pitchfork Z$. ■

Corollary 5.3. *Let $f : M \to \mathbb{R}^\ell$ be smooth, $Z \subset \mathbb{R}^\ell$ be open, at most $\partial M \neq \emptyset$. Then for almost all $s \in S$ the mappings f_s with*

$$f_s(x) = f(x) + s$$

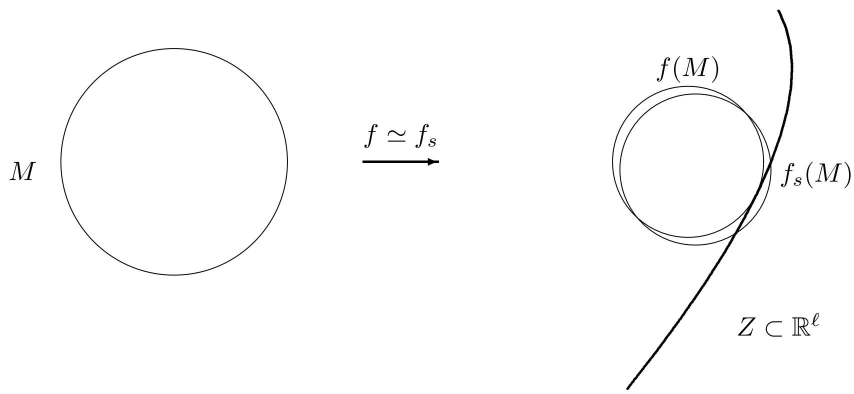
are transversal to Z.

Proof. Consider $F : M \times S \to \mathbb{R}^\ell$ defined by $F(x, s) = f(x) + s$. Then $dF_{(x,s)} = (df_x, id_{\mathbb{R}^\ell})$, where we identift TS_s with $\mathbb{R}^\ell$. Thus F is a submersion. A similar argument applies to ∂F. Thus for $Z \subset \mathbb{R}^\ell$ arbitrarily we have $F, \partial F \pitchfork Z$ and the result follows from 5.2. ■

Example. Let $S = D^\ell$ the open ball in $\mathbb{R}^\ell$. Let $f : M \to \mathbb{R}^\ell$ be a smooth map. Then f_s as above is smoothly homotopic to f for each $s \in S$. Just consider the homotopy

$$(x, t) \mapsto f_{ts}(x)$$

Thus each smooth map $f : M \to \mathbb{R}^\ell$ is *smoothly homotopic* to a smooth map, which is transversal to Z.



In the rest of Chapter 5 we will globalize the above result to the more general situation of $Z \subset N$.

Theorem 5.4. *Let $M^m \subset \mathbb{R}^k$ be a smooth manifold. Then*

$$TM := \{(x, v) \in M \times \mathbb{R}^k | v \in TM_x\}$$

is a smooth manifold of dimension $2m$ in $\mathbb{R}^k \times \mathbb{R}^k = \mathbb{R}^{2k}$. Moreover, if $f : M \to N$ is a smooth map then so is

$$df : TM \to TN$$

defined by

$$df(x, v) := (f(x), df_x(v))$$

Proof. Let $\varphi : U \to M$ be a parametrization with $U \subset H^m$ open. Then we define a parametrization

$$H^{2m} \supset \mathbb{R}^m \times U \to M \times \mathbb{R}^k \subset \mathbb{R}^{2k}$$

by

$$(v, u) \mapsto (\varphi(u), d\varphi_u(v))$$

It is easy to show that this is a parametrization. Let $f : M \to N \subset \mathbb{R}^\ell$ be a smooth map. Extend f smoothly by a map $F : W \to \mathbb{R}^\ell$ with $W \subset \mathbb{R}^k$ open and $x \in W$. Then $TW = W \times \mathbb{R}^k \subset \mathbb{R}^{2k}$ is open and

$$dF : TW \to \mathbb{R}^{2\ell}$$

defined by

$$dF(x, v) = (F(x), dF_x(v))$$

is smooth. But $(x, v) \mapsto dF_x(v)$ is a smooth extension of *df* on some open subset of $\mathbb{R}^{2k}$. ■

Definition 5.5. An r-dimensional *vector bundle* (of finite type) over a space $X \subset \mathbb{R}^k$ is a space

$$\xi \subset X \times \mathbb{R}^\ell$$

such that the following holds:
(a) Let $p_1 : X \times \mathbb{R}^\ell \to X$ be the projection, and let $\pi = \pi_\xi$ be the restriction of p_1 to ξ. Then for each $x \in X$ the space $\pi^{-1}(x) = \{x\} \times \xi_x$ with $\xi_x \subset \mathbb{R}^\ell$ is an r-dimensional vector subspace of $\mathbb{R}^\ell$.
(b) π is *locally trivial*: For each $x \in X$ there exists a neighborhood U and a homeomorphism $\tau = \tau_U$, a *local trivialization*, in the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tau} & U \times \mathbb{R}^r \\ {\scriptstyle \pi|}\downarrow & & \downarrow{\scriptstyle p_1} \\ U & = & U \end{array}$$

such that

$$\xi_y \to \{y\} \times \xi_y \xrightarrow{\tau|} \{y\} \times \mathbb{R}^r \to \mathbb{R}^r$$

is an isomorphism of vector spaces for all $y \in U$. ξ is called briefly just an r*-bundle over* X and $\pi^{-1}(U) =: \xi|U$. ξ is called a *smooth* r*-bundle* if all τ are diffeomorphisms.

Examples. (i) The trivial bundle $X \times \mathbb{R}^\ell$ over a space X
(ii) The *open Moebius band*, see Example 4.23., is a subset in $\mathbb{R}^3$, which is a line bundle (i. e. 1-bundle) over the circle, which is not trivial.

Remark 5.6. If ξ is a smooth r-bundle over a smooth manifold M then we can assume that the *trivializations* $\tau = \tau_U$ define parametrizations in the following way. If $\varphi : U' \to U$ is a parametrization then we have the commutative diagram:

$$\begin{array}{ccccc} \pi^{-1}(U) & \xrightarrow[\cong]{\tau} & U \times \mathbb{R}^k & \xleftarrow[\cong]{} & U' \times \mathbb{R}^k \\ {\scriptstyle \pi|}\big\downarrow & & {\scriptstyle p_1}\big\downarrow & & \big\downarrow \\ U & = \!= \!= & U & \xleftarrow{\varphi} & U' \end{array}$$

where $U' \subset \mathbb{R}^m$ open and thus $U' \times \mathbb{R}^k \subset \mathbb{R}^{m+k}$ open.

Theorem 5.7. *$TM \subset M \times \mathbb{R}^k$ is a smooth m-bundle over M^m for each smooth manifold $M^m \subset \mathbb{R}^k$.*

Proof. Let $\pi : TM \to M$ be the projection. Then

$$\pi^{-1}(x) = \{x\} \times TM_x = TM_x \subset \mathbb{R}^k$$

is a vector subspace of $\mathbb{R}^k$ of dimension m. Let $x \in M$ and $U \subset M$ be domain of a coordinate system thus $\varphi : U' \xrightarrow{\cong} U$ for $U' \subset H^m$ open. Then

$$\tau : \pi^{-1}(U) = \{(x, v) \in U \times \mathbb{R}^k | v \in TM_x\} \to U \times \mathbb{R}^m$$

will be defined by

$$\tau(x, v); +(x, w)$$

where $\varphi(u) = x$ for $u \in U'$, $d\varphi_u(w) = v$. Since

$$\tau | \pi^{-1}(x) : \{x\} \times TM_x \to \{x\} \times \mathbb{R}^m$$

is defined by $d\varphi_u^{-1}$ condition (b) of 5.5. is also satisfied. ■

Given a bundle ξ over X we will write in general for $Y \subset X$

$$\xi | Y := \pi^{-1}(Y).$$

Theorem 5.7. *For $M^m \subset N^k \subset \mathbb{R}^r$ smooth manifolds*

$$\nu(M, N) := \{(x, v) \in TN | v \in \nu(M, N)_x\} \subset TN | M \subset M \times \mathbb{R}^r$$

is a smooth $(k - m)$-bundle over M.

Recall that $\nu(M, N)_x := \{v \in TN_x | v \perp TM_x\}$.
We will write $\nu(M)$ for $\nu(M, \mathbb{R}^k$ if $M \subset \mathbb{R}^k$ is given.
We need the following

Lemma 5.8. *For $A : \mathbb{R}^k \to \mathbb{R}^\ell$ a linear map let $A^t : \mathbb{R}^\ell \to \mathbb{R}^k$ be defined by*

$$< Av, w >=< v, A^t w >$$

for all $v \in \mathbb{R}^k$ and $w \in \mathbb{R}^\ell$. Thus if $A = (a_{ij})_{i,j}$ then $A^t = (a_{ji})_{i,j}$. Then A is surjective $\Longrightarrow A^t(\mathbb{R}^\ell) = (ker(A))^\perp$ and A^t is injective.

