ON THE RELATION GAP AND RELATION LIFTING PROBLEM

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Abstract

This article surveys results in connection with the relation gap problem, the relation lifting problem, and the geometric realization problem. These three problems lie in the intersection of combinatorial group theory and 2-dimensional homotopy theory. We present key examples that lie at the heart of these problems.

1 Three problems

Let G be a group. A generating set $\mathcal{G} = \{g_1, ..., g_n\}$ defines an epimorphism

$$\phi_{\mathcal{G}} : F(x_1, ..., x_n) \to G,$$

that sends x_i to g_i, $i = 1, ..., n$. Let $N_{\mathcal{G}}$ be the kernel of $\phi_{\mathcal{G}}$. A fundamental problem in combinatorial group theory is to determine a minimal normal generating set for $N_{\mathcal{G}}$. If it is clear which generating set is used we will drop the subscript and simply write N instead of $N_{\mathcal{G}}$. When we say

$$F/N = \langle x_1, ..., x_n \mid r_1 = 1, ..., r_m = 1\rangle$$

is a presentation for a group G, we mean that F is a free group on $\{x_1, ..., x_n\}$, N is the normal closure of $r_1, ..., r_m$ in F, and G is isomorphic to F/N. The conjugation action of F on N provides a $\mathbb{Z}G$-module structure on $N_{ab} = N/[N, N]$. This module is the relation module of the presentation F/N of G. We have

$$d_F(N) \geq d_G(N/[N, N]) \geq d(N/[F, N]) = n - tfr(H_1(G)) + d(H_2(G)).$$

Here $d_F(-)$ denotes the minimal number of F-group generators, $d_G(-)$ denotes the minimal number of G-module generators, $d(-)$ denotes the minimal number of generators, and $tfr(-)$ denotes the torsion free rank. The chain of inequalities follows from the exact sequence

$$H_2(G) \to H_2(Q) \to \mathbb{Z} \otimes_{\mathbb{Z}G} H_1(H) \to H_1(G) \to H_1(Q) \to 0,$$

associated with an exact sequence of groups $1 \to H \to G \to Q \to 1$ (see Brown [5]). We say the presentation F/N has a relation gap if

$$d_F(N) - d_G(N/[N, N]) > 0.$$

The presentation is called efficient if $d_F(N) = d(N/[F,N])$.

Relation gap problem: Does there exist a finite presentation F/N with a relation gap?

Infinite relation gaps are known to exists for finitely generated groups. See Bestvina and Brady [2].

Relation lifting problem: Given set $s_1[N,N], ..., s_m[N,N]$ of relation module generators, do there exists defining relators $r_1, ..., r_m$ (i.e. elements that normally generate N), such that $r_i[N,N] = s_i[N,N]$, $i = 1, ...m$?

We refer to the elements r_i as lifts of the generators $s_i[N,N]$, $i = 1, ..., m$. The relation lifting problem arose in work of C. T. C. Wall [31]. M. Dunwoody [8] provided an example where lifting is not possible. We will provide more details on Dunwoody's construction in a later section.

Let $\mathcal{G}$ be a finite generating set of a group G and $F/N_{\mathcal{G}}$ be the associated presentation. Since the relation module $N_{\mathcal{G}_{ab}}$ is isomorphic to $H_1(\Gamma_{\mathcal{G}})$, where $\Gamma_{\mathcal{G}}$ is the Cayley graph of G associated with the generating set $\mathcal{G}$, a choice of relation module generators $\mathcal{M}$ gives rise to an partial resolution $\mathcal{K}_{\mathcal{M}}$ of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$

$$\mathbb{Z}G^m \stackrel{\partial_2}{\to} \mathbb{Z}G^n \stackrel{\partial_1}{\to} \mathbb{Z}G \stackrel{\epsilon}{\to} \mathbb{Z} \to 0,$$

where m is the number of elements in $\mathcal{M}$. We call such a partial resolution $\mathcal{K}_{\mathcal{M}}$ an algebraic 2-complex for G. We say an algebraic 2-complex $\mathcal{K}$ for G is geometrically realizable if it is chain homotopy equivalent to an algebraic 2-complex $\mathcal{C}(\tilde{K})$ that arises as the augmented chain complex of the universal covering $\tilde{K}$ of a 2-complex K with fundamental group G.

Geometric realization problem: Does there exist an algebraic 2-complex that is not geometrically realizable?

