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THE Σ^3-CONJECTURE FOR METABELIAN GROUPS

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Abstract

The Σ^3-conjecture for metabelian groups is proved in the split extension case.

1. *Introduction*

A *metabelian* group G is an extension of abelian groups M and Q (M being the kernel and Q the quotient). G is uniquely given, up to isomorphism, by the abelian group Q, the Q-module M (by conjugation in G), and the cohomology class of the extension in $H^2(Q, M)$. For finitely generated metabelian groups, Bieri and Strebel [**11**] defined a certain subset $\Sigma(M)$ of the sphere S^{n-1}, where n is the torsion-free rank of Q, that was shown to contain complete information about the finite presentability of G. Since then this geometric invariant has undergone quite an evolution. Geometric invariants $\Sigma^m(G)$ can be defined for any group of type F_m and have been studied extensively for various types of group. Recently the theory of geometric invariants has been further generalised by the introduction of invariants associated with groups acting on non-positively curved spaces [**3**–**5**]. Nevertheless, two fundamental conjectures about metabelian groups remain unsettled, the FP_m-conjecture and the Σ^m-conjecture. The FP_m-conjecture states that $\Sigma^1(G)$ contains complete information about the higher finiteness properties of the metabelian group G, and not just finite presentability. The Σ^m-conjecture, loosely formulated, states that the complement of $\Sigma^m(G)$ is obtained from the complement of $\Sigma^1(G)$ by a simple process involving the taking of convex hulls.

This paper is concerned with the Σ^3-conjecture. We show that the Σ^3-conjecture holds in the split case, where G is the split-extension of the Q-module M by Q. The validity of the Σ^2-conjecture in the split case has been established already by the second author in [**17**], and recently the general non-split case has been proved in [**14**].

We also shed some light on the question of how the two conjectures are related. In fact, the method of proof presented here is modeled on the techniques developed in paper [**8**] by Bieri and the first author and the idea of the diameter of cells introduced in [**17**]. The main point of [**8**] was the construction of a classifying $K(M, 1)$-complex for the Q-module M with Q-finite 3-skeleton (under the assumption that M is 3-tame), whereas here we construct a classifying $K(M, 1)$-complex for the Q_χ-module M, where $\chi : Q \longrightarrow \mathbb{R}$ is a non-trivial (that is, non-zero) character and Q_χ is the positive submonoid of Q with respect to χ. In a sense one could say that the Σ^m-conjecture is a monoid version of the FP_m-conjecture.

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In the following we give precise definitions, statements of the conjectures and our results. Suppose that G is a finitely generated group. Then the set

$$S(G) = \{[\chi] = \mathbb{R}_{>0}\chi \mid \chi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\}\}$$

is called the character sphere of G. Note that $\mathrm{Hom}(G, \mathbb{R})$ is an n-dimensional real vector space, where n is the torsion-free rank of the abelianization of G. More precisely there is an isomorphism between $\mathbb{R}^n$ and $\mathrm{Hom}(G, \mathbb{R})$ via

$$u \longmapsto \chi_u = (u, -),$$

where $(,)$ denotes the standard inner product on $\mathbb{R}^n$. This sets up a bijection between the unit sphere S^{n-1} in $\mathbb{R}^n$ and the character sphere of G.

Let M be a (left) $\mathbb{Z}[G]$-module. Then the homological Σ^m-invariant is defined by Bieri and Renz [**10**] to be

$$\Sigma^m(G, M) = \{[\chi] \mid M \text{ is of homological type } FP_m \text{ over } \mathbb{Z}[G_\chi]\} \subseteq S(G),$$

where $G_\chi = \{g \in G \mid \chi(g) \geqslant 0\}$. The invariants $\Sigma^m(G)$ are mentioned in [**10**, Chapter 6] and should be thought of as the homotopical version of the invariants $\Sigma^m(G, \mathbb{Z})$. More details about $\Sigma^m(G)$ can be found in the introduction of [**21**].

If S is a subset of $\mathbb{R}^n$, then we denote by $\mathrm{conv}(S)$ the convex hull of S and by $\mathrm{conv}_{\leqslant m}(S)$ we denote $\bigcup \mathrm{conv}(C)$, where the union is taken over all subsets C of S containing no more than m points.

FP_m-CONJECTURE [**6**]. Suppose that G is a finitely generated metabelian group. Then G is of homological type FP_m if and only if

$$0 \notin \mathrm{conv}_{\leqslant m}(\mathbb{R}_{>0}\Sigma^1(G)^c).$$

Here $\Sigma^m(G)^c$ is the complement of $\Sigma^m(G)$ in $S(G)$ and

$$\mathbb{R}_{>0}\Sigma^m(G)^c = \{\chi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\} \mid [\chi] \in \Sigma^m(G)^c\} \subseteq \mathrm{Hom}(G, \mathbb{R}) \simeq \mathbb{R}^n.$$

Σ^m-CONJECTURE (Bieri). Suppose that G is a metabelian group of homological type FP_m. Then

$$\mathbb{R}_{>0}\Sigma^m(G)^c = \mathbb{R}_{>0}\Sigma^m(G, \mathbb{Z})^c = \mathrm{conv}_{\leqslant m}(\mathbb{R}_{>0}\Sigma^1(G)^c).$$

Let us review the status of these conjectures. The FP_m-conjecture is true for $m = 2$ [**11**], for $m = 3$ in the split case [**8**], for G of finite Prüfer rank [**1**], and for the case when M is torsion and of Krull dimension 1 [**13**, **15**]. The direction ($\Rightarrow$) of the FP_m-conjecture is true in the split case [**15**, **16**] (see also [**22**] for M torsion-free), and in the case when M is torsion [**15**, **16**].

The Σ^m-conjecture is known to hold for $m = 2$ (see [**14**, **17**]), for G of finite Prüfer rank [**20**, **21**], and for the case when M is torsion and of Krull dimension 1 [**18**]. Furthermore, the inclusion

$$\mathrm{conv}_{\leqslant m}(\mathbb{R}_{>0}\Sigma^1(G)^c) \subseteq \mathbb{R}_{>0}\Sigma^m(G)^c$$

has been established [**18**] for metabelian groups for which the extension splits or M is torsion.

In this paper we obtain the Σ^3-conjecture in the split extension case as a corollary of the following.

THEOREM 1.1. *Let G be a split extension of abelian groups M and Q that is of type FP_3. Suppose that χ is a real non-trivial character of G such that $\chi(M) = 0$ and*

$$\chi \notin \mathrm{conv}_{\leqslant 3}(\mathbb{R}_{>0}\Sigma^1(G)^c).$$

Then $[\chi] \in \Sigma^3(G, \mathbb{Z})$.

