A Remark on the Polyhedrality Theorem for the Σ-Invariants of Modules over Abelian Groups

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1 Introduction

1.1 Background. Throughout this note Q stands for a finitely generated multiplicative Abelian group of torsion-free rank n, R for a commutative ring with 1, and M for a finitely generated RQ-module. The **geometric invariant** Σ_M of M was introduced in [BS 80/81]. It can be viewed as a subset of the $\mathbb{R}$-vector space of all (additive) charakters of Q, $Q^* = \mathrm{Hom}(Q, \mathbb{R}) \cong \mathbb{R}^n$, as follows: For every character $\chi : Q \to \mathbb{R}$ one considers the submonoid $Q_\chi = \{q \in Q \mid \chi(q) \geq 0\}$ of Q and puts

$$\Sigma_M := \{\chi \mid M \textit{ is finitely generated over } RQ_\chi\}.$$

Note that $0 \in \Sigma_M$. It is often convenient to work with the complement Σ_M^c of Σ_M in Q^*.

The geometric invariant Σ_M has been investigated for two reasons. Firstly, if R is a Dedekind domain, then Σ_M turns out to be a polyhedral[1] subset of Q^*. This rather subtle fact was conjectured, for R a field, by G.M. Bergman [Be 71] and established by John R.J. Groves and the first author in [BG 84]; it opens the possibility for computations and imposes arithmetic restrictions on automorphisms of M. Secondly, for $R = \mathbb{Z}, \Sigma_M$ contains interesting information on the (metabelian) groups G which are extensions[2] of M by Q. In [BS 80] it is proved that G has a finite presentation if and only if $\Sigma_M \cup -\Sigma_M = Q^*$. A number of attempts have been undertaken to extend this result to a charakterization of the higher dimensional finiteness property

[1] i.e., a finite union of finite intersection of (open) vector half spaces.

[2] i.e., G fits into a short exact sequence $M \rightarrowtail G \twoheadrightarrow Q$.

that G be of type[3] FP_m for $m > 2$, and they all revolve around the following

***FP*$_{\mathrm{m}}$*− Conjecture:*[4]** *G is of type FP_m if and only if $0 \in Q^*$ is not in the convex hull of m points of Σ_M^c.*

1.2 The purpose of this short note is to establish a stronger version of the Polyhedrality Theorem for Σ_M of [BG 84]. This sheads new light on the tomography method introduced in [BG 84]. Our result can be applied in the proof of the FP_3−Conjecture for the semi-direct product $G : M \rtimes Q$ which will appear in a subsequent paper — though, just as in the proof of the FP_2−Theorem in [BS 80], one can get away with argument using the compactness of the sphere of directions in $\mathrm{Hom}(Q, \mathbb{R})$ to prove this FP_3−result.

To state our result we recall that the invariant Σ_M has the following description in terms of the centralizer[5] $C(M)$ of M in the group ring RQ. For each $\lambda \in RQ$ we consider the *support* of λ in Q , denoted supp(λ); this is the set of all $q \in Q$ with non-zero coefficient in λ. If $\chi \in Q^*$ we put

$$\chi_*(\lambda) := \inf \chi(\mathrm{supp}(\lambda)),$$

which we interprete as ∞ if $\lambda = 0$. Proposition 2.1 of [BS 80] asserts that

$$\Sigma_M = \bigcup_{\lambda \in C(M)} \{\chi \mid \chi_*(\lambda) > 0\}\,. \tag{1}$$

We improve this to

Theorem. *If R is a Dedekind domain and M a finitely generated RQ-module then there is a finite set of centralizers $\Lambda \subseteq C(M)$ with*

$$\Sigma_M = \bigcup_{\lambda \in \Lambda} \{\chi \mid \chi_*(\lambda) > 0\}\,.$$

Remarks. 1) Formally the Theorem implies immediately that Σ_M is a rational polyhedral subset of Q^*, i.e., the Polyhedrality Theorem. However, we will make heavy use of both the methods and some preliminary results of [BG 84].

2) Our Theorem seems to be implicitely contained in the paper [BrG 98] which defines Σ−invariants in a more general setting.

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[3] A group of G is of type FP_m, if the trivial G-module $\mathbb{Z}$ admits a free resolution $\mathbf{F} \twoheadrightarrow \mathbb{Z}$ with finitely generated m-skeleton. For metabelian groups G it is known, by [BS 80], that FP_2 is equivalent to finite presentability.

[4] The conjecture appeared in [BG 82] and [Bi 81].

[5] $C(M) = \{\lambda \in RQ \mid \lambda m = m \text{ for all } m \in M\}$.

2 Reduction steps

2.1 Reduction to M cyclic. Throughout the remainder of the paper we write $I = \mathrm{Ann}_{RQ}(M)$ for the annihilator ideal of M in RQ. Since $C(M) = 1 + I$ it is clear from (1) that $\Sigma_M = \Sigma_{RQ/I}$ depends only on the annihilator ideal of M and hence we may as well assume that $M = RQ/I$.