A relation gap in a finite presentation F/N would certainly provide a set of relation module generators that can not be lifted. If in addition $d_F(N) - d(F)$ is minimal among all finite presentations of G one can construct an algebraic 2-complex that is not realizable. Indeed, if $\mathcal{M}$ is a minimal generating set for the relation module $N/[N,N]$, then the Euler characteristic of $\mathcal{K}_{\mathcal{M}}$ is smaller that the Euler characteristic of any finite 2-complex with fundamental group G. Geometric realization has been studied by F. E. A. Johnson and his students in connection with the $D(2)$-problem. See [25].

2 Examples concerning the relation gap question

Example 1: Consider the presentation

$$F/N = \langle a, b, c, d \mid [a, b] = 1, a^m = 1, [c, d] = 1, c^n = 1 \rangle$$

of the group $G = (\mathbb{Z}_m \oplus \mathbb{Z}) \star (\mathbb{Z}_n \oplus \mathbb{Z})$, where m and n are relatively prime. Note that $d(N/[F, N]) = 4 - 2 + 1 = 3$. Epstein asked [10] if this presentation is efficient. Gruenberg and Linnell showed [12] that $d_G(N/[N, N]) = 3$.

Example 2: The following examples were constructed by Bridson and Tweedale [3]. They are very much in the spirit of the Epstein example above, but an unexpectedly small set of relation module generators can be seen more easily. Let Q_n be the group defined by $\langle a, b, x \mid a^n = 1, b^n = 1, [a, b] = 1, xax^{-1} = b \rangle$. This groups is an HNN-extension of $\mathbb{Z}_n \times \mathbb{Z}_n$ where the stable letter x conjugates one factor into the other. Note that $\langle a, x \mid a^n = 1, [a, xax^{-1}] = 1 \rangle$ also presents Q_n. Let $\rho_n(a, x) = [xax^{-1}, a]a^{-n}$ and let $q_n = (n + 1)^n - 1$. Then

$$F/N = \langle a, x, b, y \mid a^m = 1, [a, xax^{-1}] = 1, b^n = 1, [b, yby^{-1}] = 1 \rangle$$

is a presentation of the free product $Q_m \star Q_n$. Bridson and Tweedale show that the relation module N_{ab} is generated by $\rho_m(a, x)[N, N]$, $\rho_n(b, y)[N, N]$, and $a^m b^{-n}[N, N]$.

Other articles by Bridson and Tweedale that address the relation gap are [4] and [13].

Example 3: The following construction first appeared in [16]. Let F_1/N_1 and F_2/N_2 be finite presentations of groups G_1 and G_2, respectively. Let H be a finitely generated subgroup of both G_1 and G_2 and let F/N be the standard presentation of the amalgamated product $G = G_1 \star_H G_2$. One can show that

$$d_G(N_{ab}) \leq d_{G_1}(N_{1ab}) + d_{G_2}(N_{2ab}) + d_H(IH),$$

where IH is the augmentation ideal of H. Denote by H^n the n-fold direct product $H \times ... \times H$. Cossey, Gruenberg, and Kovacs [7] showed that $d_{H^n}(IH^n) = d_H(IH)$ in case H is a finite perfect group. Since $d(H^n) \to \infty$ as $n \to \infty$ one can produce arbitrarily large generation gaps $d(H^n) - d_{H^n}(IH^n)$. This leads to unexpectedly small generating sets for the relation module N_{ab} for for presentations F/N of $G_1 \star_{H^n} G_1$. The hope is that the amalgamated product shifts the generation gap into a relation gap.

3 Dunwoody's counter example to relation lifting [8]

Consider the presentation $F/N = \langle a, b \mid a^5 = 1 \rangle$ of the group $G = \mathbb{Z}_5 \star \mathbb{Z}$. Note that $(1 - a + a^2)(a + a^2 - a^4) = 1$, so $1 - a + a^2$ is a non-trivial unit in $\mathbb{Z}G$. Hence so is