COROLLARY 1.2. *The Σ^3-conjecture holds true in the split case.*

Let us show how the corollary follows from Theorem 1.1. First we remark that $\Sigma^m(G) = \Sigma^m(G, \mathbb{Z})$ for split-metabelian groups of type F_m. This follows from the validity of the Σ^2-conjecture established in [**17**] for $m = 2$ and for $m \geqslant 3$ from the general Hurewicz type formula $\Sigma^m(G, \mathbb{Z}) \cap \Sigma^2(G) = \Sigma^m(G)$. Thus all we have left to show is

$$\mathbb{R}_{>0}\Sigma^3(G, \mathbb{Z})^c = \mathrm{conv}_{\leqslant 3}(\mathbb{R}_{>0}\Sigma^1(G)^c).$$

Since the direction $\supseteq$ is known from previous work [**18**], we are left with $\subseteq$. Therefore suppose that $[\chi] \notin \mathrm{conv}_{\leqslant 3}(\mathbb{R}_{>0}\Sigma^1(G)^c)$. If $\chi(M) \neq 0$, then it is known [**19**, Theorem C] that $[\chi] \in \Sigma^m(G)$ for all $m \geqslant 1$. If $\chi(M) = 0$, then Theorem 1.1 tells us that $[\chi] \in \Sigma^3(G, \mathbb{Z})$.

2. *Geometric properties of Σ^0 and Σ^1*

The geometric structure of $\Sigma^1(G)$ for general finitely generated groups G is unknown except that it is an open subset of $S(G)$. In fact for every m the geometric invariants $\Sigma^m(G, A)$ and $\Sigma^m(G)$ are open in $S(G)$ [**10**]. In the case when G is metabelian, $\Sigma^1(G)^c$ is a rationally defined spherical polyhedron, that is, a finite union of finite intersections of closed half subspheres, where every subsphere is defined by a rational point in the unit sphere $S(G)$ [**7**]. In particular, the rational points in $\Sigma^1(G)^c$ form a dense subset. It is interesting to note that there are examples of subgroups G of the group of the PL homeomorphisms of the interval $[0, 1]$ for which $\Sigma^1(G)^c$ has only one or two points, neither of which is necessarily rational [**12**, Theorem 8.1].

In the rest of this section we consider the invariant $\Sigma_M(Q) = \Sigma^0(Q, M)$, where Q is a free abelian group of rank n and M is a finitely generated (left) $\mathbb{Z}[Q]$-module. We view Q as the lattice $\mathbb{Z}^n$ in the Euclidean space $\mathbb{R}^n$ equipped with the standard inner product $(\,,)$ and the corresponding norm $|\;|$. $\mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{R})$ is identified with $\mathbb{R}^n$, where $v \in \mathbb{R}^n$ corresponds to the homomorphism sending an element q from Q to (q, v). For an extension G of M by Q, the map $\pi_* : S(Q) \longrightarrow S(G)$ induced by the projection $\pi : G \longrightarrow Q$ gives a bijection between $\Sigma_M^c(Q) = S(Q) \setminus \Sigma_M(Q)$ and $\Sigma^1(G)^c$. Although it is not obvious from the definition, $\Sigma_M(Q)$ does not depend on the whole structure of M as a $\mathbb{Z}[Q]$-module but only on the annihilator of M in $\mathbb{Z}[Q]$ [**11**, Proposition 2.1]. This fact was recently refined in [**9**], where it was shown that a finite subset of the centralizer of M in $\mathbb{Z}[Q]$ determines $\Sigma_M(Q)$.

PROPOSITION 2.1 [**9**]. *Suppose that M is a finitely generated $\mathbb{Z}Q$-module. Then there exists a finite subset Λ of the centraliser of M in $\mathbb{Z}Q$ and some $v > 0$ such that for every $[\mu] \in \Sigma_M(Q)$, there is an element λ in Λ with*

$$\min\{\mu(q) \mid q \in \mathrm{supp}\,\lambda\} > v.$$

COROLLARY 2.2 [**11**, Lemma 1.1]. *Let Λ be the set and v be the positive real number given by Proposition 2.1. Then there is a positive integer $\rho_1(v)$ such that for $x \in \mathbb{R}^n$ with $|x| \geqslant \rho_1(v)$, $x/|x| \in -\Sigma_M(Q)$, there is a $\lambda \in \Lambda$ such that $x + \operatorname{supp} \lambda$ is a subset of the open ball with centre the origin and radius $|x| - v/2$.*

We finish this section with a corollary that follows immediately from the polyhedral structure of $\Sigma^1(G)^c$ and the fact that for split extension metabelian groups G of type FP_m, the trivial character is not in $\mathrm{conv}_{\leqslant m}(\mathbb{R}_{>0}\Sigma^1(G)^c)$ [**15**].

COROLLARY 2.3. *Suppose that M is a finitely generated $\mathbb{Z}[Q]$-module, χ is a non-trivial real character of Q and*

$$\{0, \chi\} \cap \mathrm{conv}_{\leqslant m}(\mathbb{R}_{>0}\Sigma_M(Q)^c) = \varnothing.$$

Then there exists a positive real number ϵ depending on χ with the following property. Suppose that $(x_1, x_2, \ldots, x_m)$ is an m-tuple of non-trivial real characters of Q such that, for some m-tuple $(\mu_1, \ldots, \mu_m)$ of non-trivial real characters of Q with $[\mu_i] \in \Sigma^c_M(Q)$, the angle between x_i and μ_i is at most ϵ for every $i \leqslant m$. Then $x_1, \ldots, x_m, -\chi$ lie in an open half subspace of $\mathbb{R}^n$.

3. *Constructing a complex for M*

A *standard $(Q - K(M,1))$-complex* is a $K(M,1)$-complex with a single 0-cell with a Q-action that is free on cells of dimension greater than zero and that makes the fundamental group into a Q-module isomorphic to M. Similarly, we define a *standard $(Q_\chi - K(M,1))$-complex* substituting, in the above definition, Q by Q_χ.

THEOREM 3.1. *Suppose that X is a standard $(Q_\chi - K(M,1))$-complex with Q_χ-finite k-skeleton. Then $[\chi]$ is contained in $\Sigma^k(G)$.*

Proof. Let $\tilde{X}$ be the universal covering of X and let $\tilde{*}$ be a vertex that maps to the unique vertex $*$ of X under the covering projection. The Q_χ-action on X lifts uniquely to a Q_χ-action on $\tilde{X}$, fixing the vertex $\tilde{*}$, and one uses the M-action on $\tilde{X}$ by covering transformations to construct a G_χ-action, where G_χ is the split extension of M by Q_χ. If $\tilde{x}$ is a point in $\tilde{X}$ and $(m, q) \in G_\chi$, then simply define

$$(m, q)\tilde{x} = m(q\tilde{x}).$$

One can check that this makes $\tilde{X}$ into a G_χ-complex with vertex stabilizers all isomorphic to Q_χ and trivial stabilizers of higher-dimensional cells. If $k = 2$, then we consider an obvious Q-extension X_0 of $\tilde{X}$ on which $G = M \rtimes Q$ acts freely on all edges and 2-cells, the vertex stabilizers are isomorphic to Q, and the action is cocompact. Then, as $\tilde{X}$ is simply connected, the main result of [**21**] implies the theorem.

In general, for $k \geqslant 3$, we use the formula $\Sigma^k(G) = \Sigma^k(G, \mathbb{Z}) \cap \Sigma^2(G)$ and reduce the theorem to $[\chi] \in \Sigma^k(G, \mathbb{Z})$. Now we consider the cellular chain complex of $\tilde{X}$

$$\mathscr{F} : F_k \longrightarrow \ldots \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow \mathbb{Z}[M] \longrightarrow \mathbb{Z} \longrightarrow 0,$$

and note that it has the property that all F_i for $i \leqslant k$ are free $\mathbb{Z}[G_\chi]$-modules of finite rank. In particular, the augmentation ideal $\mathrm{Aug}\,\mathbb{Z}[M]$ is of homological type FP_{k-1} over $\mathbb{Z}[G_\chi]$, where M acts on the augmentation ideal via multiplication and

Q_χ via its action on M, that is, $[\chi] \in \Sigma^{k-1}(G, \mathrm{Aug}\,\mathbb{Z}[M])$. The result now follows from the following.