Proof. $A^t w = 0 \Longrightarrow < Av, w >=< v, A^t w >= 0$ for all $v \in \mathbb{R}^k \Longrightarrow w \perp A(\mathbb{R}^k) = \mathbb{R}^\ell \Longrightarrow w = 0$, and thus A^t is injective. Also $Av = 0 \Longrightarrow < Av, w >=< v, A^t w >= 0$ for all $w \in \mathbb{R}^\ell \Longrightarrow A^t(\mathbb{R}^\ell) \perp (ker(A) \Longrightarrow A^t(\mathbb{R}^\ell) \subset ker(A)^\perp$. It follows $dim(ker(A)^\perp) = \ell$. ■

Proof of 5.7. We have $M^m \subset N^k$, $x \in M$. Then there is an open set $\tilde{U} \subset N^k$ and a submersion

$$\rho : \tilde{U} \to \mathbb{R}^\ell,$$

$\ell = k - m$ such that $\rho^{-1}(0) = M \cap \tilde{U} =: U \subset M$, see proof of 3.24. Then

$$\nu(U, N) = \nu(M, N) \cap (TN|U) \subset \nu(M, N)$$

is an open subset. Let $y \in U$. We know $d\rho_y : TN_y \to \mathbb{R}^\ell$ is onto with kernel TM_y. Consider $d\rho_y^t : \mathbb{R}^\ell \to \nu(M, N)_y$. Then define

$$\psi : U \times \mathbb{R}^\ell \to \nu(U, N)_y$$

by

$$\psi(y, v) = (y, d\rho_y^t(v)).$$

This defines a parametrization of $\nu(U, N)$. We have

$$dim(\nu(M, N)) = dim(M) + \ell = m + (k - m) = k.$$

Consider the projection $\pi : \nu(M, N) \to M$ with

$$\pi^{-1}(x) = \{x\} \times \nu(M, N)_x \subset \mathbb{R}^k,$$

which is a liner subspace. For $x \in M$ and U as bove we have commutative diagram

$$\begin{array}{ccc} \nu(U, N) & \xrightarrow{\tau} & U \times \mathbb{R}^\ell \\ {\scriptstyle \pi|}\downarrow & & {\scriptstyle p_1}\downarrow \\ U & = & U \end{array}$$

where

$$\tau(y, w) := (y, (d\rho_y^t)^{-1}(w)). \qquad ■$$

Remark. For each immersion $f : M^m \to N^n$ there can be defined a normal bundle $\nu(f)$, which is a $n-m$-bundle over M.

Example. Let $f : M^m \to GL(\ell) := \{A \in M(\ell) | A \text{ is invertible}\}$ be smooth. (Recall that $M(\ell) \cong \mathbb{R}^{\ell^2}$ is the smooth manifold of $\ell \times \ell$-matrices.) Let $x_0 \in M$ such that $f(x_0)$ is the identity matrix I_ℓ. Define

$$\xi := \bigcup_{x \in M} \{x\} \times \xi_x$$

where

$$\xi_x := f(x) \cdot (\mathbb{R}^r \times \{0\}) \subset \mathbb{R}^\ell.$$

It can be shown that $\xi \subset M \times \mathbb{R}^{ell}$ is a smooth r-bundle.

Let $H_{\ell,r}$ denote the set of r-dimensional subspaces of $\mathbb{R}^\ell$.

Lemma 5.9. *There is a bijective map*

$$H_{\ell,r} \to G_{\ell,r} := \{A \in M(\ell) | A^t = A, A^2 = A, tr(A) = r\} \subset M(\ell),$$

and $G_{\ell,r}$ is a smooth manifold of dimension $r(\ell - r)$.

Proof. Let $H \subset \mathbb{R}^\ell$ be a linear subspace of dimension r. Let $A_H : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the *uniquely determined* orthogonal projection onto H. Thus for an orthonormal bases $v_1, \ldots, v_r$ of H and $v_{\ell+1}, \ldots, v_r$ of $H^\perp$ let

$$A_H(v) := < v, v_1 > v_1 + \ldots + < v, v_r > v_r.$$

Note that $\mathbb{R}^\ell = H \oplus H^\perp$. Then $A_H^2 = A_H$ because $A_H(\mathbb{R}^\ell) = H$ and $A_H | H = id$. With respect to the basis above the linear map A_H is represented by the symmetric matrix in block form

$$A'_H = \begin{pmatrix} I_r & 0_{r,\ell-r} \\ 0_{\ell-r,r} & 0_{\ell-r,\ell-r} \end{pmatrix}$$

with trace r. The matrix representative with respect to the canonical basis $(e_1, \ldots, e_r)$ is also symmetric because any two orthonormal bases are related by orthogonal matrices and the change of matrix has the form $A \mapsto BAB^t$.

Now conversely let $A \in G_{\ell,r}$ be given. Then define

$$H := A(\mathbb{R}^\ell) \subset \mathbb{R}^\ell.$$

Since A is symmetric it can be diagonalized in an orthonormal basis, and the matrix with respect to this basis is

$$\begin{pmatrix} I_r & 0_{r,\ell-r} \\ 0_{\ell-r,r} & 0_{\ell-r,\ell-r} \end{pmatrix}$$

(Note that $A^2 = A$ implies that the eigenvalues of A can only be) and 1.) This proves the bijection. Thus we can think of $G_{\ell,r}$ as the *space of r-planes in* $\mathbb{R}^\ell$.

Now consider the *projection*

$$\mathfrak{p} : V_{\ell,r} \to G_{\ell,r},$$

which is defined by mapping

$$(v_1, \dots, v_r) \mapsto span(v_1, \dots, v_r)$$

It is not hard to see that $\iota \circ \mathfrak{p}$ is smooth where ι is the inclusion $G_{\ell,r} \subset \mathbb{R}^{\ell^2}$.

We will show that $G_{\ell,r}$ is a smooth manifold of dimension $r(\ell - r)$. (It can be proved that with respect to this smooth structure, $\mathfrak{p}$ is a submersion. Here the idea is that two r-tuples span the same linear subspace iff there is a corresponding automorphism of $span(v_1, \dots, v_r)$. These automorphisms correspond to the general linear group $GL(r)$. Because the preimage of a regular value is diffeomorphic to $GL(r)$ then

$$dim(GL(r)) = r^2 = dim(V_{\ell,r}) - dim(G_{\ell,r}) = \ell \cdot r - dim(G_{\ell,r})$$

corresponding to the dimension formula

$$dim(G_{\ell,r}) = \ell \cdot r - r^2 = r(\ell - r).)$$

Now let $H = span(v_1, \dots, v_r)$ be an r-dimensional linear subspace and let $v_{r+1}, \dots, v_\ell$ be a basis of $H^\perp$. For nonnegative integers s, t let $M(s,t) \subset \mathbb{R}^{st}$ denote the smooth manifold of all real $(r \times s)$-matrices. Consider the map:

$$\delta : M(r, \ell - r) \to G_{\ell,r}$$

defined by

$$(\lambda_{ij})_{\substack{1 \le i \le r \\ r+1 \le j \le \ell}} \mapsto span(v_i + \sum_{j>r} \lambda_{ij} v_j | 1 \le i \le r)$$

We want to prove that this map is a diffeomorphism in a neighborhood of $0_{r,\ell-r} \in M(r, \ell - r)$ and thus defines a parametrization of $G_{\ell,r}$ at H.

Without restriction we will prove this for $H = span(e_1, \dots, e_r)$. Consider a subspace H' close to $span(e_1, \dots, e_r)$. We can write $H' = span(w_1, \dots, w_r)$ with $w_i = (a_i, b_i)$ and $a_i \in \mathbb{R}^r$ and $b_i \in \mathbb{R}^{\ell - r}$ such that $(b_1, \dots b_r)^t$ is close to $0_{r,\ell-r}$. Then there is a uniquely determined $A \in GL(r)$ such that

$$A(a_1, \dots, a_r) = (e_1, \dots, e_r)$$

We define $A(b_1, \dots, b_r)^t =: ((\lambda_{ij})_{i,j}) \in M(r, \ell - r)$. It is easy to see that $\delta((\lambda_{ij})_{i,j}) = H'$ and thus δ is onto. If H' is close to H then $(\lambda_{ij})_{i,j}$ is uniquely determined by H. In fact we can define $(w_1, \dots, w_r)$ spanning H' by orthogonal projection of $(e_1, \dots, e_r)$ onto H'.