$\alpha = (1 - a + a^2)b$. It follows that

$$\alpha \cdot a^5[N, N] = (ba^5b^{-1})(aba^{-5}b^{-1}a^{-1})(a^2ba^5b^{-1}a^{-2})[N, N] = s[N, N]$$

is a generator for the relation module. This generator can not be lifted. For suppose that $\langle\langle a^5 \rangle\rangle = \langle\langle r \rangle\rangle$ and $r[N, N] = s[N, N]$. One relator group theory implies that $r = wa^{\pm 5}w^{-1}$, for some $w \in F$. Using the well known resolution for $\mathbb{Z}_5$ (see Brown [5], Chapter I, Section 6) one can show that $N_{ab} \cong \mathbb{Z}G/\mathbb{Z}G\langle a - 1 \rangle \cong A$, where A is the free abelian group with basis the elements of G that end in $b^{\pm 1}$ (this can also be seen by directly inspecting the Cayley graph Γ for the generating set $\{a, b\}$.) The isomorphism $N_{ab} \to A$ sends $s[N, N]$ to $b - ab + a^2b$ and sends $r[N, N]$ to a single basis element in A. Hence $r[N, N] \neq s[N, N]$.

Dunwoody's construction uses non-trivial units in $\mathbb{Z}G$ and one relator group theory. We do not know an example of a torsion free group where relation lifting fails. Also note that the algebraic 2-complex $\mathcal{K}$ that one obtains from the relation module generator $s[N, N]$ is geometrically realizable. In fact it is chain homotopically equivalent to the cellular chain complex of the universal covering of the standard 2-complex K built from the presentation $F/N = \langle a, b \mid a^5 = 1 \rangle$ of G. Thus the example does not provide a negative answer to the geometric realization question.

It is easy to give examples of relation module generators that are not defining relators but can be lifted. Let

$$F/J = \langle a, b, c, d, x \mid a = 1, b = 1, c = 1, d = 1 \rangle$$

be a presentation of the infinite cyclic group. Let $s_1 = b[b, a]$, $s_2 = c[c, b]$, $s_3 = d[d, c]$, $s_4 = a[a, b]$. The elements $s_i[J, J]$, $i = 1, 2, 3, 4$, generate the relation module J_{ab}, but the s_i, $i = 1, 2, 3, 4$, do not normally generate J. Indeed, notice that $s_1 = 1$ can be rewritten as $aba^{-1} = b^2$. Rewriting the other relations $s_i = 1$ in a similar way the presentation $F/N = \langle a, b, c, d, x \mid s_1 = 1, s_2 = 1, s_3 = 1, s_4 = 1 \rangle$ turns into the presentation

$$F/N = \langle a, b, c, d, x \mid aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \rangle,$$

which presents the free product $H \star \mathbb{Z}$, where $H = \langle a, b, c, d \rangle$ is the Higman group on four generators. Higman groups H_n on $n \geq 2$ generators are defined similarly [19]. It is known that H_n is trivial for $n = 2, 3$ and infinite (in fact aspherical, see Gersten [11]) for $n \geq 4$. All H_n are perfect and do not have proper subgroups of finite index. In particular F/N does not present the infinite cyclic group. Since $s_1[J, J] = b[J, J]$, $s_2[J, J] = c[J, J]$, $s_3[J, J] = d[J, J]$, $s_4[J, J] = a[J, J]$, the relation module generators $s_i[J, J]$, $i = 1, 2, 3, 4$ can be lifted.

The following observation is often useful when constructing examples of presentations that are interesting in view of relation lifting.

Lemma 3.1 *Let $F/N = \langle x_1, ..., x_n \mid s_1, ..., s_m \rangle$ be a presentation of a group G and suppose $P = J/N$ is a perfect normal subgroup. Let Q be the quotient G/P. Then F/J is a presentation for Q and the relation module J_{ab} is generated by $s_1[J, J], ..., s_m[J, J]$.*

Proof. Since $P_{ab} = J/N[J, J] = 1$ we have $J = N[J, J]$. The result follows.

4 Stabilization

All examples encountered so far are free products, except for Example 3 in Section 2, where the group is a free product amalgamated over a finite group. Free products can have unexpectedly small presentations. See Hog-Angeloni, Metzler, Lustig [20]. In his work on distinguishing homotopy and simply homotopy type for 2-complexes, Metzler [27] showed that Whitehead torsion elements can be topologically realized if one allows stabilization using copies of the complex $K = \langle a, b \mid [a, b], a^2, b^4 \rangle$. See also Hog-Angeloni, Metzler [22] and [24], Chapter XII. These stabilization techniques can also be applied to closing the relation gap. See [14], [15], and also [16].