LEMMA 3.2. *For* $k \geqslant 1$,

$$\Sigma^k(G, \mathbb{Z}) \cap \{[\chi] \mid \chi(M) = 0\} = \Sigma^{k-1}(G, \mathrm{Aug}\,\mathbb{Z}[M]) \cap \{[\chi] \mid \chi(M) = 0\}.$$

Proof. In this proof we consider only real characters χ of G such that $\chi(M) = 0$. By definition, $[\chi] \in \Sigma^k(G, \mathbb{Z})$ if and only if $\mathbb{Z}$ is of type FP_k over $\mathbb{Z}[G_\chi]$. Using the dimension shifting argument [**2**, Proposition 1.4] for the short exact sequence

$$0 \to \mathrm{Aug}\,\mathbb{Z}[G_\chi] \to \mathbb{Z}[G_\chi] \to \mathbb{Z} \to 0,$$

we see that $\mathbb{Z}$ is FP_k over $\mathbb{Z}[G_\chi]$ if and only if $\mathrm{Aug}\,\mathbb{Z}[G_\chi]$ is of type FP_{k-1} over $\mathbb{Z}[G_\chi]$.

Note that since Q is abelian,

$$\Sigma^\infty(Q, A) = \bigcap_{j \geqslant 1} \Sigma^j(Q, A) = \Sigma^0(Q, A)$$

for every finitely generated $\mathbb{Z}[Q]$-module A. In particular, $S(Q) = \Sigma^\infty(Q, \mathbb{Z})$ and $\mathbb{Z}$ is of type FP_∞ over $\mathbb{Z}[Q_\chi]$, although $\mathbb{Z}[Q_\chi]$ is not necessarily Noetherian. By the dimension shifting argument for the short exact sequence

$$0 \to \mathrm{Aug}\,\mathbb{Z}[Q_\chi] \to \mathbb{Z}[Q_\chi] \to \mathbb{Z} \to 0,$$

the augmentation ideal $\mathrm{Aug}\,\mathbb{Z}[Q_\chi]$ is of type FP_∞ over $\mathbb{Z}[Q_\chi]$ and then the induced module

$$\Big\uparrow_{\mathbb{Z}[Q_\chi]}^{\mathbb{Z}[G_\chi]} \mathrm{Aug}\,\mathbb{Z}[Q_\chi]$$

is FP_∞ over $\mathbb{Z}[G_\chi]$. Consider the short exact sequence

$$0 \to \Big\uparrow_{\mathbb{Z}[Q_\chi]}^{\mathbb{Z}[G_\chi]} \mathrm{Aug}\,\mathbb{Z}[Q_\chi] \simeq \mathbb{Z}[G_\chi] \cdot \mathrm{Aug}\,\mathbb{Z}[Q_\chi] \xrightarrow{\tau} \mathrm{Aug}\,\mathbb{Z}[G_\chi] \to \mathrm{Aug}\,\mathbb{Z}[M] \to 0,$$

where τ is the inclusion map. By the dimension shifting argument, $\mathrm{Aug}\,\mathbb{Z}[G_\chi]$ is FP_{k-1} over $\mathbb{Z}[G_\chi]$ if and only if $\mathrm{Aug}\,\mathbb{Z}[M]$ is FP_{k-1} over $\mathbb{Z}[G_\chi]$, that is, $[\chi] \in \Sigma^{k-1}(G, \mathrm{Aug}\,\mathbb{Z}[M])$. This completes the proof. □

We will next construct the 3-skeleton X of a $(Q_\chi - K(M, 1))$-complex in almost complete analogy to the construction of a

$$(Q - K(M, 1)^3)\text{-complex}$$

given in [**8**].

Since $[\chi]$ is assumed to be in $\Sigma_M(Q)$, we can choose a resolution of the Q_χ-module M by finitely generated Q_χ-modules

$$\mathbb{Z}B \xrightarrow{\partial} \mathbb{Z}A \to M \to 0,$$

where A and B are free Q_χ-sets with finitely many orbits. Fix sets of orbit representatives A_0, B_0 for A, B, respectively.

It is convenient to put an ordering on A and B that is compatible with the Q_χ-action. Such an ordering can be obtained from an ordering of Q_χ and an ordering on the fixed finite sets A_0, B_0 of orbit representatives.

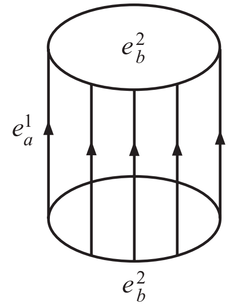

FIGURE 1. *Prism* 3-*cell.*

Let Z be the subset of the product of spheres $\prod_{a\in A} S_a^1$ consisting of all tori $S_{a_1}^1 \times \ldots \times S_{a_k}^1$ with $a_1 < \ldots < a_k$. Now Q_χ acts in the obvious way on the product of spheres and makes Z into a cubical Q_χ-complex. Note that Z is a $K(\mathbb{Z}A, 1)$-complex with a single vertex $*$ and that the 1-cells e_a^1 of Z are in one-to-one correspondence with the elements of A. We write e_{a_1,a_2}^2 for $e_{a_1}^1 e_{a_2}^1$. For every $b \in B$ we attach to Z a 2-cell e_b^2 as follows. If

$$\partial(b) = \sum_{i=1}^{k} n_i a_i,$$

where $a_1 < \ldots < a_k$, use the attaching path $w = e_{a_1}^{n_1} \ldots e_{a_k}^{n_k}$ (we use the ordering on A to obtain a uniquely defined edge path from a sum on the a_i). Q_χ acts on the set of new 2-cells by $qe_b^2 = e_{qb}^2$.

For every pair (b, a) we furthermore attach a 3-cell $e_{a,b}^3$, as shown in Figure 1. This 3-cell is the product of the 2-cell e_b^2 and the 1-cell a. We refer to such 3-cells as prisms.

Again, Q_χ acts on this set of 3-cells by acting on the labels $qe_{a,b}^3 = e_{qa,qb}^3$. We call the resulting Q_χ-complex Y. The action induces Q_χ-module structures on all homotopy groups and turns the fundamental group into a Q_χ-module isomorphic to M.