Examples 5.10. (a) $P^{m-1} := G_{m,1}$ is an $m-1$-dimensional smooth manifold denoted the *projective space of lines in* $\mathbb{R}^m$.
(b) Let $f : X \to G_{\ell,r}$ be a continuous map. Then

$$\xi_f := \bigcup_{x \in X} \{x\} \times f(x) \subset X \times \mathbb{R}^\ell$$

is an r-bundle (this is not obvious!). Conversely for each bundle $\xi \subset X \times \mathbb{R}^\ell$ there exists the *Gauss map*

$$f_\xi : X \to G_{\ell,r}$$

mapping $x \in X$ to ξ_x.
(c) There is a *canonical* r-bundle $\gamma_r \subset G_{\ell,r} \times \mathbb{R}^\ell$ over $G_{\ell,r}$ defined by

$$\gamma_r := \bigcup_{H \in G_{\ell,r}} \{H\} \times H.$$

Definition 5.11. (a) A continuous map

$$F : \xi \to \eta$$

between bundles ξ over X and η over Y is called *linear over* f if there is a commutative diagram of continuous maps

$$\begin{array}{ccc} \xi & \xrightarrow{F} & \eta \\ {\scriptstyle \pi_\xi}\downarrow & & {\scriptstyle \pi_\eta}\downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

such that $F|\xi_x : \xi_x \to \eta_{f(x)}$ is a linear map. If $F|\xi_x$ is a vector space isomorphism for all $x \in X$ then F is called a *bundle map over* f.
noindent (b) If $X = Y$ and $f = id$ as in (a) then a linear map is a *bundle homomorphism*, and a bundle map is a *bundle isomorphism*.

Examples 5.12. (a) Let $f : M \to N$ be smooth. Then $df : TM \to TN$ is a linear map over f. If f is a diffeomorphism then df is a bundle map.
(b) If $f : M \to N$ is smooth and $Z \subset N$ with $f \pitchfork Z$. Then

$$\begin{array}{ccc} \nu(f^{-1}(Z), M) & \xrightarrow{df|} & \nu(Z, N) \\ \downarrow & & \downarrow \\ f^{-1}(Z) & \xrightarrow{f|} & Z \end{array}$$

is a bundle map.

(c) If $\xi \to X$ is an arbitrary r-bundle with Gauss map f_ξ then there is a bundle map $F_\xi : \xi \to \gamma_r$ over f_ξ, i. e. a commutative diagram

$$\begin{array}{ccc} \xi & \xrightarrow{F_\xi} & \gamma_r \\ \downarrow & & \downarrow \\ X & \xrightarrow{f_\xi} & G_{\ell,r} \end{array}$$

(d) If there is an isomorphism $\xi \to \eta$ over X then ξ and η are *equivalent bundles*.

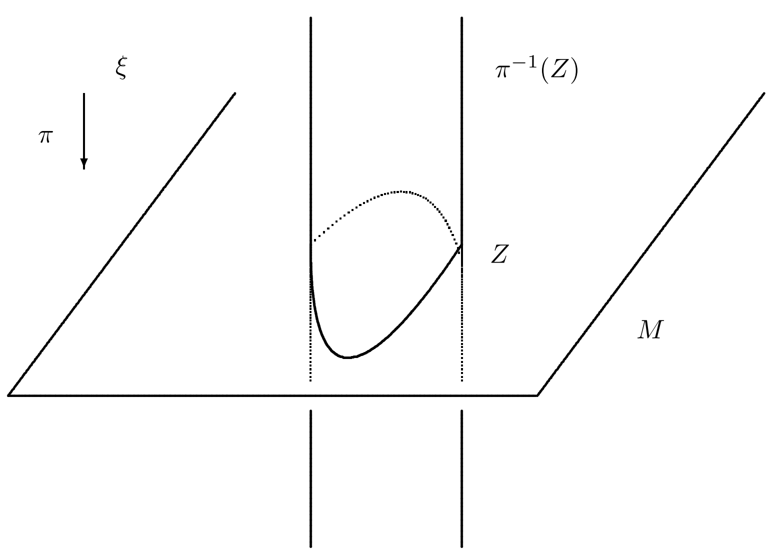
Lemma 5.13. *For each smooth bundle ξ over M the projection $\pi : \xi \to M$ is a submersion.*

Proof. Consider the local trivialization:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tau_U} & U \times \mathbb{R}^r \\ {\scriptstyle \pi|}\downarrow & & {\scriptstyle p_1}\downarrow \\ U & = & U \end{array}$$

Since τ_U is a diffeomorphism and p_1 is a submersion, π is a submersion. ■

Corollary. If $Z \subset M$ is a smooth manifold then $\pi^{-1}(Z) \subset \xi$ is a smooth manifold.



Theorem 5.14 (partition of unity). *Let $X \subset \mathbb{R}^k$ be a space and (U_α) be*

a covering of X by open sets $U_\alpha \subset X$. Then there is a sequence of smooth functions

$$\theta_i : X \to \mathbb{R}$$

such that the following holds:
(i) $0 \le \theta_i(x) \le 1$ *for all* $x \in X$, $i \in \mathbb{N}$,
(ii) *For each $x \in X$ there exists a neighborhood $V(x)$ such that $\theta_i|V(x) \equiv 0$ for almost all $i \in \mathbb{N}$ (local finiteness)*
(iii) *For all $i \in \mathbb{N}$ there exists α and a closed set $V_i \subset U_\alpha$ such that $\theta_i|X \setminus V_i \equiv 0$.*
(iv) $\sum_{i\in\mathbb{N}} \theta_i(x) = 1$ *for all* $x \in X$.
The family $(\theta_i)_{i\in\mathbb{N}}$ is called a partition of unity subordinate to the covering (U_α).

For the proof see Michael Spivak, Calculus on Manifolds, 3.11.

Definition 5.15. (a) An open covering (U_α) of a space X is called *locally finite* if for each $x \in X$ there exists a neighborhood V such that $\{\alpha | V \cap U_\alpha \neq \emptyset\}$ is finite.
(b) A covering (V_β) *refines* a covering (U_α) if for each α there exists β such that $V_\beta \subset U_\alpha$.

Corollary 5.16. *Each covering (U_α) of a space X by open sets admits a countable locally finite refinement by open sets.*

Proof. For a partition of unity (θ_i) subordinate to the partition (U_α) consider $V_i := \theta^{-1}(\mathbb{R} \setminus \{0\})$. Then (V_i) refines (U_α) and is locally finite by 5.14 (ii). ■

Lemma 5.17. *Let $f : M \to N$ be smooth, $Z \subset M$ a submanifold and $f : Z \to f(Z)$ a diffeomorphism such $df_x : TM_x \to TN_{f(x)}$ is an isomorphism for all $x \in Z$. Then there is an open neighborhood U of Z in M such that*

$$f|U : U \to f(U)$$

is a diffeomorphism.

Proof. There is a neighborhood U_0 of Z such that $df_x : TM_x \to TN_{f(x)}$ is an isomorphism for all $x \in U_0$. (Define $U_0 := \cup_{x\in Z} U_x$, such that the claim holds for each U_x using 3.7.) For each $y \in f(Z)$ choose a neighborhood V_y and a diffeomorphism $g_y : V_y \to U'_y \subset U_0$ inverse to f. Let (V_i) be a locally finite refinement of (V_y) by open sets, and $g_i : V_i \to U_i$ be the restriction of the corresponding g_y (with out restriction $i \in \mathbb{N}$). Let

$$W_i := \{y \in U_i | g_i(y) = g_j(y) \text{ for } y \in U_i \cap U_j\} \subset U_i$$

and

$$W := \bigcup_{i\in\mathbb{N}} W_i$$

with $g : W \to M$ defined by $g|W_i = g_i|W_i$ smooth. For fixed i consider for each $j \in \mathbb{N}$

$$\{y \in U_i \cap U_j | g_i(y) = g_j(y)\}.$$

This set is both closed and open in $U_i \cap U_j$. In fact if we have y_0 with $g_i(y_0) = g_j(y_0) = x_0 \in U_0$ the it follows from the invertibility of df_{x_0} that there exists a neighborhood U' of x_0 such that $f| : U' \to f(U')$ is a diffeomorphism. In particular the inverse function is unique thus $g_i|f(U') = g_j|f(U')$. For $y_0 \in f(Z)$ fixed let $W_{ij}(y_0)$ be that component $\{y \in U_i \cap U_j | g_i(y) = g_j(y)\}$, which contains $y_0 \in U_i \cap U_j$. Let

$$W(y_0) := \bigcap_{\{(i,j)|y_0 \in U_i \cap U_j\}} W_{ij}(y_0) \subset W,$$

open in W. So let $U := \cup_{y_0 \in f(Z)} W(y_0)$. Then $U \subset W$, so g is defined on U and is open neighborhood of $f(Z)$. ■

Let $X \subset \mathbb{R}^k$ be a space and $\varepsilon : X \to \mathbb{R}_+ := \{t \in \mathbb{R} | t > 0\}$ be a smooth map. Then let

$$X(\varepsilon) := \{y \in \mathbb{R}^k | \exists x \in X \text{ with } ||y - x|| < \varepsilon(x)\}$$

be the *ε-neighborhood of X in $\mathbb{R}^k$*.

