# On Primeness of Labeled Oriented Trees

Jens Harlander and Stephan Rosebrock

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**Abstract**

Knot complements are aspherical. Whether this extends to ribbon disc complements, or, equivalently, to standard 2-complexes of labeled oriented trees, remains unresolved. It is known that prime injective labeled oriented trees are diagragramtically reducible, that is, aspherical in a strong combinatorial sense. We show that arbitrary prime labeled oriented trees need not be DR. We conjecture that all injective labeled oriented trees are aspherical and prove the conjecture under natural conditions. [1]

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## 1 Introduction

This article is concerned with the Whitehead conjecture, which states that a subcomplex of an aspherical 2-complex is aspherical. See Bogley [2] and Rosebrock [11] for surveys. The conjecture originally arose in the context of knot theory. The Wirtinger presentation of a knot gives rise to a 2-complex that is a subcomplex of a contractible 2-complex. Thus, an affirmative answer to the conjecture implies the asphericity of knot complements in the 3-sphere. Labeled oriented trees, LOTs for short, are a way to record presentations that generalize Wirtinger presentations for knots. They play a central role in the work on the Whitehead conjecture. Results of Howie [4] imply that the finite case of the Whitehead conjecture reduces, up to the Andrews-Curtis conjecture, to the statement that LOT presentations are aspherical.

A *labeled oriented graph* (LOG) is an oriented graph on vertices $\{1, ..., n\}$, where each oriented edge is labeled by a vertex. Associated with it comes a presentation on generators $x_1, \ldots, x_n$ in one-to-one correspondence with the vertices.

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For an edge with initial vertex $i$, terminal vertex $j$ and label $k$ we add a relation $x_ix_k = x_kx_j$. We refer to such a presentation as a LOG-presentation. Associated with a labeled oriented graph $P$ comes a LOG-presentation, a LOG-complex $K(P)$, the standard 2-complex associated with the presentation, and a LOG-group $G(P)$, the group defined by the presentation. We say a labeled oriented graph is *aspherical* if its associated LOG-complex is aspherical. A labeled oriented tree (LOT) is a labeled oriented graph where the underlying graph is a tree. A labeled oriented interval (LOI) is a labeled oriented graph where the underlying graph is an interval. A labeled oriented forest (LOF) is a labeled oriented graph where the underlying graph is a forest. A *sub-LOG* $Q$ of a labeled oriented graph $P$ is a subgraph of $P$ such that each edge label of $Q$ is a vertex label of $Q$. A sub-LOG $Q$ is *proper* if it contains at least one edge but is not all of $P$. A labeled oriented graph is *prime* if it does not contain a connected proper sub-LOG. It is called *injective* if each vertex label occurs at most once as an edge label.

A labeled oriented graph is called *compressed* if every edge contains 3 different labels. It is called *boundary reducible* if there is a boundary vertex label that does not occur as edge label and *boundary reduced* otherwise. A labeled oriented graph is called *interior reducible* if there is a vertex with two adjacent edges with the same label that either point away or towards that vertex. A labeled oriented graph which is boundary reduced, interior reduced and compressed is called *reduced*. Any labeled oriented tree can be transformed into a reduced labeled oriented tree. The homotopy type of the associated 2-complex remains unchanged under this transformation.

If a labeled oriented graph arises from a knot diagram then all the definitions just given can be interpreted in terms of the knot. For example injective corresponds to alternating, performing compressions and interior reductions correspond to performing certain Reidemeister moves in the knot diagram.

Let $K$ be a finite 2-complex. In this article $K$ will always be the standard 2-complex of a finite presentation. A *surface diagram* over $K$ is a piecewise linear map $f\colon C \to K$, where $C$ is a cell decomposition of a closed orientable surface and $f$ carries open cells of $C$ homeomorphically to open cells of $K$. If $C$ is a 2-sphere then $f$ is called a *spherical diagram*. A cell of $C$ will be labeled by the cell of $K$ it maps to under $f$. The 1-cells also get their orientation from the 1-cells of $K$. In this way $C$ itself carries all the information of the map $f\colon C \to K$ and we often speak of the "diagram $C$". A surface diagram $f\colon C \to K$ is called *reducible* if there is a pair of 2-cells in $C$ having a boundary edge $t$ in common and being mapped by $f$ onto the same 2-cell in $K$ by folding over $t$. The surface diagram is called *reduced* if it is not reducible. A 2-complex $K$ is called *diagrammatically reducible* (DR) if each spherical diagram over $K$ is reducible (or, equivalently, if there does not exist a reduced spherical diagram over $K$). A labeled oriented graph is called DR if the corresponding standard-2-complex is DR. A DR 2-complex is aspherical.