LEMMA 3.3. *The second homotopy module $\pi_2(Y)$ is a finitely generated Q_χ-module.*

Proof. Consider the composition of Q_χ-module maps

$$\pi_2(Y) \xrightarrow{h} H_2(Y) \xrightarrow{j} H_2(Y, Z) = \mathbb{Z}B \xrightarrow{\partial} H_1(Z) = \mathbb{Z}A,$$

where h is the Hurewicz map and j is induced by inclusion. One shows exactly as in [**8**] (by combinatorial homotopy arguments) that the composition $j \circ h$ is injective. We next show that the image of this composition is the kernel of the boundary map $\partial : \mathbb{Z}B \longrightarrow \mathbb{Z}A$. Since $j(H_2(Y)) = \ker(\partial)$, it is clear that $j \circ h(\pi_2(Y)) \subseteq \ker(\partial)$. Let us show the other inclusion. Suppose that $\beta = \sum n_i b_i$ is an element in $\ker(\partial)$. Then, by construction of the cells e_b^2, we see that the boundary w of the union (planar diagram)

$$d_1 = \bigcup n_i e_{b_i}^2$$

(see Figure 2) represents the trivial element of $H_1(Z) = \pi_1(Z) = \mathbb{Z}A$. Thus there exists a contracting disc d_2 in Z with $\partial(d_2) = \partial(d_1)$. Now the union $d_1 \cup -d_2$ makes a sphere that represents an element α of $\pi_2(Y)$ such that $j \circ h(\alpha) = \beta$.

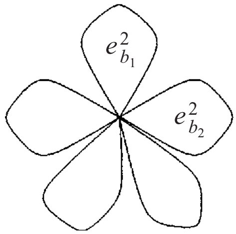

FIGURE 2. *Planar diagram* d_1.

Since M is finitely generated as a Q_χ-module (we assumed that $[\chi]$ was in $\Sigma_M(Q)$), it is actually of type FP_∞. In particular, the kernel of ∂ is a finitely generated Q_χ-module. □

Lemma 3.3 shows that we can attach finitely many Q_χ-orbits of 3-cells to Y to obtain the 3-skeleton X of a $(Q_\chi - K(M,1))$-complex.

4. *Support geometry and pushing cells*

Define the support of the 0-cell $*$ in X to be the subset of Q consisting of the trivial element. The support of a 1-cell e^1_a of X, $a = qa_0$, $a_0 \in A_0$, $q \in Q$ is defined to be $\operatorname{supp}(e^1_a) = \{q\}$. If c is a higher-dimensional cell of X, then we define

$$\operatorname{supp}(c) = \{q \in Q \mid q \text{ is in the support of a 1-cell occurring in } \partial(c)\}.$$

In [8], the diameter of a cell was defined to be the minimal radius of a ball in $\mathbb{R}^n$ that contains the support of the cell. However, this notion is not suitable for working over the halfspace $\mathbb{R}^n_\chi = \{r \in \mathbb{R}^n \mid \chi(r) \geqslant 0\}$.

Let $I(m, z_0)$ be a cube with edge length $2m$ and centre $(m + z_0)[\chi]$, where z_0 is a fixed positive real number, and is such that one of its walls is perpendicular to $[\chi]$. At this stage we do not pose any restrictions on z_0, but later on we will encounter technical points that can be resolved only when z_0 is sufficiently large. Now let $B(m, z_0)$ be the union of all closed balls with radius z_0 and center in $I(m, z_0)$ (see Figure 3).

We define the *global diameter* of a cell c in X to be

$$\min\{m \in \mathbb{R} \mid \operatorname{supp}(c) \text{is a subset of some } \mathbb{R}^n_\chi\text{-translate of } B(m, z_0)\}.$$

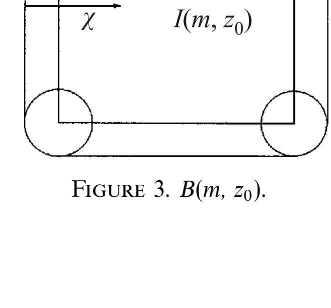

FIGURE 3. $B(m, z_0)$.

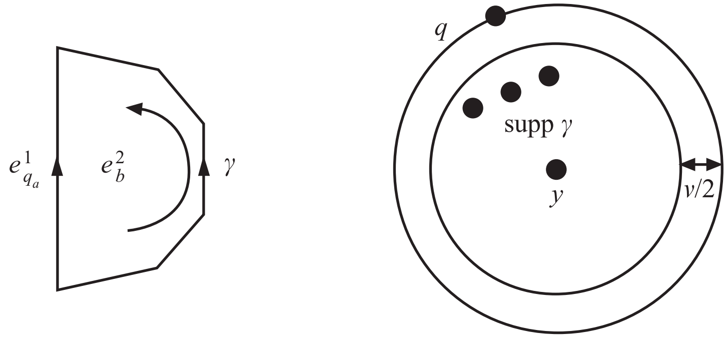

FIGURE 4. *Pushing 2-cell and its support.*

Note that since every $B(m, z_0)$ can only contain finitely many integral points, the set of diameters of cells of X is a discrete subset of $\mathbb{R}_{\geqslant 0}$.

The construction of X depends on the choice of the free Q_χ-sets A and B. Here we impose some restrictions on set B. Fix some positive real number ϵ given by Corollary 2.3 for $m = 3$; the positive real number v is given by Proposition 2.1 and $\rho_1(v)$ is given by Corollary 2.2. According to Proposition 2.1, there is a special finite subset Λ of the centralizer of M in $\mathbb{Z}[Q]$. For every $\lambda \in \Lambda$ and every $\mu \in \Sigma_M(Q)$ there is an element q_λ from $\operatorname{supp} \lambda \cup \{1\}$ such that

$$\min\{\mu(Q) \mid q \in \operatorname{supp} \lambda \cup \{1\}\} = \mu(q_\lambda).$$

Note that $q_\lambda^{-1} \in \mathbb{Z}Q_\chi$. Since $q_\lambda^{-1}(1-\lambda)a$ is sent to the trivial element under $\mathbb{Z}A \longrightarrow M$, we may assume that for every $(\lambda, a) \in \Lambda \times A_0$ there exists a $b_{\lambda,a} \in B$ such that $\partial(b_{\lambda,a}) = q_\lambda^{-1}(1-\lambda)a$.

Geometrically, this means that if q is an element of Q_χ, y is a point of $\mathbb{R}^n_\chi$ such that the closed ball with centre y and radius $|q - y| = z_0 \geqslant \rho_1(v)$ is inside $\mathbb{R}^n_\chi$ and $(q - y)/(|q - y|) \in -\Sigma_M(Q)$, then there is a 2-dimensional cell e^2_b in X that gives a homotopy between the edge e^1_{qa} and a path γ with support $q \operatorname{supp}(\lambda)$ for some $\lambda \in \Lambda$ given by Corollary 2.2 (see Figure 4). In particular, the support of the path γ is in the open ball with centre y and radius $|q - y| - v/2$. We call these special cells 'pushing 2-cells'. Note that we have already imposed a restriction on z_0 by assuming that $z_0 \geqslant \rho_1(v)$.

Let us next construct 3-dimensional pushing cells. For this it is convenient to consider a Q-enlargement $\hat{X}$ of X that we define inductively as follows. Define $\hat{X}^{(0)} = X^{(0)} = *$. Suppose that $\mathscr{C}_i$ is a set of Q_χ-orbit representatives for the i-dimensional cells in X, $i = 1, 2, 3$. Suppose that $\hat{X}^{i-1}$ is defined. For $c \in \mathscr{C}_i$, $q \in Q - Q_\chi$, attach a cell qc to $\hat{X}^{i-1}$ along $q\partial(c)$. We extend the notion of cell support to $\hat{X}$. Here the support of a cell of $\hat{X}$ is a finite subset of Q.