Example. Let $X \to \mathbb{R}_+$ be the constant map to $\varepsilon > 0$ then

$$X(\varepsilon) = \{y \in \mathbb{R}^k | d(y, X) < \varepsilon\}$$

Lemma 5.18. *Let $X \subset \mathbb{R}^k$ be compact and U a neighborhood of X (i. e. U is neighborhood of each point $x \in X$). Then there exists $\varepsilon > 0$ such that $X(\varepsilon) \subset U$.*

We leave this as an exercise.

Lemma 5.19. *Let $M \subset \mathbb{R}^k$ be a smooth manifold and U a neighborhood of M. Then there exists a smooth function $\varepsilon : M \to \mathbb{R}_+$ such that $M(\varepsilon) \subset U$. For M compact we can assume that ε is constant by* 5.18.

Proof. Let (U_α) be a covering of M by open sets with $cl_M U_\alpha$ compact. Then by 5.18 there exist $\varepsilon_\alpha > 0$ such that $U_\alpha(\varepsilon_\alpha) \subset U$. Let θ_i be a partition of unity subordinate to (U_α). Let $U_i := \theta_i^{-1}(\mathbb{R} \setminus \{0\}) \subset U_\alpha$ for some α, and all $i \in \mathbb{N}$. Let

$$\varepsilon := \sum_{i\in\mathbb{N}} \theta_i \varepsilon_i.$$

Let $y \in M(\varepsilon)$. Then there exists $x \in M$ such that $||y - x|| < \varepsilon(x)$. Note that $\{j \in \mathbb{N} | x \in U_j\}$ is finite. Let $i_x \in \mathbb{N}$ be chosen such that $\varepsilon_{i_x} = \max\{\varepsilon_j | x \in U_j\}$. It follows that $\varepsilon(x) \leq \sum_{i \in \mathbb{N}} \theta_i(x)\varepsilon_{i_x} = \varepsilon_{i_x}$ and thus $y \in U_{i_x}(\varepsilon(i_x)) \subset U$. ■

Theorem 5.20 (ε-neighborhood theorem). *Let $M \subset \mathbb{R}^k$ with $\partial M = \emptyset$. Then there exists a smooth function*

$$\varepsilon : M \to \mathbb{R}_+,$$

(constant for M compact) and a submersion

$$\pi : M(\varepsilon) \to M$$

with $\pi | M = id$.

Proof. Let $h : \nu(M) \to \mathbb{R}^k$ be defined by $h(x, v) = x + v$, where $\nu(M)$ is the normal bundle of the inclusion $M \subset \mathbb{R}^k$. Let $(x, 0) \in M \times 0 \subset \nu(M)$. Then there is the commutative diagram:

$$\begin{array}{ccc} T\nu(M)_{(x,0)} & = & T(M \times 0)_{(x,0)} + \{0\} \times \nu(M)_x \\ {\scriptstyle dh_{(x,0)}}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathbb{R}^k_x & = & TM_x \oplus \nu(M)_x \end{array}$$

Here $T\nu(M)_{(x,0)} \subset \mathbb{R}^k \times \mathbb{R}^k$, and the right vertical arrow is an isomorphism because $dim(\nu(M)) = k$. Moreover, h is regular on $M \times 0 : M \times 0 \xrightarrow{\cong} M$. So by 5.17. there is a neighborhood U of $M \times 0$ in $\nu(M)$, which is mapped by h diffeomorphically onto a neighborhood of M in $\mathbb{R}^k$. By 5.19. each such neighborhood contains a neighborhood of the form $M(\varepsilon)$. Then with $h^{-1}| : M(\varepsilon) \to \nu(M)$ we can let $\pi := \pi_{\nu(M)} \circ (h^{-1}|)$ be the submersion we want. Here $\pi_{\nu(M)}$ is the projection of the normal bundle. ■

Corollary 5.21. *Let $f : M \to N$ be smooth with $\partial N = \emptyset$ and $N \subset \mathbb{R}^\ell$. Then there exists a smooth map*

$$F : M \times D^\ell \to N$$

such that $F(x, 0) = f(x)$ and for each $x \in M$ the map $s \mapsto F(x, s)$ is a submersion $D^\ell \to N$. (In particular F and ∂F are submersions.)

Proof. Let $\varepsilon, N(\varepsilon)$ and π be as in 5.20., and

$$F : M \times D^\ell \to N$$

be defined by

$$F(x, s) = \pi(f(x) + \varepsilon(f(x)s)).$$

Then $F(x,0) = \pi(f(x)) = f(x)$. For fixed x the mapping

$$s \mapsto f(x) + \varepsilon(f(x)s)$$

is a submersion $D^\ell \to \mathbb{R}^\ell$, and π is a submersion. Thus also $s \mapsto F(x,s)$ is a submersion. ■

Theorem 5.22. *Let $f : M \to N$ be smooth, $Z \subset N$ with $\partial Z = \partial N = \emptyset$. Then there exists a smooth map $g : M \to N$ smoothly homotopic to f with $g \pitchfork Z$ and $\partial g \pitchfork Z$.*

Proof. For F as in 5.21. we have $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$ for almost all $s \in D^\ell$ (by 5.2.). But each f_s is homotopic to f using

$$I \times M \to N$$

defined by

$$(t,x) \mapsto F(x,ts).$$

■

Lemma 5.23. *Let $X \subset \mathbb{R}^k$ be a space with $A, B \subset X$ closed and $A \cap B = \emptyset$. Then there exists a continuous function $\lambda : X \to [0,1]$ with $\lambda|A \equiv 0$ and $\lambda|B \equiv 1$. In particular there are open neighborhoods U and V of A and B with $U \cap V = \emptyset$.*

Proof. Let

$$\lambda(x) := \frac{1}{2}\left(\frac{d(x,A) - d(x,B)}{d(x,A) + d(x,B)} + 1\right).$$

Then λ is continuous. Notice that

$$x \in A \Longleftrightarrow d(x,A) = 0$$

respectively

$$x \in B \Longleftrightarrow d(x,B) = 0.$$

Let $U := \lambda^{-1}[0,\varepsilon)$ and $V : +\lambda^{-1}(1-\varepsilon,1])$. ■

Corollary 5.24. *Let $A \subset X$ be closed with open neighborhood U. Then there exists a smooth function $\lambda : X \to [0,1]$ with $\lambda|(X \setminus U) \equiv 1$ and $\lambda(x) = 1$ for all x in a neighborhood of A.*

Proof. A and $X \setminus U$ are disjoint and closed. Let V, W be disjoint open neighborhoods of $A, X \setminus U$ by 5.23. Then $W \supset X \setminus U \Longrightarrow X \setminus W \subset U$ is closed, and $V \subset X \setminus W \Longrightarrow \overline{V} \subset X \setminus W \subset U$. Now we can argue as follows: If we have found $V \supset A$ with $\overline{V} \subset U$ then let $(\theta_i)_i$ be a partition of unity subordinate to the covering $\{U, X \setminus \overline{V}\}$. Then

$$\lambda := \sum_{\theta_i|\overline{V}\equiv 0} \theta_i$$

is smooth. If $x \in V$ then $\theta_i(x) \neq 0 \Longrightarrow \theta_i|\overline{V} \neq 0$. Thus θ_i does not appear in λ or $\theta_i(x) = 0$.Let $x \in X \setminus U$. Then $\theta_i(x) = 0$ is not important. If $\theta_i(x) \neq 0 \Longrightarrow \theta_i^{-1}(\mathbb{R} \setminus 0) \subset X \setminus U \Longrightarrow \theta_i|\overline{V} \cong 0$. Thus all θ_i actually appear in the above sum. This implies:

$$\lambda(x) = \sum \theta_i(x) = 1.$$

■

Definition 5.25. Let $f : M \to N$ be smooth, $Z \subset N$. Then f is *transversal to Z along $C \subset M$* if

$$df_x(TM_x) + TZ_{f(x)} = TN_{f(x)}$$

for all $x \in f^{-1}Z) \cap C$.

Theorem 5.26 (Extension theorem). *Let $f : M \to N$ be smooth, $\partial N = \emptyset$. Let $Z \subset N$ be a closed submanifold without boundary. Let $C \subset M$ be a closed subset. Let $f \pitchfork Z$ along C and $\partial f \pitchfork C$ along $C \cap \partial M$. Then there is a smooth map $g : M \to N$ smoothly homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$ and $g = f$ on a neighborhood of C.*

Proof. The idea is to modify the function F from 5.21.