The starting point for this article is the following result.

**Theorem 1.1** *(Huck/Rosebrock 2001 [6]): If a labeled oriented forest is compressed and injective and does not contain a boundary reducible sub-LOT, then it is DR. In particular, an injective compressed prime labeled oriented forest is DR.*

We show that this result does not hold if one removes the injectivity hypothesis.

**Theorem 1.2** *There does exist a reduced prime labeled oriented interval that is not DR.*

Examples of reduced non-prime labeled oriented trees that are not DR have been constructed before (see [9]). Here we produce the first examples that are prime.
We believe that the prime condition in Theorem 1.1 is not needed. In particular, we conjecture that all injective labeled oriented trees are aspherical. The best we can do so far is the following:

**Theorem 1.3** *If the boundary vertices of each proper interior boundary reducible sub-LOI $Q$ of an injective compressed labeled oriented interval $P$ generate a free group of rank 1 or 2 in $G(Q)$, then $P$ is aspherical.*

Here $Q$ being *interior* means that the boundary vertices of $Q$ are interior vertices of $P$.
We thank the referee for his helpful comments.

# 2 Proof of Theorem 1.2

Given a surface diagram $f\colon M \to K$ over a LOG-complex $K$ we can draw its dual by replacing the square 2-cells by crossings. We undercross when labels change. The process is depicted for a single 2-cell of $M$ in Figure 1. We de-

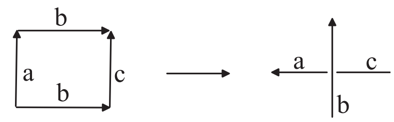


Figure 1: Dualizing a surface diagram

fine an orientation of the link by requiring that when traveling along the link we encounter the 1-cells as pointing to the left. This leads to an oriented link projection $L$ on the surface $M$ which contains all the information of the diagram.

Consider the reduced prime labeled oriented interval $P$ shown in Figure 2. Figure 3 depicts a link projection on a 2-sphere that is a reduced spherical diagram over $P$. Thus $P$ is not DR. This proves Theorem 1.2. □

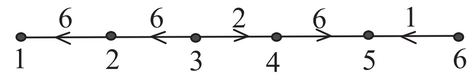


Figure 2: A non-DR LOI

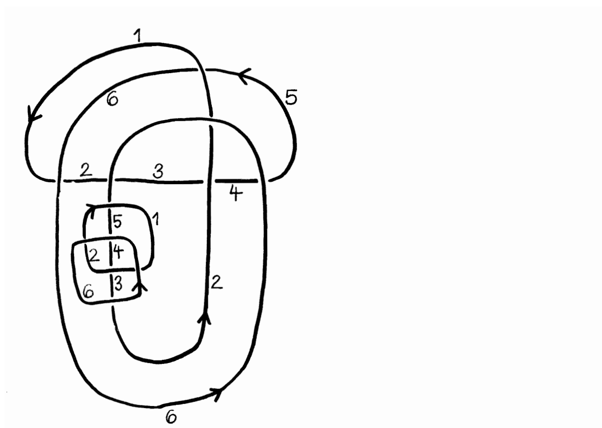

Figure 3: Spherical diagram

This example is the smallest known non-DR example over a reduced LOI. It is not difficult to see that the 2-complex $K(P)$ collapses via simple homotopy moves to a 2-complex with a single 2-cell. Thus $K(P)$ is aspherical (see Lyndon [7]). It can also be seen that $G(P)$ is infinite cyclic. The diagram was found by computer search, as were many others. See [12] for more labeled oriented trees that are not DR.