If μ is a character and d is a real number, then define $\hat{X}_{\mu \geqslant d}$ to be the subcomplex on cells c such that

$$\operatorname{supp}(c) \subseteq \{q \in Q \mid \mu(Q) \geqslant d\}.$$

Note that $X = \hat{X}_{\chi \geqslant 0}$. The existence of pushing 2-cells yields the following result (see [**8**, Lemma 6.1]).

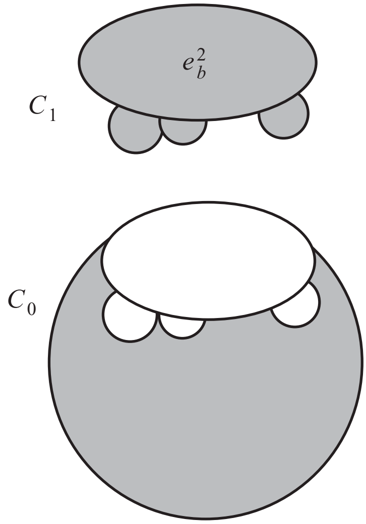

FIGURE 5. *Pushing 3-cell with boundary* $-C_1 \cup C_0$.

LEMMA 4.1. *For every* $[\mu] \in \Sigma_M(Q)$ *and real number d, the inclusion of* $\hat{X}_{\mu \geqslant d}$ *in* $\hat{X}$ *induces a monomorphism of* Q_μ*-modules*

$$\pi_1(\hat{X}_{\mu \geqslant d}) \longrightarrow \pi_1(\hat{X}).$$

We finish this section with the definition of some special pushing 3-cells that play an important role in the proof of the main result.

Suppose that w is the boundary path of a 2-cell e_b^2, $b \in B$, of $\hat{X}$, with edges $e_{a_i}^1$ with support q_i. Assume that the support $\{q_1, \dots, q_m\}$ of e_b^2 is inside $v_0 + B(\rho, z_0)$ for some $v_0 \in \mathbb{R}^n_{\chi \geqslant 0}$, and that at least one q_i is on the boundary of $v_0 + B(\rho, z_0)$. We assume further that for every q_i that is on the boundary of $v_0 + B(\rho, z_0)$,

$$\chi_i = \frac{q_i - y_i}{|q_i - y_i|} \in -\Sigma_M(Q),$$

where y_i is the projection of q_i to $v_0 + I(\rho, z_0)$, that is, y_i is the unique point from $v_0 + I(\rho, z_0)$ such that the distance between y_i and q_i is z_0. Let I_w be the subset of $\{1, \dots, m\}$ for which the latter holds, $Y_w = \{y_i \mid i \in I_w\}$, and suppose that the support of w is in $U_w(z_0)$ (this holds if z_0 is sufficiently large), where $U_w(r)$ is the union over $i \in I_w$ of all closed balls with centre y_i and radius r. Now, to every edge $e_{a_i}^1$ with $i \in I_w$, a pushing cell $e_{b_i}^2$ is glued with the property that $\operatorname{supp}(e_{b_i}^2) \setminus \{q_i\}$ is in the open ball with centre y_i and radius z_0. Let us call this union of 2-cells C_1 (see top of Figure 5). Note that for our fixed e_b^2, there are only finitely many such configurations C_1. The boundary path $\hat{w}$ of C_1 has support in $\bigcup_{r<z_0} U_w(r)$, so is strictly inside $v_0 + B(\rho, z_0)$. Our aim is to show that if z_0 is sufficiently large, then the path $\hat{w}$ has a contracting disc C_0 with support entirely contained in the interior of $v_0 + B(\rho, z_0)$.

Using the fact that $\Sigma_M(Q)$ is an open subset of $S(Q)$, if z_0 is sufficiently large, then for every q_i from the support of w the direction $\mu_{q_i} = (q_i - y)/(|q_i - y|) \in -\Sigma_M(Q)$, where y is $(1/|I_w|)\sum_{i \in I_w} \bar{y}_i$, $\bar{y}_i$ is a point with integral coordinates in the intersection of the closed ball with centre q_i and radius z_0 with the closed ball with centre y_i and radius $\sqrt{n} \leqslant z_0$, and the average direction $\mu = \sum \mu_{q_i}/|\sum \mu_{q_i}|$ is in $-\Sigma_M(Q)$. By Lemma 4.1, we know that $\hat{w}$ is contractible in $\hat{X}_{-\mu \geqslant d}$, where $d = \min\{-\mu(Q) \mid q \in \operatorname{supp} \hat{w}\}$. If z_0 is sufficiently large, the fact that $\hat{w}$ is contractible in the halfspace implies that it is contractible by a disc C_0 with support in $U_w(z_0')$, $z_0' < z_0$, and is hence in the interior of $v_0 + B(\rho, z_0)$. Add a free Q-orbit of 3-dimensional cells Qc to $\hat{X}$, where c is the 3-cell with boundary $-C_0 \cup C_1$ as shown in Figure 5. Note that the construction of C_0 depends on C_1 (or rather its boundary $\hat{w}$) and the direction μ (which depends on the choice of $v_0 + B(\rho, z_0)$). However, only finitely many directions μ can arise for our fixed e_b^2, because every $\bar{y}_i$ is an integer lattice point in the ball of radius z_0 with center q_i. Thus, for a fixed C_1, we have built only finitely many contracting discs C_0. We call these added 3-cells 'pushing 3-cells'. They can be used to push the boundary of C_1 in the $-\mu$ direction. We proceed in this fashion for each e_b^2, $b \in B$ and $v_0 + B(\rho, z_0)$. Note that Q acts cofinitely on the Q-orbits generated by all possible C_1. In the end, we have added Q-finitely many pushing 3-cells to $\hat{X}$. We denote that bigger complex also by $\hat{X}$ and the half $\hat{X}_{\chi \geqslant 0}$ by X. From now on we will work with this bigger $(Q_\chi - K(M,1)^{(3)})$-complex.

5. *Proof of the main result*

In this section we prove Theorem 1.1 by showing that $[\chi] \in \Sigma^3(G)$. Suppose that $\chi : G \longrightarrow \mathbb{R}$ is a character, $\chi(M) = 0$, and $\chi \notin \operatorname{conv}_{\leqslant 3}(\mathbb{R}_{>0}\Sigma^1(G)^c)$. Consider the $Q_\chi - K(M,1)^{(3)}$ X constructed before. In order to show that $[\chi] \in \Sigma^3(G)$, we have to reduce it down to a Q_χ-finite one so that we can apply Theorem 3.1. For $\rho > 0$, let X_ρ be the subcomplex of X on cells of global diameter less than or equal to ρ. Clearly X_ρ is a Q_χ-finite complex. The goal of this section is to show that for some ρ_0 the complex X_{ρ_0} is a

$$\left(Q_\chi - K(M,1)^{(3)}\right)\text{-complex.}$$

Let $\{\rho_0 < \rho_1 < \rho_2 < \ldots\}$ be the set of diameters of cells of X bigger than or equal to some big real number ρ_0 (occurring as diameter) that we will specify as we go along. We aim to construct cellular maps $r_i : X_{\rho_i} \longrightarrow X_{\rho_{i-1}}$, which is the identity on $X_{\rho_{i-1}}$, $i = 1, 2, \ldots$. The main theorem follows quickly once these maps are constructed. Indeed, let

$$f_i = r_i \circ \ldots \circ r_1 : X_{\rho_i} \longrightarrow X_{\rho_0},$$

and let $r : X \longrightarrow X_{\rho_0}$ be the union of the f_i. Then r is a retraction and hence the inclusion $X_{\rho_0} \subseteq X$ induces monomorphisms on the homotopy groups. Thus $\pi_2(X_{\rho_0}) = 1$. Since both X and X_{ρ_0} have the same 1-skeleton (1-cells have global diameter 0), the inclusion induces a Q_χ-isomorphism on fundamental groups.