Claim 1. *$f \pitchfork Z$ in a neighborhood U of C.*

Proof of Claim 1. Let $x \in C$, $x \notin f^{-1}(Z)$. Z is closed. It follows that $M \setminus f^{-1}(Z)$ is a neighborhood of x where $f \pitchfork Z$. For $x \in f^{-1}(Z)$ let W be an open neighborhood of $f(x)$ and $\phi : W \to \mathbb{R}^k$ be a smooth submersion such that $f \pitchfork Z$ at $x' \in f^{-1}(Z) \Longleftrightarrow \phi \circ f$ is regular at x. But $\phi \circ f$ is regular at $x \Longrightarrow$ it is regular in a neighborhood $\Longrightarrow f \pitchfork Z$ on a neighborhood U of C. This proves Claim 1. ■

Now let $\lambda : M \to [0, 1]$ with $\lambda|(M \setminus U) \equiv 1$ and $\lambda \equiv 0$ on a neighborhood of C in U. Let $\tau := \lambda^2$. Because

$$d\tau_x = 2\lambda(x)d\lambda_x$$

we have

$$\tau(x) = 0 \Longrightarrow d\tau_x = 0.$$

Let $F : M \times D^\ell \to N$ be as in 5.21. and define

$$G : M \times D^\ell \to N$$

by

$$G(x,s) := (F(x\tau(x),s).$$

Claim 2. *$G \pitchfork Z$ and $\partial G \pitchfork Z$.*

Proof of Claim 2: Let $(x,s) \in G^{-1}(Z)$. First suppose that $\tau(x) \neq 0$. The mapping $r \mapsto G(x,r), D^\ell \to N$ is a submersion because it is composition of the diffeomorphism $r \mapsto \tau(x)r$ with the submersion $r \mapsto F(x,r)$. This implies that G is regular at (x,s). Thus $G \pitchfork Z$ in (x,s). Now suppose that $\tau(x) = 0$. Let

$$m : M \times D^\ell \to M \times D^\ell$$

be defined by

$$m(x,s) = (x, \tau(x)s).$$

Then

$$dm_{(x,s)}(v,w) = (v, \tau(x)w + d\tau_x(v)s)$$

for

$$(v,w) \in TM_x \times TD^\ell_s = TM_x \times \mathbb{R}^\ell.$$

To prove this let γ, ρ be paths in M, D^ℓ with $\gamma(0) = x, \gamma'(0) = v$ and $\rho(0) = s$, $\rho'(0) = w$. Then

$$\begin{aligned} dm_{(x,s)}(v,w) = &\left.\frac{d}{dt}\right|_{t=0} m \circ (\gamma(t), \rho(t)) = \left.\frac{d}{dt}\right|_{t=0} (\gamma(t), \tau(\gamma(t))\rho(t)) \\ =&(\gamma'(0), \tau(\gamma(0)\rho'(0) + \tau'(\gamma(0))\gamma'(0)\rho(0)) \\ =&(v, \tau(x)w + d\tau_x(v)s) \end{aligned}$$

Then $G = F \circ m$ and $\tau(x) = 0 = d\tau_x$ implies

$$dG_{(x,s)}(v,w) = dF_{m(x,s)} \circ dm_{(x,s)}(v,w) = dF_{(x,0)}(v,0).$$

We know that $F|M \times 0 = f$ and thus

$$dG_{(x,s)}(v,w) = df_x(v).$$

But $\tau(x) = 0 \Longrightarrow x \in U \Longrightarrow f \pitchfork Z$ at x. Since $im(dG_{(x,s)} = im(df_x)$ is follows that $G \pitchfork Z$ at (x,s). Similarly $\partial G \pitchfork Z$. This proves Claim 2. ■

Now by 5.2. there is some s such that we have for the mapping $g(x) = G(x,s)$ that $g \pitchfork Z$ and $\partial g \pitchfork Z$, $g \simeq f$. If $x \in C$ then $\tau(x) = 0$, and it follows that

$$g(x) = G(x,s) = F(x,0) = f(x).$$

■

Corollary 5.27. *Let $f : M \to N$ be smooth with $\partial f \pitchfork Z$. Then there exists $g : M \to N$ smoothly homotopic to f such that $\partial f = \partial g$.*

Proof. $\partial M \subset M$ is closed. ■

Chapter 6

Intersection numbers, vector fields and Euler characteristic.

Throughout the following we will have condition (S): *M, N, Z are smooth oriented manifolds without boundary, M compact and $Z \subset N$ closed, and $dim(M) + dim(Z) = dim(N)$.*

A smooth compact manifold without boundary is also called a *closed* manifold. Thus in condition (S) the manifolds M and Z are closed.

Suppose $f : M \to N$ is smooth with $f \pitchfork Z$. Then $f^{-1}(Z) \subset M$ is a finite set of oriented points. The orientation number $\sigma(x)$ for $x \in f^{-1}(Z)$ is $+1$ respectively -1 if the direct sum orientation on $TN_{f(x)}$ given by

$$df_x(TM_x) \oplus TZ_{f(x)} = TN_{f(x)}$$

agrees respectively does not agree with the orientation of $TN_{f(x)}$ given by the orientation of N.

Definition 6.1. The *intersection number of f and Z* is

$$I(f, Z) = \sum_{x \in f^{-1}(Z)} \sigma(x) \in \mathbb{Z}.$$

Without any given orientations there is still defined

$$I_2(f, Z) = \sum_{x \in f^{-1}(Z)} 1 = |f^{-1}(x)| \pmod 2 \in \mathbb{Z}_2.$$

Theorem 6.2. *Let $M = \partial W$ for some oriented smooth manifold W, and suppose that $f : M \to N$ extends to a smooth map $G : W \to N$ then $I(f, Z) = 0$. Correspondingly $I_2(f, Z) = 0$ if W is not necessarily orientable.*

Proof. By 5.27. there exists an extension F with $\partial F = f$ and $F \pitchfork Z$. Then $F^{-1}(Z)$ is a smooth oriented 1-dimensional manifold with boundary $f^{-1}(Z)$. It follows from 4.26 (ii) and $(\partial F)^{-1}(Z) = (-1)^{codim_N Z} \partial(F^{-1}(Z))$ (see 4.28) that $I(f, Z) = 0$. ■

Remark. The sign discussion in the proof of 6.3. is based on the following observations: Let $A \subset F^{-1}(Z)$ be an arc with $\partial A = a \cup b$. We need to argue that $\sigma(a) - \sigma(b) = 0$. Now TA_x is oriented by $v(x) \neq 0$ where

$$dF_x(TA_x^\perp) \oplus TZ_{F(x)} = TN_{F(x)}$$

$$TA_x^\perp \oplus v(x)\mathbb{R} = TW_x$$

Then $v(x) \in HW_x$ at precisely one of a, b. This follows from the fact if $v(x)$ on one side of the arc is derivative of a path running into A, and $v(x) \neq 0$ all along the arc, then $v(x)$ will be derivative of a path running out of A on the other side (to make the argument precise you could use a function on A, which is locally constant thus constant because A is connected). Note that $TA_x^\perp$ for $x = a$ and $x = b$ can be replaced by TM_x and the two conditions above become:

$$df_x(TM_x) \oplus TZ_{f(x)} = TN_{f(x)}$$

$$TM_x \oplus v(x)\mathbb{R} = TW_x.$$

Theorem 6.3. *Homotopic maps $M \to N$ for M closed have the same intersection numbers.*

Proof. Let f_0 and f_1 be smoothly homotopic, both $\pitchfork Z$. Let $F : I \times M \to N$ be a smooth homotopy. By 6.2. it follows that $I(\partial F, Z) = 0$. We know that $\partial(I \times M) = M_1 - M_0$ and $\partial F|M_0 = f_0$, $\partial F|M_1 = f_1$ thus

$$\partial F^{-1}(Z) = f_1^{-1}(Z) - f_0^{-1}(Z)$$

as oriented manifolds. It follows that

$$I(\partial F, Z) = I(f_1, Z) - I(f_0, Z) = 0.$$

■

Now let $g : M \to N$ be a smooth map not necessarily $\pitchfork Z$. Let f be smoothly homotopic to g and $\pitchfork Z$. Such a smooth function g exists by 5.22. Then

$$I(g, Z) := I(f, Z)$$

is well-defined. In fact it does not depend on the choice of f by 6.3.

If $g : M \subset N$ is the inclusion of a submanifold then we write $I(M, Z) := I(g, Z)$.

Definition 6.4. Two smooth maps $f : M \to N$ and $g : Z \to N$ are *transversal* if

$$df_x(TM_x) + dg_x(TZ_z) = TN_y$$

for all $x \in M$ and $z \in Z$ with $f(x) = y = g(z)$ holds. Notation: $f \pitchfork g$.