# 3 Proof of Theorem 1.3

A simple way to construct labeled oriented graphs that contain interior sub-LOIs is the following: Let $P'$ be a labeled oriented graph and let $Q$ be a labeled oriented interval with boundary vertices $b$ and $b'$. Cut $P'$ at an interior vertex $a$ such that there are two vertices $a$ and $a'$ coming from cutting at $a$, and such that $a'$ has valence 1 in the resulting labeled oriented graph $P''$. Edge labels are not changed. Identify $b$ with $a$ and $b'$ with $a'$. Edges in $Q$ labeled $b$ and $b'$ are relabeled $a$ and $a'$, respectively. This produces a labeled oriented graph $P$ that contains $Q$ as a sub-LOT. Since $P'$ is obtained from $P$ by collapsing $Q$ we use the more intuitive notation $P' = P/Q$ and $P'' = P - Q$. We say that $P$ *is obtained from* $P/Q$ *by inserting the labeled oriented interval* $Q$.

**Lemma 3.1** *If* $P/Q$ *is a labeled oriented graph that is DR, then* $P - Q$ *is DR as well.*

**Proof:** Note that there is no edge in $P - Q$ labeled by $a'$. Thus removing the edge $e$ that contains the boundary vertex $a'$ gives a labeled oriented graph $R$ that is a sub-LOG of $P/Q$ and $K(R)$ is obtained from $K(P - Q)$ by collapsing one 2-cell. As a subcomplex of the DR complex $K(P/Q)$, $K(R)$ is DR. And since $K(R)$ is obtained from $K(P-Q)$ by collapsing a 2-cell, the complex $K(P-Q)$ is also DR. Indeed, any spherical diagram $f: C \to K(P - Q)$ having the 2-cell corresponding to $e$ in its image is reducible along an edge labeled $a'$ and $f$ is a diagram over the DR complex $K(R)$ otherwise. □

**Theorem 3.2** *Let* $Q$ *be a labeled oriented interval with boundary vertices* $b$ *and* $b'$*. Assume that these vertices generate a free group of rank one or two in* $G(Q)$*. Let* $P$ *be obtained from a labeled oriented forest* $P/Q$ *by inserting* $Q$*. If* $P/Q$*,* $P - Q$*, and* $Q$ *are aspherical, then so is* $P$*.*

**Remark:** If $P/Q$ in the the above theorem is assumed to be DR, then asphericity of $P - Q$ is implied by Lemma 3.1.

We give a proof of Theorem 3.2 after recalling needed results on amalgamated products. Suppose $A$ and $B$ are groups and $C$ is a subgroup of both $A$ and $B$. We can build a presentation for the amalgamated product $G = A *_C B$ in the following way. Choose presentations $\langle X \mid R\rangle$ and $\langle X' \mid R'\rangle$ for $A$ and $B$, respectively. Choose a presentation $\langle Y \mid S\rangle$ for $C$ and denote by $t_y$,and $t'_y$ words in $X$ and $X'$ that represent $y \in C$ as an element of $A$ and $B$, respectively. Then $\langle X, X' \,|R, R', t_y = t'_y, y \in Y\rangle$ presents the amalgamated product $G$. Let $K$ be the standard 2-complex associated with this presentation. Note that an element $s \in S$ gives rise to a word $w_s$ in the $t_y$ that represents the trivial element in $A$. Hence we can construct a Van Kampen diagram $D_s$ over $\langle X \mid R\rangle$ with boundary word $w_s$. Similarily, we construct a diagram $D'_s$ over $\langle X' \mid R'\rangle$ with boundary word $w'_s$. We can attach 2-cells with boundary words $t_y = t'_y$ to $D_s$ to obtain a diagram with boundary word $w'_s$. If we glue it to the diagram $-D'_s$ we obtain a spherical diagram $E_s$ over $K$. The following result can be found in Baik, Pride [1].

**Lemma 3.3** *The second homotopy module* $\pi_2(K)$ *is generated by spherical diagrams over* $\langle X \mid R\rangle$*, spherical diagrams over* $\langle X' \mid R'\rangle$*, together with the spherical diagrams* $E_s$*,* $s \in S$*.*

Let $P$ be a labeled oriented forest with components $C_1, ..., C_n$. For every component $C_i$ choose a vertex $a_i$ contained in $C_i$. Let $H$ be the subgroup generated by the vertices $a_1, ..., a_n$ in $G(P)$.