2.2 Reduction to the case when $I = P$ prime. From now on we assume the ring R is Noetherian. Then RQ is also Noetherian so that there are only finitely many prime ideals $P_1, P_2, ..., P_k$ in RQ which are minimal over I. By [BS 81], Theorem 1.1, we know that

$$\Sigma_M = \Sigma_{RQ/P_1} \cap ... \cap \Sigma_{RQ/P_k}.$$

Assume now the Theorem holds for each $M_i = RQ/P_i$, and write Λ_i for the corresponding finite subset of $C(M_i) = 1 + P_i$. Then, given any $\chi \in \Sigma_M$, we find centralizers $\lambda \in \Lambda_i$ with $\chi_*(\lambda_i) > 0$ for each $i = 1, ..., k$. The product $\mu = \prod_{i=1}^{k}(1 - \lambda_i)$ is in the intersection $P_1 \cap ... \cap P_k = \sqrt{I}$ and hence there is some $m \in \mathbb{N}$, depending only on $\lambda_1, ..., \lambda_k$, with $\mu^m \in I$. It follows that $\lambda = 1 - \mu^m \in C(M) = 1 + I$ with $\chi_*(\lambda) > 0$.

Since each Λ_i is finite, one has a uniform choice for the exponent m and finds a finite subset $\Lambda \subseteq C(M)$ containing all the elements λ as constructed above. This shows that the assertion of the Theorem holds for M.

2.3 Reduction to $R \cap P = 0$. Let us write Σ_M^R for Σ_M in order to emphasize the ground ring. If $\mathfrak{a}$ is an ideal of R with $\mathfrak{a}M = 0$ then M can be viewed as an $(R/\mathfrak{a})Q$-module, and $\Sigma_M^R = \Sigma_M^{R/\mathfrak{a}}$. Moreover, every $\overline{\lambda} \in (R/\mathfrak{a})Q$ centralizing M can be represented by some $\lambda \in C(M)$ with $\mathrm{supp}(\lambda) = \mathrm{supp}(\overline{\lambda})$. This shows that it suffices to prove the Theorem for the case when M is torsion free as an R-module, i.e. $R \cap P = 0$.

2.4 Reduction to $Q \cap C(M) = 1$ Let us write $\Sigma_M(Q)$ for Σ_M in order to emphasize the group Q. If $Z := Q \cap C(M)$ then M can be viewed as an $R(Q/Z)$-module. We identify $(Q/Z)^*$ with the subspace $\{\chi | \chi(z)\} = 0$ of Q^* and observe, using (1), that the complement of $(Q/Z)^*$ in Q^* is contained in Σ_M. In fact,

$$\Sigma_M(Q) = \Sigma_M(Q/Z) \cup (Q^* - (Q/Z)^*).$$

Using generators of Z as particular centralizers $\lambda \in C(M)$ one observes readily that it suffices to prove the Theorem for Q/Z ; hence we may assume $Q \cap C(M) = 1$.

We are now reduced to prove the Theorem in the case when the RQ-module M has the form $M = RQ/P$ with P a prime ideal of RQ, and both R and Q are embedded in M. In others words, $A = RQ/P$ is a domain, $R \subseteq A$ a subring and $Q \subseteq U(A)$ a group of units of a which generates A as an R-algebra, $A = R[Q]$.

3 The Tomography Lemma

3.1 Characters induced by valuations. As above $A = RQ/P$ is an domain containing both R and Q. Let $k \subseteq K$ denote the field of fractions of $R \subseteq A$. Following [BG 84] we fix a valuation[6] $v : R \to \mathbb{R}_\infty$ and write $\Delta_A^v(Q) \subseteq Q^*$ for the set of all characters on Q which are induced by a valuation on A which extends v, i.e.,

$$\Delta_A^v(Q) = \{\chi = w|_Q \mid w : A \to \mathbb{R}_\infty, \text{ with } w|_R = v\}.$$

For simplicity we assume that $v^{-1}(\infty) = 0$, so that v can also be regarded as a valuation on the field of fractions k. It is then non-obvious but a very convenient fact that to compute $\Delta_A^v(Q)$ one can restrict attention the valuations w on the field of fractions K of A, i.e., we have

Proposition 1 $\qquad \Delta_A^v(Q) = \Delta_K^v(Q).$

This is Corollary 6.3 of [BG 84]. Whereas it was crucial, in the proof of polyhedrality in [BG 84], to use the "normalized" sets $\Delta_K(Q)$, it will be more convenient, for the purpose of this paper, to use

$$\Delta_K^R(Q) := \bigcup_v \{\Delta_K^v(Q) \mid v(R) \geq 0\}$$

Note that if v is a valuation on R so is every positive multiple of v; hence $\Delta_K^R(Q)$ is a conical subset of Q^*. From Theorem 8.1 of [BG 84] and Proposition 3.1 above we find

Proposition 2 $\qquad \Sigma_A^c = \Delta_K^R(Q). \quad \square$

3.2 Tomography is a good descriptive name for the geometric method introduced in § 4 of [BG 84] as the tool to prove polyhedrality and related results on $\Delta_A^v(Q)$: One observes that for every subgroup $U \leq Q$ the restriction map $\mathrm{res}_U : Q^* \to U^*$ maps $\Delta_A^v(Q)$ onto $\Delta_{R[U]}^v(U)$, since every valuation