Theorem 4.1 *Given a finite presentation F/N. Then there exists $k \geq 0$ such that*

$$F/N \star \langle a, b \mid a^2 = 1, b^2 = 1, [a, b] = 1 \rangle \star ... \star \langle a, b \mid a^2 = 1, b^2 = 1, [a, b] = 1 \rangle$$

(k copies) does not have a relation gap.

Here is the main idea of the proof. Suppose the relation module is generated by $r_1[N, N], ..., r_m[N, N]$. Then N is normally generated by $r_1, ..., r_m$ together with a finite set of elements of the form $[s, t]$, where $s, t \in N$. For simplicity assume that $N = \langle\langle r_1, ..., r_m, [s, t] \rangle\rangle$. Now note that

$$\langle x_1, ..., x_n, a, b \mid r_1 = 1, ..., r_m = 1, [s, t] = 1, a^2 = 1, b^2 = 1, [a, b] = 1 \rangle$$

presents the same groups as

$$\langle x_1, ..., x_n, a, b \mid r_1 = 1, ..., r_m = 1, s = a^2, t = b^2, [a, b] = 1 \rangle.$$

Indeed, since a and b commute, the squares a^2 and b^2 also commute, hence s and t commute. So the relation $[s, t] = 1$ holds. Since $N = \langle\langle r_1, ..., r_m, [s, t] \rangle\rangle$, we get $s = 1$ and $t = 1$, and hence $a^2 = 1$ and $b^2 = 1$. The commutator relation $[s, t] = 1$ got "absorbed", the second presentation makes due with one less relator than the first presentation.

We will illustrate this method further by working through Dunwoody's example considered in Section 3. For economical reasons we use the notation $x * y = xyx^{-1}$. Let $r = a^5$ and let $s = (b*r)(ab*r^{-1})(a^2b*r) \in N = \langle\langle r \rangle\rangle$. We know that $\langle a, b \mid s = 1 \rangle$ is not a presentation for $G = \mathbb{Z}_5 * \mathbb{Z}$. But since s generates the relation module, we can add commutators of relators to obtain a presentation. We need to add enough commutators to be able to do the calculation $(a+a^2-a^4)\cdot s[N,N] = b\cdot r[N,N]$(based on $(1-a+a^2)(a+a^2-a^4)b = b$) in N. We claim that

$$\langle a, b \mid s = 1, [r, b*r] = 1, [b*r, ab*r] = 1, [b*r, a^2b*r] = 1\rangle$$

does present G. Note that $(a*s)(a^2*s)(a^4*s^{-1})$ is

$$(ab*r)(a^2b*r^{-1})(a^3b*r)(a^2b*r)(a^3b*r^{-1})(a^4b*r)(a^6b*r^{-1})(a^5b*r)(a^4b*r^{-1}).$$

Since $b*r$ commutes with $ab*r$, it follows that a^ib*r commutes with $a^{i+1}b*r$. And since $b*r$ commutes with a^2b*r it follows that a^ib*r commutes with $a^{i+2}b*r$. Thus the above expression becomes

$$(ab*r)(a^6b*r^{-1})(a^5b*r).$$

Now note that $a^5b*r = r*(b*r) = b*r$ because r commutes with $b*r$. It follows that $a^6b*r = a*(a^5*r) = ab*r$. Thus our expresion becomes

$$(ab*r)(ab*r^{-1})(b*r) = b*r.$$

Thus $b*r$ and hence r defines the trivial element in the group defined by the presentation above. Hence we have a presentation of $G = \mathbb{Z}_5 * \mathbb{Z}$ as claimed. It follows that the presentation of generators

$$a, b, c, d, e, f, g, h$$

and relations

$$s = 1, c^2 = r, d^2 = b*r, e^2 = b*r, f^2 = ab*r, g^2 = b*r, h^2 = a^2b*r, [c,d] = [e,f] = [g,h] = 1$$

is a presentation for $(Z_5 \star \mathbb{Z}) \star (\mathbb{Z}_2 \times \mathbb{Z}_2) \star (\mathbb{Z}_2 \times \mathbb{Z}_2) \star (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