**Lemma 3.4** *The subgroup $H$ is free of rank $n$.*

**Proof:** The result is a consequence of Stallings' Theorem [13]. First note that the abelianized group $G(P)_{ab}$ is free abelian of rank $n$ and that the $a_i$ form a basis. Furthermore, since $K(P)$ is a subcomplex of a contractible 2-complex, we have $H_2(K(P)) = 0$ and thus $H_2(G(P)) = 0$. Let $\phi\colon F = F(x_1, ..., x_n) \to G(P)$ be the homomorphism of the free group on $x_1, ..., x_n$ to $G(P)$ that sends $x_i$ to $a_i$. This homomorphism induces an isomorphism on the first and second homology and hence, by Stallings, we have an isomorphism $\bar{\phi}\colon F/F_m \to G(P)/G(P)_m$ for all $m \geq 1$. Here $G_m$ is the $m$-th term in the lower central series (that is $G_1 = [G, G]$ and, inductively, $G_k = [G, G_{k-1}]$). Since the free group $F$ is residually nilpotent (see [8], Proposition 3.3, page 14) that is $\bigcap_{i=1}^{\infty} F_i = 1$, we see that $\phi$ is injective. □

Let $P$ be a labeled oriented tree obtained from a labeled oriented tree $P/Q$ by inserting a labeled oriented interval $Q$ (see the beginning of this section). Let $b$ and $b'$ be the boundary vertices of $Q$. Let $N$ be the kernel of the homomorphism $F(b, b') \to G(Q)$ that sends $b$ to $b$ and $b'$ to $b'$ and let $S$ be a set of normal generators for $N$. Let $H = \langle b, b' \mid S\rangle$. Note that the map $H \to \pi_1(K(Q)) = G(Q)$ is injective. Note further that $S$ is a set of words in $\{b, b'\}^{\pm 1}$. Let $S'$ be the corresponding set of words in $\{a, a'\}^{\pm 1}$ obtained by replacing each $b$ by $a$ and each $b'$ by $a'$ and let $H' = \langle a, a' \mid S'\rangle$. Let $L(P - Q)$ be the standard 2-complex obtained from $K(P - Q)$ by adding 2-cells with boundary words $s' \in S'$.

**Lemma 3.5** *If both $L(P - Q)$ and $K(Q)$ are aspherical and the map $H' \to \pi_1(L(P - Q))$ is injective, then $K(P)$ is aspherical.*

**Proof:** Glue $K(Q)$ to $L(P - Q)$ by identifying the edges $b = a$ and $b' = a'$ and denote the result by $K(Q) \cup L(P - Q)$. By the Van-Kampen Theorem we have $\pi_1(K(Q) \cup L(P - Q)) = \pi_1(K(Q)) *_H \pi_1(L(P - Q))$, where $H = \langle b, b' \mid S\rangle = \langle a, a' \mid S'\rangle$ is the subgroup generated by the two loops in the intersection. Note that $K(Q) \cup L(P - Q)$ is a standard 2-complex built for the amalgamated product as described before we stated Lemma 3.3. Hence that lemma applies and, because we assumed that both $K(Q)$ and $L(P - Q)$ are aspherical, we see that $\pi_2(K(Q) \cup L(P - Q))$ is generated by spherical diagrams $E_s$, $s \in S$. Attach a 3-cell to $K(Q) \cup L(P - Q)$ for each $s' \in S'$ to obtain a 3-complex with trivial second homotopy group. Note that each of these 3-cells contains the 2-cell with boundary word $s'$ exactly once. Performing simple homotopy moves on this 3-complex by first collapsing the 3-cells across the 2-cells with boundary word $s'$ and then collapsing the 2-cells with boundary word $a = b$ and $a' = b'$ across the edges $b$ and $b'$, respectively, we obtain the 2-complex $K(P)$. Thus $\pi_2(K(P)) = 0$. □