Now suppose (S) holds for M, N, Z. Then the above sum has to be a direct sum and df_x, dg_z are isomorphisms. The *local intersection number* $\sigma(x, z)$ is defined to be $+1$ respectively -1 if the direct sum orientation given by

$$df_x(TM_x) \oplus dg_z(TN_z) = TN_y$$

agrees respectively disagrees with the orientation given by N.

Definition 6.5. For M, N, Z, f, g as above let

$$I(f, g) := \sum_{(x,z) \in M \times N, f(x) = g(z)} \sigma(x, z)$$

be the *intersection number of f and g.*

Remark. If $g : Z \subset N$ is inclusion of a submanifold then $I(f, g) = I(f, Z)$.

In order to show that the sum in 6.5. is finite we prove the following:

Theorem 6.6.

$$f \pitchfork g \Longleftrightarrow (f \times g) \pitchfork \Delta,$$

where

$$f \times g : M \times Z \to N \times N$$

is defined by

$$(f \times g)(x, z) = (f(x), g(z))$$

and

$$\Delta \subset N \times N$$

is the diagonal. Moreover,

$$I(f,g) = (-1)^{dim(Z)} I(f \times g, \Delta).$$

The result follows with $U = df_x(TM_x)$, $W = dg_z(TZ_z)$ and $V = TN_{f(x)}$ from the following result, which is proved by a tedious calculation.

Lemma 6.7. *Let $U, W \subset V$ be vector subspaces. Then*

$$U \oplus W = V \Longleftrightarrow (U \times W) \oplus \Delta = V \times V,$$

where $\Delta \subset V \times V$ is the diagonal. Moreover: Suppose U, W are oriented and V is oriented by the direct sum, Δ is oriented by the usual isomorphism $V \to \Delta, v \mapsto (v,v)$. Then the product orientation of $V \times V$ agrees with the direct sum orientation of $(U \times W) \oplus \Delta$ iff $dim(W)$ is even.

Now for $f : M \to N$ and $g : Z \to N$ arbitrary smooth maps (not necessarily $\pitchfork$) we define:

$$I(f,g) := (-1)^{dim(Z)} I(f \times g, \Delta).$$

Theorem 6.8. *If $f_0 \simeq f_1$ and $g_0 \simeq g_1$ then $I(f_0, g_0) = I(f_1, g_1)$.*

Proof. $f_t \times g_t$ is a homotopy between $f_0 \times g_0$ and $f_1 \times g_1$. Thus the resulkt follows from 6.3. ■

Corollary 6.10. *If $dim(M) = dim(N)$ and N is connected then $I(f, \{y\}) \in \mathbb{Z}$ does not depend on $y \in N$. Then*

$$deg(f) := I(f, \{y\})$$

is called the Brower degree of f.

Proof. Let i_0, i_1 be inclusions of a point with image y_0, y_1. Then

$$I(f, \{y_0\}) = I(f, i_0) = I(f, y_1) = I(f, \{y_1\}).$$

It is an exercise to prove $i_0 \simeq i_1$. ■.

Examples 6.11. (i) The Brower degree of a map $f : M \to N$ with $dim(M) = dim(N)$ counts for regular values the preimages with signs. For example the degree of $f_r : S^1 \to S^1$ defined by $f_r(z) = z^r$ is r for all $r \in \mathbb{Z}$. Generally it follows from Exercises 4.4. that for each $r \in \mathbb{N}$ there exists a map $f_r : M \to S^m$,

$m = dim(M)$ such that $deg(f) = r$. Using a composition with a reflection $S^m \to S^m$ (which has degree -1) it follows from 6.12 (b) below that there exist maps $M \to S^m$ of arbitrary integer degree.
(ii) The identity map $id : M \to M$ has degree $+1$ thus is not homotopic to a constant map for $dim(M) > 0$, which has degree 0.

The following is immediate from the definitions.

Theorem 6.12. (a) *Suppose* $f : M \to N$ *and* $g : Z \to N$ *with* $dim(M) + dim(Z) = dim(N)$. *Then*

$$I(f, g) = (-1)^{dim(M)dim(Z)} I(g, f)$$

(b) *Suppose* $f : M \to N$ *and* $g : N \to W$ *with* $dim(M) = dim(N) = dim(W)$. *Then*

$$deg(f \circ g) = deg(f)deg(g).$$

Definition 6.13. Let $\pi : \xi \to M$ be a smooth r-bundle. A *section of* ξ is a smooth map

$$\mathfrak{s} : M \to \xi$$

such that

$$\pi \circ \mathfrak{s} = id_M,$$

i.e. $\mathfrak{s}(x) \in \xi_x$ for all $x \in M$.

An example is the zero-section $\mathfrak{z} : M \to M \times \{0\} \subset \xi$.

Definition 6.14. An *orientation of a bundle* $\xi \subset X \times \mathbb{R}^\ell$ is a family of orientations $(\sigma_x)_{x \in X}$ of orientations σ_x of ξ_x such that there are local parametrizations

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tau} & U \times \mathbb{R}^m \\ \downarrow & & \downarrow \\ U & = & U \end{array}$$

for which the isomorphisms $\tau|\xi_x : \xi_x \to \mathbb{R}^m$ carry the σ_x to the standard orientation of $\mathbb{R}^m$. A bundle ξ admitting an orientation is called an *oriantable* bundle.

Example 6.14. M is an orientable manifold of dimension m iff TM is an orientable m-bundle over M.

Lemma 6.15. *Let ξ be an orientable smooth r-bundle over the oriented manifold M. Then the total space $\xi \subset M \times \mathbb{R}^\ell$ is an orientable manifold of dimension $dim(M) + r$ (with coordinate systems given by suitable trivializations).*

Proof. We have $T\xi_{(x,v)} = TM_x \times \xi_x$ (consider paths to prove this, $v \in \xi_x$). Let $\alpha = (v_1, \ldots, v_m)$ be an oriented basis for TM_x and $\beta = (w_1, \ldots, w_r)$ is an oriented basis for ξ_x according to the bundle orientation of ξ. Then

$$sign(\alpha \times 0, 0 \times \beta) = sign(\alpha)sign(\beta)$$

similarly to the product orientation of manifolds. ■

Definition 6.16. Let ξ be an oriented m-bundle over the oriented closed manifold M with $dim(M) = m$. Then

$$\chi(\xi) := I(\mathfrak{z}, M \times 0) = I(\mathfrak{z}, \mathfrak{z})$$

is called the *Euler characteristic of* ξ.

Example. For M oriented and closed,

$$\chi(M) := \chi(TM) \in \mathbb{Z}$$

is the *Euler characteristic of the manifold* M. If $f : M \to N$ is an oriented diffeomorphism of oriented manifolds then $\chi(M) = \chi(N)$. If $f : M \to N$ is an immersion of an orientable with $dim(N) = 2dim(M)$ and orientable normal bundle then $\chi(f) := \chi(\nu(f))$ For an arbitrary manifold and immersion without any orientatability assumptions $\chi(M)_2$ and $\chi(f)_2$ are defined in $\mathbb{Z}_2$.

Theorem 6.17. *Let $\xi \to M$ be a smooth oriented m-bundle over a smooth oriented closed manifold of dimension m. Let m be odd. Then $\chi(\xi) = 0$.*

Proof. Suppose that m is odd. Then

$$I(\mathfrak{z}, \mathfrak{z}) = (-1)^{dim(M)dim(M)} I(\mathfrak{z}, \mathfrak{z}) = -I(\mathfrak{z}, \mathfrak{z}),$$

which implies $2I(\mathfrak{z}, \mathfrak{z}) = 0$. ■

In particular $\chi(M) =)$ for each oriented closed manifold of odd dimension. For example $\chi(S^1) = \chi(S^3) = 0$.

Theorem 6.18. *Let $\xi \to M$ be a smooth oriented m-bundle over a smooth oriented closed manifold of dimension m. Suppose ξ has a smooth section $\mathfrak{s}$ with $\mathfrak{s}(x) \neq 0$ for all $x \in M$ (i. e. $\mathfrak{s}$ is nonsingular). Then $\chi(\xi) = 0$. In the nonorientable case $\chi(\xi)_2 = 0$.*

Proof. Each two sections $\mathfrak{s}_0, \mathfrak{s}_1$ are smoothly homotopic. In fact, for $0 \leq t \leq 1$ define

$$\mathfrak{s}_t(x) := t\mathfrak{s}_1(x) + (1-t)\mathfrak{s}_0(x) \in \xi_x$$

for all $x \in M$. For $\mathfrak{s}$ without any zeroes obviously

$$I(\mathfrak{s}, M \times 0) = 0,$$

thus also $I(\mathfrak{z}, M \times 0) = 0$ by 6.3. ■

Definition 6.20. A *vector field* on $M \subset \mathbb{R}^k$ is a smooth map

$$\mathfrak{v} : M \to \mathbb{R}^k$$

such that $\mathfrak{v}(x) \in TM_x$ for all $x \in M$.