5.1. *A geometric lemma*

The construction of the maps r_i relies on the following geometric lemma.

Lemma 5.1. *Suppose that M is a finitely generated Q-module and χ is a real character of Q such that*

$$\{0, \chi\} \cap \operatorname{conv}_{\leqslant m}\left(\mathbb{R}_{>0}\Sigma_M^c(Q)\right) = \varnothing$$

for some $m \geqslant 2$. Then there exist positive numbers ϵ_0, $\tilde{\rho}$ and $\tilde{z}_0$ with the following property.

Suppose that $Y = \{q_1, \dots, q_s\}$ *is a subset of* $B(\rho, z_0)$ *of global diameter* $\rho \geqslant \tilde{\rho}$, $z_0 \geqslant \tilde{z}_0$, *and for every* q_i *from the boundary of* $B(\rho, z_0)$ *define* y_i *to be the projection of* q_i *to* $I(\rho, z_0)$ (*that is,* y_i *is the unique element from* $I(\rho, z_0)$ *such that the distance between* y_i *and* q_i *is* z_0) *and set* $\chi_i = q_i - y_i$. *Assume further that one of the following holds:*

(1) $s = m$.

(2) $\{q_2, \dots, q_s\}$ *is the support of some element* e_b.

Then, for some i_0,

$$[\chi_{i_0}] \in -\Sigma_M(Q),$$

and for every $[\mu] \in S(Q)$ *such that the angle between* μ *and* χ_{i_0} *is at most* ϵ_0,

$$[\mu] \in -\Sigma_M(Q).$$

Proof. (1) We consider first the case when $s = m$. Set $\epsilon_0 = \epsilon$, where ϵ is the positive real number given by Corollary 2.3. We assume that the lemma does not hold and for every χ_i there is a $[\mu_i] \notin -\Sigma_M(Q)$ such that the angle between μ_i and χ_i is at most ϵ. By Corollary 2.3, the set X of all $x_i = -\chi_i$ together with $-\chi$ lies in an open halfspace; in particular, there is a $v \in \mathbb{R}^n$ with arbitrary small length such that

$$|x| > |v + x| \quad \text{for every } x \in X = \{x_1, \dots, x_m, -\chi\}.$$

Then, for sufficiently small positive real number α and every q_i from the boundary of $B(\rho, z_0)$,

$$q_i - v \in B(\rho, z_0 - \alpha/2) \subseteq B(\rho - \alpha/2, z_0)$$

and $0 < |-\chi|^2 - |-\chi + v|^2 = 2(v, \chi) - |v|^2$, so $v \in \mathbb{R}^n_\chi$. Furthermore, if α is sufficiently small and the length of v is sufficiently small for all q_i that are not on the boundary of $B(\rho, z_0)$, we have $q_i \in v + B(\rho - \alpha/2, z_0)$. In particular, $\{q_1, \dots, q_m\}$ is a subset of $v + B(\rho - \alpha/2, z_0)$ and its global diameter is at most $\rho - \alpha/2$, a contradiction.

(2) Now we consider the second case and set $\epsilon_0 < \epsilon$, where ϵ is the positive real number given by Corollary 2.3. Assume that the lemma does not hold and, as in the first part, for every χ_i there is a $[\mu_i] \notin -\Sigma_M(Q)$ such that the angle between μ_i and χ_i is at most ϵ_0. Define I to be the index set of all $q_i \in \partial B(\rho, z_0)$, $i \geqslant 2$. For z_0 sufficiently large and ϵ_0 sufficiently small, the elements of $\{\mu_i\}_{i \in I}$ are very close. As ρ is the diameter of the set $\{q_1, q_2, \dots, q_s\}$ for $\tilde{\rho}$ sufficiently big, q_1 is on the boundary of $B(\rho, z_0)$. Then χ_1 is defined, $\mu_1 \in -\Sigma^c_M(Q)$, and for $i \in I$ we have $\operatorname{conv}_{\leqslant 2}\{-\mu_1, -\mu_i\} = \operatorname{conv}\{-\mu_1, -\mu_i\}$ is close to $\operatorname{conv}\{-\mu_1, -\mu_i\}_{i \in I}$. In other words, for sufficiently large $\tilde{\rho}, \tilde{z}_0$ and sufficiently small ϵ_0, the fact that $\Sigma^c_M(Q)$ is a polyhedron together with $\{0, \chi\} \cap \operatorname{conv}_{\leqslant 2}\{-\mu_1, -\mu_i\} \subseteq \{0, \chi\} \cap \operatorname{conv}_{\leqslant 2}\Sigma^c_M(Q) = \emptyset$ implies that $\{0, \chi\} \cap \operatorname{conv}\{-\mu_1, -\mu_i\}_{i \in I} = \emptyset$ and $\{0, \chi\} \cap \operatorname{conv}\{-\chi_1, -\chi_i\}_{i \in I} = \emptyset$. Then there exists a $v \in \mathbb{R}^n$ with arbitrary small length such that

$$|x| > |v + x| \quad \text{for every } i \in I,\ x \in X = \{-\chi_1, -\chi_i, -\chi\},$$

and we can continue exactly as in the previous case. □

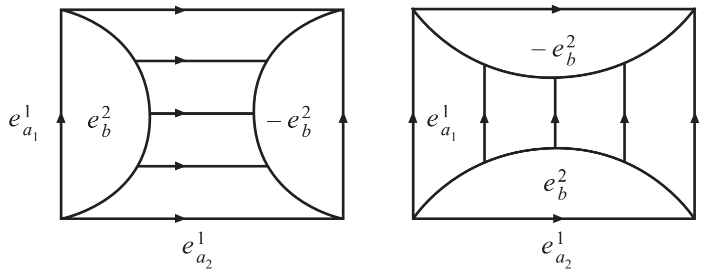

FIGURE 6. *Image of a square.*

5.2. *Construction of the map r_i*

We start this subsection with the definition of ρ_0. Edges have global diameter 0 and all cells other than squares, cubes and prisms have bounded global diameter, so we can choose ρ_0 big enough so that X_{ρ_0} contains all cells except big squares, big cubes and big prisms. Furthermore, we want $\rho_0 \geqslant \tilde{\rho}$, where $\tilde{\rho}$ is given by Lemma 5.1. We define r_i on a generator of the Q_χ-orbit of cells (of dimension 1 or more) and then extend the map in the unique way that will make it commute with the Q_χ-action.