**Proof of Theorem 3.2:** Assume first that the subgroup $H = \langle b, b'\rangle$ of $G(Q)$ is free of rank two where $b, b'$ are the boundary vertices of $Q$. Then $N$, the kernel

of the map $F(b, b') \to H$ is trivial, hence $S$ and $S'$ can be chosen to be empty. So $L(P-Q) = K(P-Q)$. Thus $L(P-Q)$ and $K(Q)$ are aspherical. Note that $H' = \langle a, a' \mid - \rangle$ is free of rank 2, hence the map $H' \to \pi_1(L(P-Q)) = \pi_1(K(P-Q))$ is injective by Lemma 3.4. Now Lemma 3.5 implies the result.
Next assume that $H$ is free of rank one. Then $H \to G(Q) \to G(Q)_{ab}$ is an isomorphism since $G(Q)_{ab}$ is free of rank one, generated by any vertex in $Q$, in particular the vertex $b$. Moreover, since $b = b'$ in $G(Q)_{ab}$, we have $b = b'$ in $H$. Thus we can take $S = \{bb'^{-1}\}$ and $S' = \{aa'^{-1}\}$. In $L(P-Q)$ we collapse the 2-cell with boundary word $a = a'$ across the edge $a'$. This simple homotopy move turns $L(P-Q)$ into $K(P/Q)$. Since we assumed $K(P/Q)$ to be aspherical, we see that $L(P-Q)$ is aspherical. The complex $K(Q)$ is aspherical by hypothesis. Consider the composition

$$H' = \langle a, a' \mid a = a' \rangle \to \pi_1(L(P-Q)) \to \pi_1(K(P/Q)) \to H_1(K(P/Q)).$$

Now $H_1(K(P/Q)) = G(P/Q)_{ab}$ is cyclic, generated by any vertex in $P/Q$, in particular by the vertex $a$. This shows that the composition is an isomorphism, and hence that $H' \to \pi_1(L(P-Q))$ is injective. Now Lemma 3.5 implies the result. $\square$

**Proof of Theorem 1.3:** Let $\mathcal{W}$ be the set of all injective compressed labeled oriented forests where the underlying graph is a disjoint union of intervals, and where the boundary vertices of all proper interior boundary reducible sub-LOIs $Q$ generate a free group in $G(Q)$. Note that if $P'$ is a sub-LOF of $P \in \mathcal{W}$, then $P' \in \mathcal{W}$.
We first show that if $Q$ is a maximal proper interior boundary reducible sub-LOI in $P \in \mathcal{W}$, then $P/Q \in \mathcal{W}$. The labeled oriented forest $P/Q$ is obtained by collapsing $Q$ in $P$ to a single vertex. If $a, a'$ are the boundary vertices of $Q$, then at least one of them, say $a'$, does not occur as an edge label in $P-Q$. Label the vertex that $Q$ collapses to with the letter $a$. The labeled oriented forest $P$ is obtained from $P/Q$ by inserting $Q$. Now suppose that $Q'$ is a proper interior boundary reducible sub-LOI of $P/Q$. If $Q'$ contains the vertex $a$, then inserting $Q$ leads to a proper interior boundary reducible sub-LOI $Q''$ in $P$ that contains $Q$. This contradicts maximality of $Q$. Hence $Q'$ does not contain $a$ and hence is also a proper interior boundary reducible sub-LOI of $P$. It follows that the boundary vertices of $Q'$ generate a free group in $G(Q')$. This shows that $P/Q$ is contained in $\mathcal{W}$.
We will show that every $P$ in $\mathcal{W}$ is aspherical by induction on $v(P)$, the number of vertices. If $v(P) = 1$, then $P$ consists of a single vertex and hence is aspherical. Assume that $v(P) = n > 0$. If $P$ does not contain a proper boundary reducible sub-LOI, then it is aspherical by Theorem 1.1. If $P$ does contain a proper boundary reducible sub-LOI, then it contains a maximal such sub-LOI $Q$. We first address the case where $Q$ is not interior. Then $P$ is the union of $Q$ and $P-Q$, the intersection being a vertex $a$. Both $Q$ and $P-Q$ are contained in $\mathcal{W}$ and contain fewer that $n$ vertices. It follows that both $Q$ and $P-Q$ are

aspherical, in particular torsion-free. Hence the vertex $a$ generates an infinite cyclic group in both $G(Q)$ and $G(P-Q)$. Asphericity of $P$ now follows from Lemma 3.3. Next we consider the case where $Q$ is interior. We know that $Q$, $P-Q$, and $P/Q$ are all contained in $\mathcal{W}$, each containing fewer than $n$ vertices. It follows by induction that $Q$, $P-Q$ and $P/Q$ are all aspherical. Since the boundary vertices of $Q$ generate a free subgroup of $G(Q)$ it follows from Theorem 3.2 that $P$ is aspherical. □