5 One relator groups and another example of Dunwoody's

Let $F/N = \langle x_1, ..., x_n \mid r\rangle$ be a presentation of a torsion-free one-relator group G. Then the relation module N_{ab} is isomorphic to $\mathbb{Z}G$. Let α and β be left module generators of $\mathbb{Z}G$. Let $\partial_{\alpha,\beta} : \mathbb{Z}G \oplus \mathbb{Z}G \to N_{ab}$ be the homomorphism that sends $e_1 = (1,0)$ to $\alpha \cdot r[N,N]$ and $e_2 = (0,1)$ to $\beta \cdot r[N,N]$. Since the relation module N_{ab} is isomorphic to the kernel of $\partial_1 : \mathbb{Z}G^n \to \mathbb{Z}G$ that sends the basis element e_i to $x_i - 1$, $i = 1, ..., n$, we obtain an algebraic 2-complex $\mathcal{K}_{\alpha,\beta}$

$$\mathbb{Z}G \oplus \mathbb{Z}G \stackrel{\partial_{\alpha,\beta}}{\to} \mathbb{Z}G \oplus \mathbb{Z}G \stackrel{\partial_1}{\to} \mathbb{Z}G \stackrel{\epsilon}{\to} \mathbb{Z} \to 0.$$

This construction provides easy access to examples relevant for relation lifting and geometric realization.

Example (M. Dunwoody [9]): Let G be the trefoil group presented by $\langle a, b \mid a^2 = b^3 \rangle$. Then $\alpha = 1 + a + a^2$ and $\beta = 1 + b + b^2 + b^3$ generate the left module $\mathbb{Z}G$. In order to see this observe that $(a-1)\alpha = a^3 - 1$ and $(b-1)\beta = b^4 - 1$. Since a^3 and b^4 generate G (simply note that $(a^3)^3(b^4)^{-3} = a$ and $(a^3)^2(b^4)^{-2} = b$), the elements $(a-1)\alpha$ and $(b-1)\beta$ generate the augmentation ideal. Thus $(a-1)\alpha$, $(b-1)\beta$, $\beta - \alpha$, and hence α and β, generate $\mathbb{Z}G$. It follows that $\alpha \cdot r[N,N] = (r)(ara^{-1})(a^2ra^{-2})[N,N]$ and $\beta \cdot r[N,N] = (r)(brb^{-1})(b^2rb^{-2})(b^3rb^{-3})[N,N]$ generate the relation module, where $r = a^2b^{-3}$. One obtains an algebraic 2-complex $\mathcal{K}_{\alpha,\beta}$. Dunwoody shows in [9] that $H_2(\mathcal{K}_{\alpha,\beta})$ is stably-free but not free. In particular $\mathcal{K}_{\alpha,\beta}$ is not chain homotopically equivalent to the chain complex of the universal covering of $\langle a, b \mid a^2b^{-3} \rangle \vee S^2$. The algebraic 2-complex $\mathcal{K}_{\alpha,\beta}$ is geometrically realizable. Dunwoody shows that

$$\langle a, b \mid (r)(ara^{-1})(a^2ra^{-2}) = 1, (r)(brb^{-1})(b^2rb^{-2})(b^3rb^{-3}) = 1 \rangle$$

is indeed a presentation of G. This provided the first example of different homotopy types of 2-complexes K and L with the same fundamental group G and Euler characteristic $\chi(K) = \chi(L) = \chi_{min}(G) + 1$. For finite groups different homotopy types can occur only at the minimal Euler characteristic level. See [24], Chapter III. Other examples similar to Dunwoody's are known [17]. Later M. Lustig [23] showed that there are infinitely many distinct homotopy types for G on the Euler characteristic $\chi_{min}(G) + 1$.

6 Algebraic 2-complexes for the Klein bottle group

The homotopy classification of 2-complexes with fundamental group G is complete in case G is free of rank n, or G is free abelian of rank 2. In the first case $(S^1 \vee ... \vee S^1) \vee S^2 \vee ... \vee S^2$ (n copies of S^1) is a complete list, and in the second case $(S^1 \times S^1) \vee S^2 ... \vee S^2$ is a complete list. This follows from the fact that the homotopy type of a 2-complexe K is assembled from $\pi_1(K)$, $\pi_2(K)$, and the k-invariant $\kappa \in H^3(\pi_1(K), \pi_2(K))$. See Chapter II in [24]. For G free or $G = \mathbb{Z} \times \mathbb{Z}$, the cohomology group $H^3(G, M) = 0$ for all $\mathbb{Z}G$-modules M. The second homotopy module $\pi_2(K)$ is stably free since the cohomological dimension of G is less or equal to two, and hence free by results of Cohn [6] for free groups (see also Hog-Angeloni [21] for a short topological proof) and Quillen [29] (independently, Suslin) for free abelian groups. It follows that the homotopy type is determined by the Euler characteristic.