Theorem 6.21. *The set of vector fields on M is in one-to-one correspondence with the set of sections of TM. Moreover, $\mathfrak{s} : M \to TM$ is nonsingular iff $\mathfrak{v}$ has no zeroes, i. e. $\mathfrak{v}(x) \neq 0$ for all $x \in M$.*

Proof. For $\mathfrak{v} : M \to \mathbb{R}^k$ let $\mathfrak{s}_\mathfrak{v} : M \to TM$ be defined by

$$\mathfrak{s}_\mathfrak{v}(x) := (x, \mathfrak{v}(x)).$$

For $\mathfrak{s} : M \to TM$ define $\mathfrak{v}_\mathfrak{s} : M \to \mathbb{R}^k$ by

$$\mathfrak{v}_\mathfrak{s} := p_2 \circ \iota \circ \mathfrak{s},$$

where ι is the inclusion $TM \subset M \times \mathbb{R}^k$ and p_2 is the projection $M \times \mathbb{R}^k \to \mathbb{R}^k$ onto the second factor. ■

Let $U \subset \mathbb{R}^m$ be an open set and $\mathfrak{v} : U \to \mathbb{R}^m$ be a smooth vector field with isolated zero at $x_0 \in U$. Then for all $\varepsilon > 0$ sufficiently small the map

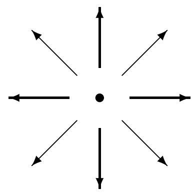
$$\overline{\mathfrak{v}} : S^{m-1} \ni x \mapsto \frac{\mathfrak{v}(x_0 + \varepsilon x)}{||\mathfrak{v}(x_0 + \varepsilon x)||} \in S^{m-1}$$

is well-defined and smooth. Let

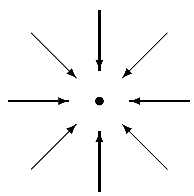
$$index(\mathfrak{v}, x_0) := deg(\overline{\mathfrak{v}}) \in \mathbb{Z}.$$

be the *index of the vector field* $\mathfrak{v}$ *at* x_0. Note that the index does not depend on ε use a suitable homotopy top show this).

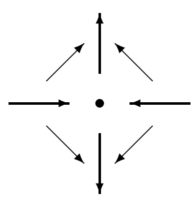
The following are the typical situations in $\mathbb{R}^m$ (pictures in $\mathbb{R}^2$):



$$\overline{\mathfrak{v}} = id \quad \Longrightarrow \quad index(\mathfrak{v}, 0) = 1$$



$$\overline{\mathfrak{v}} = -id \quad \Longrightarrow \quad index(\mathfrak{v}, 0) = (-1)^m$$



$$\overline{\mathfrak{v}} = r|S^{m-1} \quad \Longrightarrow \quad index(\mathfrak{v}, 0) = -1$$

Here

$$r = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \\ 0 & 0 & \dots & -1 \end{pmatrix}$$

is the *standard reflection* in $\mathbb{R}^m$.

Definition 6.21. Let $\mathfrak{v}$ be a vectorfield on M and let $\mathfrak{w}$ be a vetor field on N. We say $\mathfrak{v}$ *is related to* $\mathfrak{w}$ *under the smooth map* $f : M \to N$ if

$$\mathfrak{w}(f(x)) = df_x(\mathfrak{v}(x))$$

holds for all $x \in M$.

Lemma 6.22. *Each orientation preserving diffeomorphism f of $\mathbb{R}^m$ is smoothly* *isotopic to* $id_{\mathbb{R}^m}$*, i. e. there exists a smooth map*

$$F : I \times \mathbb{R}^m \to \mathbb{R}^m$$

such that

$$F(0, x) = f(x) \quad and \quad F(1, x) = x$$

for all $x \in \mathbb{R}^m$*, and for all* $t \in I$ *the map*

$$\mathbb{R}^m \ni t \mapsto F(t, x)$$

is a diffeomorphism.

Proof. Without restriction assume that $f(0) = 0$. Note that

$$df_0(x) = \lim_{t \to 0} \frac{f(tx)}{t}.$$

Let

$$G : I \times \mathbb{R}^m \to \mathbb{R}^m$$

be defined by

$$G(t, x) := \frac{f(tx)}{t} \quad \text{for} \quad 0 < t \leq 1$$

and

$$G(0, x) = df_0(x).$$

Then G defines an isotopy between f and the linear isomorphism df_0. Note that G is smooth also at $t = 0$ because by 3.19

$$f(x) = x_1 g_1(x) + \ldots + x_m g_m(x)$$

for suitable smooth functions g_i, and thus

$$F(t, x) = x_1 g_1(tx) + \ldots + x_m g_m(tx).$$

But the map $x \mapsto df_0(x)$ is smoothly isotopic into $id_{\mathbb{R}^m}$. To prove this use that the set

$$GL_+(m) := \{A \in GL(m) | det(A) > 0\}$$

is path connected (Exercise). It can be shown that isotopy is an equivalence relation (use suitable smoothing functions). This implies the claim. ■

Lemma 6.23. *Let* $U \subset \mathbb{R}^m$ *be an open set. Let* $f : U \to U'$ *be a diffeomorphism. Let*

$$\mathfrak{v}^{\mathfrak{f}} := df \circ \mathfrak{v} \circ f^{-1}$$

for a vector field $\mathfrak{v}$ *on* U*. Then for each isolated zero* z *of* $\mathfrak{v}$ *we have an isolated zero* $f(z)$ *of* $\mathfrak{v}^f$ *with*

$$index(\mathfrak{v}^f, f(z)) = index(\mathfrak{v}, z).$$

Proof.

Now let $\mathfrak{v} : M \to \mathbb{R}^k$ be a vector field for $M \subset \mathbb{R}^k$ a smooth oriented manifold. Let $\varphi : U \to M$ be an oriented parametrization of M at $x_0 \in M$ with $\varphi(0) = x_0$. Let x_0 be an isolated zero of f in $Int(M)$. Then define

$$index(\mathfrak{v}, x_0) := index(d\varphi^{-1} \circ f \circ \varphi, 0).$$

Because of 6.23. this does not depend on the choice of parametrization. Then for an arbitrary vectorfield $\mathfrak{v}$ on some oriented manifold M with only finitely many zeroes, which are all isolated, let

$$index(\mathfrak{v}) := \sum_{x_0 \text{ zero of } \mathfrak{v}} index(\mathfrak{v}, x_0)$$

Definition. Let $\mathfrak{v} : U \to \mathbb{R}^m$ be a vector field on some open set $U \subset \mathbb{R}^m$ with zero z. Then $\mathfrak{v}$ is called *nondegenerate* in $z \in U$ if $d\mathfrak{v}_z$ is an isomorphism.

Lemma 6.24. *Suppose* z *is a nondegnerate zero of* $\mathfrak{v}$*. Then*

$$index(\mathfrak{v}, z) = \pm 1.$$

Proof. We can assume that $z = 0$. Since $d\mathfrak{v}_z$ is an isomorphism it follows that $\mathfrak{v}$ is a diffeomorphism in a neighborhood U_0 of 0. Using 6.22. it can be shown that $\mathfrak{v}|U_0$ can be isotoped into the identity map or a reflection map. ■

Lemma 6.25. *Let* $\mathfrak{w} : M \to \mathbb{R}^k$ *be a vector field,* $M \subset \mathbb{R}^k$*. Then* $d\mathfrak{w}_z : TM_z \to \mathbb{R}^k$ *is a linear map with* $d\mathfrak{w}_z(TM_z) \subset TM_z$*. Considered as a linear map of* TM_z *the following holds: If the determinant* $D \neq 0$ *then* $ind(\mathfrak{w}, z) = 1$ *respectively* -1 *if* $D > 0$ *respectively* $D < 0$*.*

Here is an important example of vector fields:

Let $f : M \to \mathbb{R}$ be a smooth map. Then $df_x : TM_x \to \mathbb{R}$ thus $df_x \in TM_x^*$. (Here for each real vector space V we let V^* denote the *dual* vector space of all linear maps from V to $\mathbb{R}$.) Let $\mathfrak{v}(f) : M \to \mathbb{R}^k$ be the uniquely determined vector field defined by

$$df_x(a) =< \mathfrak{v}(f)(x), a > \in \mathbb{R}$$

for all $a \in TM_x$. The notation is

$$\mathfrak{v}(f) =: grad(f) : M \to \mathbb{R}^k$$

(defined with respect to the usual inner product $< , >$ of $\mathbb{R}^k$ where $M \subset \mathbb{R}^k$). Let z be a (nondegenerate) critical point of f. Then z is a (nondengenerate) zero of $grad(f)$. Moreover:

Lemma 6.28. *For each nondegenerate critical point z of f of index p we have*

$$index(grad(f), z) = (-1)^p.$$

Proof. Let $\varphi : U \to M$ be an oriented parametrization of M at z with $\varphi(0) = z$, $U \subset \mathbb{R}^m$ open and convex, and for $u = (u_1, \dots, u_m)$ in U we have:

$$f \circ \varphi(u) = f(z) - u_1^2 - \dots - u_p^2 + u_{p+1}^2 + \dots u_m^2.$$

Then

$$grad(f \circ \varphi)(u) = 2(-u_1, \dots, -u_p, u_{p+1}, \dots, u_m)^t.$$

It follows that

$$index(grad(f \circ \varphi, 0) = (-1)^p.$$

Note that

$$grad(f \circ \varphi)(u) = d(f \circ \varphi)_u^t$$

is determined by $< , >$ on $\mathbb{R}^m$. By definition

$$index(grad(f), z) = index(d\varphi^{-1} \circ grad(f \circ \varphi), 0),$$

where $grad(f)$ is defined by

$$df_{\varphi(u)}(d\varphi_u(a)) =< grad(f)(\varphi(u)), d\varphi_u(a) >=< d\varphi_u^t grad(f)(\varphi(u)), a >$$

for all $a \in \mathbb{R}^m$. It follows that $grad(f \circ \varphi) = d\varphi^t \circ grad(f) \circ \varphi$ differs from $d\varphi^{-1} \circ grad(f) \circ \varphi$ precisely by

$$\alpha : d\varphi^t \circ d\varphi : U \to GL_+(m).$$

But α is homotopic in $GL_+(m)$ into the constant map onto $id_{\mathbb{R}^m}$. This induces a homotpy of vector fields, which does not change the index. ∎

Lemma 6.29. *Let $U \subset \mathbb{R}^m$ be open and $\mathfrak{v} : U \to \mathbb{R}^m$ be a vector field with a nondegenerate zero z. Let $\mathfrak{s}_{\mathfrak{v}} : U \to U \times \mathbb{R}^m$ be the corresponding section of the* *trivial bundle. Then $\mathfrak{s}_{\mathfrak{v}}$ intersects the zero section $U \times 0$ transversally at z, and the intersection nuymber is*

$$(-1)^m index(\mathfrak{v}, z).$$

Proof. The intersection number of $\mathfrak{s}_{\mathfrak{v}} : U \to U \times \mathbb{R}^m$, $x \mapsto (x, \mathfrak{v}(x))$ with the zero section is given by the direct sum orientation:

$$d(\mathfrak{s}_{\mathfrak{v}})_z(\mathbb{R}^m \oplus T(U \times 0)_{(z,0)} = T(U \times \mathbb{R}^m)_{(z,0)},$$

where we identify $T(U \times \mathbb{R}^m)_{(z,0)}$ with $\mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$ and $T(U \times 0)_{(z,0)}$ with $\mathbb{R}^m \times 0$. The sum is direct because

$$(*) \qquad d(\mathfrak{s}_{\mathfrak{v}})_z(a) = (a, d\mathfrak{v}_z(a))$$

and $d\mathfrak{v}_z$ is an isomorphism. We know that $d\mathfrak{v}_z(e_1, \ldots, e_m)$ is positively respectively negatively oriented if $ind(\mathfrak{v}, z) = +1$ respectively -1. The orientation of $\mathbb{R}^{2m}$ given by $(*)$ is calculated from the following sequence of base changes:

$$\begin{array}{c} ((e_1, d\mathfrak{v}_z(e_1)), \ldots, (e_m, d\mathfrak{v}_z(e_m)), (e_1, 0), \ldots, (e_m, 0)) \\ \big\downarrow \text{positive determinant} \\ ((0, d\mathfrak{v}_z(e_1)), \ldots, (0, d\mathfrak{v}_z(e_m)), (e_1, 0), \ldots, (e_m, 0)) \\ \big\downarrow \text{positive determinant} \\ ((0, e_1), \ldots, (0, e_m)), (e_1, 0), \ldots, (e_m, 0)) \\ \big\downarrow (-1)^m \\ ((e_1, 0), \ldots, (e_m, 0)), (0, e_1), \ldots, (0, e_m)) \end{array}$$

Theorem 6.30. *Let M be a closed manifold and $\mathfrak{v} : M \to \mathbb{R}^k$ be a vector field on M with only nondegnerate zeroes $x_1, \ldots, x_r$. Then*

$$\chi(M) = \sum_{1 \leq i \leq r} index(\mathfrak{v}, x_i).$$

Proof. $\mathfrak{v}$ induces the section $\mathfrak{s}_{\mathfrak{v}} : M \to TM$. Intersection numbers and indices can be calulated using an oriented coordinate system. Let $U \subset \mathbb{R}^m$ be open and let z be the only zero of $\mathfrak{v}$ in U. Let $\tau : TM|U \overset{\cong}{U} \times \mathbb{R}^m$ be a local oriented trivialization. Consider the diagram

$$\begin{array}{ccc} TM|U & \xrightarrow[\cong]{\tau} & U \times \mathbb{R}^m \\ {\scriptstyle \mathfrak{s}_{\mathfrak{v}}|U}\uparrow & & \uparrow{\scriptstyle \mathfrak{s}} \\ U & = & U \end{array}$$

We have

$$I(\mathfrak{s}_{\mathfrak{v}}|U, U \times 0) = I(\mathfrak{s}, U \times 0).$$

But

$$\mathfrak{s}(x) = (x, d\varphi_x^{-1}(\mathfrak{v}(x)))$$

is the corresponding section over U. Let $\varphi : U' \to U$ be a parametrization with respect to $ind(\mathfrak{v}, z)$ is calulated as $ind(d\varphi^{-1} \circ \mathfrak{v} \circ \varphi, 0)$ The section of $U' \times \mathbb{R}^m$ corresponding to $d\varphi^{-1} \circ \mathfrak{v} \circ \varphi$ is precisely the section $\mathfrak{s}'$ in the following diagram:

$$\begin{array}{ccc} U \times \mathbb{R}^m & \xrightarrow{\varphi \times id} & U' \times \mathbb{R}^m \\ {\scriptstyle \mathfrak{s}}\uparrow & & \uparrow{\scriptstyle \mathfrak{s}'} \\ U & \xleftarrow[\varphi \text{ oriented}]{\cong} & U' \end{array}$$

with

$$\mathfrak{s}'(u') = (u', \mathfrak{s}(\varphi(u'))$$

and it follows that

$$I(\mathfrak{s}', U' \times 0) = I(\mathfrak{s}, U \times 0\}$$

Corollary 6.31. *Let M be a closed oriented manifold and $f : M \to \mathbb{R}$ be a Morse function with nondegnerate critical points $x_1, \ldots, x_r$ of index $p_1, \ldots, p_r$. Then*

$$\chi(M) = \sum_{1 \leq i \leq r} (-1)^{p_i}.$$

Proof. This is imediate from 6.30 and 6.28. ■

Example. $\chi(S^m) = 1 + (-1)^m$. This is easily seen by studing the restriction of the projection $(x_1, \ldots, x_{m+1}) \to x_{m+1}$ to S^m. In fact this map has precisely two crtitical points, a minimum of index 0 and a maximum of index m. In particular it follows from 6.19 that there does not exist a nonvanishing vector field on S^m for m even. This is usually called the *hairy ball theorem*.

Theorem 6.32 (Poincare-Hopf). *Let $\mathfrak{v}$ be a vector field on a closed oriented manifold M with only isolated zeroes. Then*

$$\chi(M) = \sum_{x \text{ zero of } \mathfrak{v}} index(\mathfrak{v}, x).$$

Proof. It suffices to show that we can replace $\mathfrak{v}$ by a vector field with only nondegenerate zeroes. Thus let $U \subset \mathbb{R}^m$ be open with a single zero at z. Let

$$\lambda : U \to [0,1]$$

be smooth with $\lambda|U_1 \equiv 1$ in a neighborhood U_1 of z and $\lambda|U \setminus V \equiv 0$ where $U_1 \subset V$. Then for y a regular value of $\mathfrak{v}$ with $||y||$ sufficiently small let

$$\mathfrak{w}(x) := \mathfrak{v}(x) - \lambda(x)y.$$

Then $\mathfrak{w}$ has only nondegnerate zeroes on U_1. In fact, $\mathfrak{w}(x) = 0$ for $x \in U_1$ implies $\mathfrak{v}(x) = y$, which implies that $d\mathfrak{v}_x$ is an isomorphism. For $||y||$ sufficiently small there will be no zeroes in $V \setminus U_1$ ($\overline{V} \setminus U_1$ is compact). Thus the sum of the indices in V is equal to the degree of the Gauss map $\partial V \to S^{m-1}$ thus will not be changed under a deformation of $\mathfrak{v}$ into $\mathfrak{w}$. ■