# 4 Examples

It is easy to construct examples, where the boundary of $Q$ generates a free group of rank 1. This is the case for example if $Q$ is a presentation of a knot. In fact in all known examples of non-prime reduced labeled oriented trees that are not DR, the sub-LOT $Q$ does come from a knot (see [9]). So we know that these labeled oriented trees are aspherical but not DR.

It is also not difficult to give examples of labeled oriented intervals where the boundary vertices generate a free group of rank two. Note that the 2-complex associated with a labeled oriented graph is a square complex, that is every 2-cell has four edges in its boundary. A square complex is called *non-positively curved* if the vertex link does not contain edge cycles of length shorter than four.

**Theorem 4.1** *Let $P$ be a compressed labeled oriented tree whose 2-complex $K(P)$ is a non-positively curved square complex. Let $a, b$ be vertices in $P$ that do not appear in the same relation. Then $\langle a, b\rangle$ is free of rank 2.*

**Proof:** Suppose $\langle a, b\rangle$ is not free of rank 2. Then there exists a reduced word $w$ in $\{a, b\}^{\pm 1}$ that represents the trivial element in $G(P)$. So there is a van Kampen diagram $M$ with boundary word $w$. Since $K(P)$ is non-positively curved each inner vertex of $M$ has valence at least 4, thus the curvature at every inner vertex is not positive. Since the Euler characteristic of $M$ is one, we have to have a vertex of positive curvature on the boundary by the combinatorial Gauss-Bonnet Theorem. Thus two consecutive 1-cells in the boundary of $M$ are part of the same 2-cell of $M$. Since $P$ is compressed these two 1-cells cannot have the same label. They cannot be labeled by $a$ and $b$ since $a, b$ do not appear in the same relation. This is a contradiction since $w$ consists of $a, b$ and their inverses only.□

In [10], Figure 6, conditions on a labeled oriented tree $P$ are given that imply that the corresponding LOT presentation is C(4)-T(4). These combinatorial conditions imply that the complex $K(P)$ is non-positively curved. Example 7 in [10] is an example of labeled oriented tree whose 2-complex is non-positively curved.

There are also examples, where the boundary of the sub-LOI does not generate a free subgroup. Consider the labeled oriented interval $Q$ that gives rise to the presentation $\langle a, b, c \mid ac = cb, ca = ab\rangle$. Note that the group $G(Q)$ is generated by the boundary vertices $a$ and $c$ because $b = a^{-1}ca$. The group $G(Q)$ is not free of rank one ore two. Indeed, $G(Q)$ is presented by $\langle a, c \mid a^{-1}ca = c^{-1}ac\rangle$. If we set $x = a^{-1}c$ we get the presentation $\langle a, x \mid axa^{-1} = x^2\rangle$, thus $G(Q)$ is isomorphic to the Baumslag-Solitar group $B(1, 2)$.
Note that the labeled oriented interval $Q$ above is reduced. We can add a vertex $d$ and an edge from $c$ to $d$ labeled by $b$ to obtain a compressed, boundary reducible labeled oriented interval $Q'$. Note that the groups $G(Q)$ and $G(Q')$ are isomorphic and that the subgroup generated by $a$ and $d$ is not free of rank one, and certainly not free of rank two because $B(1, 2)$ is solvable and hence does not contain free subgroups that are not cyclic.

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Jens Harlander
Department of Mathematics
Boise State University
1910 University Drive, Boise, ID 83725-155

jensharlander@boisestate.edu

Stephan Rosebrock
Institut für Mathematik und Informatik
Pädagogische Hochschule Karlsruhe
Postfach 111062
76060 Karlsruhe, Germany
rosebrock@ph-karlsruhe.de