The situation is more complicated for the Klein bottle group G. Let $F/N = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ be the standard presentation of G. Then N_{ab} is isomorphic to $\mathbb{Z}G$. Let $p(b) \in \mathbb{Z}G$ be a polynomial in b and let $q(b) = p(b^{-1})$. The elements $\alpha = a + q(b)$ and $\beta = p(b)$ generate $\mathbb{Z}G$ as a left module. Indeed,

$$\begin{aligned} (a - p(b))\alpha + p(b^{-1})\beta = (a - p(b))(a + p(b^{-1}) + p(b^{-1})p(b) &= \\ a^2 + ap(b^{-1}) - p(b)a - p(b)p(b^{-1}) + p(b^{-1})p(b) &= a^2 \end{aligned}$$

Artamonov [1] and Stafford [30] studied the K-theory of solvable groups. Their results can be used to construct exotic algebraic 2-complexes for G.

Theorem 6.1 *Let $p_n(b) = 1 + nb + nb^3$ and $q_n(b) = p(b^{-1})$, $n \in \mathbb{N}$. Then the elements $\alpha_n = a + q_n(b)$ and $\beta_n = p_n(b)$ generate $\mathbb{Z}G$ as a left module and the set $\{\mathcal{K}_{\alpha_n \cdot \beta_n}\}_{n\in\mathbb{N}}$ contains infinitely many distinct homotopy types of algebraic 2-complexes for G, each of Euler characteristic one.*

The algebraic 2-complexes $\mathcal{K}_{\alpha_n,\beta_n}$ are studied in [18]. We do not know if the relation module generators

$$\alpha_n \cdot r[N,N] = (ara^{-1})(r)(b^{-1}r^n b)(b^{-3}r^n b^3)[N,N],$$

$$\beta_n \cdot r[N,N] = (r)(br^n b^{-1})(b^3 r^n b^{-3})[N,N],$$

where $r = aba^{-1}b$, can be lifted, or if any of the complexes $\mathcal{K}_{\alpha_n,\beta_n}$ are geometrically realizable.

We conclude this article by providing some details on the work of Artamonov [1] and Stafford [30]. Given a Noetherian domain R and an automorphism $\sigma : R \to R$, one can defined the skewed Laurent-polynomial ring $S = R[x, x^{-1}, \sigma]$, where $xr = \sigma(r)x$. Stafford shows that given two elements r_1, r_2 that satisfy the properties

1. $S = Sr_1 + S(x + r_2)$,
2. $\sigma(r_1)r_2 \notin Rr_1$,

then the left ideal $K = \{s \in S \mid sr_1 \in S(x+r_2)\}$ is not generated by a single element. Note that K is isomorphic to the kernel of the S-module homomorphism $S \oplus S \to S$, sending e_1 to r_1 and e_2 to $x + r_2$. Hence $K \oplus S \cong S \oplus S$. Since K is not cyclic, it is not free.

If G is the Klein bottle group as above, then $\mathbb{Z}G = \mathbb{Z}H[a, a^{-1}, \sigma]$, where $H = \langle b \rangle$. Now one can take $r_1 = 1 + nb + nb^3$ and $r_2 = 1 + nb^{-1} + nb^{-3}$. By Stafford K_n is not free. Note that K_n is isomorphic to $H_2(\mathcal{K}_{\alpha_n,\beta_n})$, where $\mathcal{K}_{\alpha_n,\beta_n}$ is the algebraic 2-complex defined above. Artamonov shows that the set $\{K_n\}$ contains infinitely many distinct isomorphism types. His reasoning is as follows. First construct a set of primes Q such that if $p < q$ and both p and q are in Q, then $q = 1$ modulo p. Let $K_{n,p} = K_n/pK_n$. Using Stafford's construction one can show that $K_{q,p}$ is not free, but $K_{q,q}$ is free. Thus if $q_1 < q_2$, then K_{q_1} and K_{q_2} are not isomorphic because K_{q_1,q_1} is free but K_{q_2,q_1} is not free. It follows that the set $\{H_2(\mathcal{K}_{\alpha_n \cdot \beta_n})\}_{n\in\mathbb{N}}$ contains infinitely many isomorphism types and hence the set $\{\mathcal{K}_{\alpha_n \cdot \beta_n}\}_{n\in\mathbb{N}}$ contains infinitely many algebraic homotopy types.

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