Let us now define the map r_i. In the following we set $\rho = \rho_i$ and $\rho' = \rho_{i-1}$. On $X_{\rho'}$ the map r_i is the identity. This implies that r_i is the identity on all edges. Let e^2 be a square of global diameter ρ . Let

$$\operatorname{supp}(e^2) = \{q_1, q_2\},$$

where $\{q_1\}$ and $\{q_2\}$ are the supports of the edges $e^1_{a_1}$ and $e^1_{a_2}$ of the square e^2. The q_i, $i = 1, 2$, lie on the boundary of some translate $v_0 + B(\rho, z_0)$ for some $v_0 \in \mathbb{R}^n_\chi$. Let y_i be the projection of q_i in $v_0 + I(\rho, z_0)$. Now by the first case of Lemma 5.1 applied for $m = 2$, there is an i_0 such that

$$\frac{q_{i_0} - y_{i_0}}{|q_{i_0} - y_{i_0}|} \in -\Sigma_M(Q).$$

Assume first that $i_0 = 1$. Then there is a 2-dimensional pushing cell e^2_b that gives a homotopy between the edge $e^1_{a_1}$ and a path γ with support in the interior of $v_0 + B(\rho, z_0)$, that is, every square that is a product of an edge from γ with the edge $e^1_{a_2}$ has a global diameter smaller than ρ. The image of the square is shown on the left-hand side of Figure 6. It is contained in $X_{\rho'}$.

We proceed in the same fashion in the case when $i_0 = 2$. The image of the square e^2 is shown on the right-hand side of Figure 6.

It remains to extend the map

$$r_i : (X_\rho)^{(2)} \longrightarrow (X_{\rho'})^{(2)}$$

to the 3-skeleton. Define r_i to be the identity on 3-cells of diameter less than ρ. In order to show that we can extend r_i to all 3-cells e^3 in X_ρ, we need to show that the boundary

$$\partial(r_i(e^3)) = r_i(\partial(e^3))$$

(which is contained in $X_{\rho'}$) can be contracted in $X_{\rho'}$.

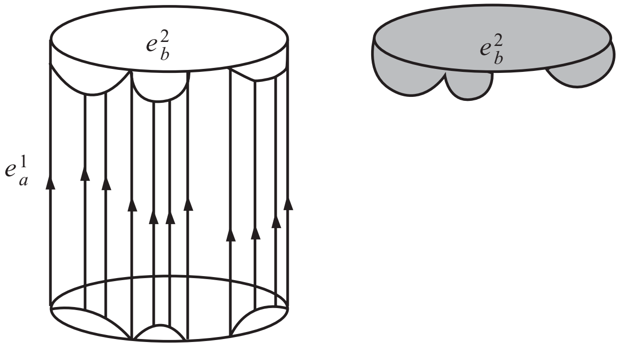

FIGURE 7. *U_0 and its top.*

Case 1: We assume that $e^3 = e^3_{a,b}$ is a prism of global diameter ρ, $\text{supp}\,(e^1_a) = \{q_1\}$, and $\text{supp}\,(e^2_b) = \{q_2, \dots, q_s\}$; hence

$$\text{supp}\,(e^3) = \{q_1, \dots, q_s\} \subseteq v_0 + B(\rho, z_0)$$

for some $v_0 \in \mathbb{R}^n_\chi$. For every q_i that is on the boundary of $v_0 + B(\rho, z_0)$, we define y_i to be the projection of q_i to $v_0 + I(\rho, z_0)$, that is, y_i is the unique element from $v_0 + I(\rho, z_0)$ such that the distance between q_i and y_i is z_0. As there are only finitely many Q_χ-orbits of cells e^2_b, that is, finitely many shapes for $\text{supp}\,(e^2_b)$, we see that for sufficiently large z_0, the point q_1 is on the boundary of $v_0 + B(\rho, z_0)$, and we assume that this is the case.

Case 1.1: Suppose that $(q_1 - y_1)/(|q_1 - y_1|) \notin -\Sigma_M(Q)$. Let I_0 be the subset of $\{2, \dots, s-1, s\}$ of all i such that the global diameter of q_1, q_i is ρ. If I_0 is not empty by the first part of Lemma 5.1 applied for $m = 2$ and $Y = \{q_1, q_i\}$, $i \in I_0$, then q_i has the property that $\chi_i = (q_i - y_i)/(|y_i - q_i|) \in -\Sigma_M(Q)$, and for every $\mu_i \in \text{Hom}\,(Q, \mathbb{R}) \setminus \{0\}$ such that the angle between μ_i and χ_i is at most ϵ_0, $[\mu_i] \in -\Sigma_M(Q)$. Fix one $i_0 \geqslant 2$ with the above property. If z_0 is sufficiently large for every $i \geqslant 2$ such that q_i is on the boundary of $v_0 + B(\rho, z_0)$ but $i \notin I_0$, then the angle between $\chi_i = (q_i - y_i)/(|q_i - y_i|)$ and χ_{i_0} is at most ϵ_0; in particular, $\chi_i \in -\Sigma_M(Q)$. If I_0 is empty, then we use the second part of Lemma 5.1 to obtain $\chi_i \in -\Sigma_M(Q)$ for all $i \geqslant 2$ when χ_i is defined.

Note that $r_i(\partial(e^3))$ is the boundary of the union U of 3-cells consisting of e^3 and some prisms e^3_{a,b_i} glued to the squares e^2_{a,a_i} (or $e^2_{a_i,a}$ if $a_i < a$) of $\partial(e^3)$ with support $\{q_1, q_i\}$, $i \in I_0$. Consider a new 3-dimensional figure U_0 obtained from U by gluing a prism e^3_{a,b_i} to the square e^2_{a,a_i} (or $e^2_{a_i,a}$) with support $\{q_1, q_i\}$ where $i \in \{2, \dots, s\} \setminus I_0$ and q_i is on the boundary of $v_0 + B(\rho, z_0)$. All elements from the support of $e^2_{b_i}$ except q_i are in the open ball with centre y_i and radius z_0 (this is possible because $[\chi_i] \in -\Sigma_M(Q)$). Note that the global diameter of e^3_{a,b_i} is smaller than ρ and so $\partial(U)$ is contractible in $X_{\rho'}$ if and only if $\partial(U_0)$ is contractible in $X_{\rho'}$.

By the construction of the additional 3-dimensional pushing cells in X given in Section 4, the top (bottom) of $\partial(U_0)$ is homotopic via a 3-dimensional pushing cell to a union C_0 of 2-cells with the property that the support of every cell in C_0 is in the interior of $v_0 + B(\rho, z_0)$, that is, no element from the support is on the boundary $v_0 + B(\rho, z_0)$ (see Figure 7 and Figure 5). Then the boundary of U_0 is the boundary

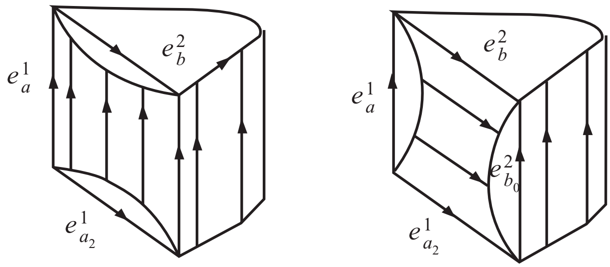

FIGURE 8. *Image of the boundary of a special prism under r_i.*

of the union of the two pushing 3-cells to the top and bottom of U_0 together with a union U_1 of 3-cells where every 3-cell in U_1 is the product of e^1_a with a cell from C_0. Because pushing 3-cells are contained in $X_{\rho'}$, the boundary of U_0 is contractible in $X_{\rho'}$ if and only if this is true for the boundary of U_1. Now note that the boundary of U_1 is the boundary of the 3-cells in the product of C_0 and e^1_a, and that these 3-cells have support contained in

$$\{\text{the interior of } v_0 + B(\rho, z_0)\} \cup \{\text{the point } q_1 \text{ from the boundary of } v_0 + B(\rho, z_0)\},$$

and are hence contained in $X_{\rho'}$.

Case 1.2: We assume that e^3 is a special prism, that is, the global diameter of only one square of $\partial(e^3)$ is ρ. Without loss of generality, we assume that the global diameter of $\{q_1, q_2\}$ is ρ, and let $e^1_{a_2}$ be the unique edge from $\partial(e^2_b)$ with support q_2. Then $r_i(\partial(e^3))$ is of one of the types shown in Figure 8.

In the first case (on the left-hand side of Figure 8), we use the method for Case 1.1. As in Case 1.1, we glue additional prisms e^3_{a,b_i} to the squares e^2_{a,a_i}, where $e^1_{a_i}$ is an edge of the boundary of e^2_b with support q_i on the boundary of $v_0 + B(\rho, z_0)$ for some $i \geqslant 3$. Thus we obtain a new figure U_0: the union of $r_i(\partial(e^3))$ with the new prisms. Note that as $(q_2 - y_2)/(|q_2 - y_2|) \in -\Sigma_M(Q)$ for sufficiently large z_0 for all q_i from $\operatorname{supp}(e^2_b) \cap \partial(v_0 + B(\rho, z_0))$, we have $(q_i - y_i)/(|q_i - y_i|) \in -\Sigma_M(Q)$. Then the top and bottom of U_0 are homotopic via a 3-dimensional pushing cell to a union C_0 of 2-dimensional cells with support strictly inside $v_0 + B(\rho, z_0)$. Then we continue as in Case 1.1.

Now we consider the second case. Note that $r_i(\partial(e^3))$ is the boundary of the union of the 3-cell e^3 and the prism that is the product of $e^2_{b_0}$ and $e^1_{a_2}$. We glue prisms to all other squares of the boundary of e^3 to obtain a union U_0 of 3-cells. The prisms we glue are products of the pushing cell $e^2_{b_0}$ with the edges from $\partial(e^2_b) \setminus e^1_{a_2}$ and so have global diameter smaller than ρ; here $e^2_{b_0}$ is the pushing cell used in the definition of $r_i(e^2_{a,a_2})$ to push e^1_a. Thus $r_i\partial(e^3)$ is contractible in $X_{\rho'}$ if and only if the boundary of U_0 is contractible in $X_{\rho'}$. Now note that the boundary of U_0 is also the boundary of a union U_1 of prisms of global diameter smaller than ρ. These prisms are products of e^2_b with edges of the boundary of $e^2_{b_0}$ different from e^1_a (see Figure 9). In particular, the boundary of U_0 is contractible in $X_{\rho'}$. Thus we can extend r_i to all special prisms in X_ρ.

Case 1.3: Suppose that $u_1 = (q_1 - y_1)/(|q_1 - y_1|) \in -\Sigma_M(Q)$. Then there is a $b_0 \in B$ such that $e^2_{b_0}$ is a pushing 2-cell that gives a homotopy between the edge e^1_a and a

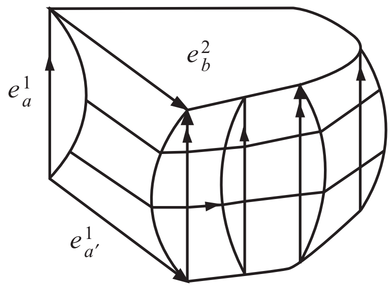

FIGURE 9. *Surrounding a special prism with prisms.*

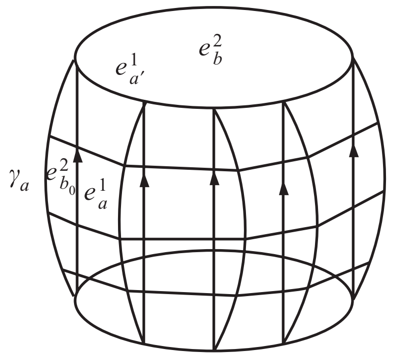

FIGURE 10. *Surrounding a prism with special prisms.*

path γ_a; the support of γ_a is in the open ball with centre y_1 and radius $z_0 - v/2$ (see Figure 10). We attach to every square $e^2_{a,a'}$ in the boundary of the prism e^3 a special prism, a product of the corresponding $e^2_{b_0}$ and $e^1_{a'}$, and obtain the union U_0 of all these 3-cells. Now note that the boundary of U_0 is also the boundary of a union U_1 of prisms of global diameter smaller than ρ. The prisms in U_1 are products of e^2_b with the edges of the path γ_a. This shows that the boundary of e^3 is the boundary of a union of special prisms from X_ρ and prisms already contained in $X_{\rho'}$. Since r_i is defined on these 3-cells, we can extend r_i to e^3.

Case 2: We assume that $e^3 = e^3_{a_1,a_2,a_3}$ is a 3-cube of global diameter ρ of support $\{q_1, q_2, q_3\}$. Furthermore, we assume that $\{q_1, q_2, q_3\} \subseteq v_0 + B(\rho, z_0)$. For every q_i on the boundary of $B(\rho, z_0)$, we define y_i to be the projection of q_i to $v_0 + I(\rho, z_0)$. Then, by the first part of Lemma 5.1 for $m = 3$, there exists for some $i_0 \in \{1, 2, 3\}$

$$\frac{q_{i_0} - y_{i_0}}{|q_{i_0} - y_{i_0}|} \in -\Sigma_M(Q).$$

Thus there is a pushing 2-cell e^2_b that gives a homotopy between the edge

$$e^1_{a_{i_0}}$$

and a path γ whose support is in the open ball with centre y_{i_0} and radius $z_0 - v/2$. Now we glue prisms $e^3_{a,b}$ to every square of the boundary of e^3 with edges with labels a and a_{i_0} to obtain a union U_0 of 3-cells (see Figure 11). Note that the prisms that we have glued on are special prisms of global diameter at most ρ, and so r_i is already defined on them by Case 1.2, previously considered. Note that the boundary

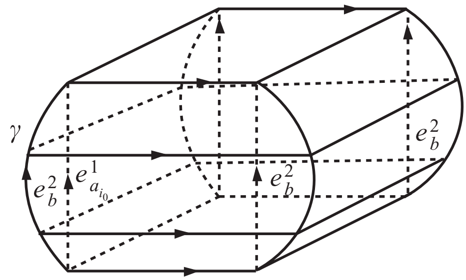

FIGURE 11. *Surrounding a cube with special prisms.*

of U_0 is also the boundary of a union U_1 of cubes of smaller global diameter. These cubes are products of the square of $\partial(e^3)$ not containing

$$e^1_{a_{i_0}}$$

with the edges from γ. This shows that $\partial(e^3)$ is the boundary of the union of 3-cells of X_ρ on which r_i is already defined. Thus we can extend r_i to e^3.

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