[6] A valuation on a commutative ring R is a function $v : R \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ satisfying $v(ab) = v(a) + v(b)$ and $v(a + b) \geq \inf\,(v(a), v(b))$ for all $a, b \in R$.

on the subfield $k(U)$ extends to a valuation on K. The key to the polyhedrality of $\Delta^v_A(Q)$ is then the following combination of Theorem 4.4 and Lemma 5.1 in [BG 84]:

Lemma 3 (Tomography Lemma). *Let*[7] m=trd(K/k). *There is a finite set $\mathcal{F}$ of direct factors of Q, each $U \in \mathcal{F}$ free Abelian of rank $m+1$ and with K finite over $k(U)$, such that*

$$\Delta^v_K(Q) = \bigcap_{U \in \mathcal{F}} \operatorname{res}_U^{-1} \Delta^v_{k(U)}(U).$$

We will need the global version of this for the conical sets $\Delta^R_K(Q)$. If R is a Dedekind domain all valuations $v : k \to \mathbb{R}_\infty$ with $v(R) \geq 0$ are equivalent to a normalized $\mathfrak{p}$-adic valuation $v_\mathfrak{p} : k \to \mathbb{R}_\infty$ for a prime ideal $\mathfrak{p}$ of R. Theorem B of [BG 84] asserts that there are only finitely many prime ideals $\mathfrak{p}$ with $\Delta^{v_\mathfrak{p}}_K(Q) \neq \Delta^0_K(Q)$. Hence applying the Tomography Lemma to this finite number of valuations $v_\mathfrak{p}$ and taking the union of the corresponding finite sets of direct factors yields a new finite set $\mathcal{F}$ of direct factors of Q, with each $U \in \mathcal{F}$ free Abelian of rank $m+1$ and K finite over k(U), such that

$$\Delta^R_K(Q) = \bigcap_{U \in \mathcal{F}} \operatorname{res}_U^{-1} \Delta^R_{k(U)}.$$

Passing to the complements in Q^* and U^* and referring to Proposition 2 yields

Lemma 4 (Tomography Lemma for Σ_A). *Let $R \subseteq A = RQ/P$ domains and $k \subseteq K$ their fields of fractions as above, with m =trd(K/k). If R is a Dedekind domain then there is a finite set $\mathcal{F}$ of direct factors of Q, each $U \in \mathcal{F}$ free Abelian of rank $m+1$ and with K finite over $k(U)$, such that*

$$\Sigma_A(Q) \;=\; \bigcup_{U \in \mathcal{F}} \operatorname{res}_U^{-1} \Sigma_{R[U]}(U).$$

Remark. $\operatorname{res}_U^{-1} \Sigma_{R[U]}(U)$ can be interpreted as the geometric invariant $\Sigma_B(Q)$ of the induced RQ-module $B = RQ \otimes_{RU} R[U]$. Since $\operatorname{Ann}_{RU}(R[U]) = P \cap RU$ and $\operatorname{Ann}_{RQ}(B) = RQ(P \cap RU)$ this follows easily by using formula (1) of § 1.

[7] trd(K/k) stands for the transcendence degree of K over k.

4 One relator modules

4.1 The conclusion of the Theorem will follow from the observation

Proposition 5. *Let $R \subseteq A = RQ/P$ be the domains above, with fields of fractions $k \subseteq K$ and m=trd(K/k). Assume that Q is free Abelian of rank $m+1$ and that R is a unique factorization domain. Then the prime ideal P is principal.*

Proof. Any basis of Q contains a transcendence basis of K over k. Hence Q has a direct product decomposition $Q = H \times gp(X)$, where H is of rank m and K is finite over $k(H)$. $\tilde{R} := RH$ is a unique factorization domain and embeds in $A = RQ/P$, i.e. $RH \cap P = 0$. We can apply the Gauss Lemma to $A \cong \tilde{R}[X]/P$ and find that P is generated by a primitive minimal polynomial in $\tilde{R}[X] = RQ$. □

Proposition 5 shows that the Tomography Lemma reduces computation of Σ_A to computing Σ_B for a finite number of cyclic one-relator modules $B = RQ/RQ(P \cap RU)$. But this is covered by Theorem 5.2 of [BS 81]. For the convenience of the reader we state this result in the case of a one-relator module $B = RQ/RQ\mu$. For this we interprete the element $\mu \in RQ$ as a function $\mu : Q \to R$ with finite support supp(μ), $\mu = \Sigma\mu(q)q$. We call an element $q \in$ supp(μ) a ***corner*** of μ if there is some $\chi \in Q^*$ with $\chi(q) < \chi(q')$ for all $q' \in$ supp $(\mu) - \{q\}$. Each corner of μ gives rise to an element $q^{-1}\mu - \mu(q) \in RQ$ with $\chi_*(q^{-1}\mu - \mu(q)) > 0$, and, if $\mu(q) \in R$ is a unit, to a centralizer

$$\lambda_q := \mu(q)^{-1}(q^{-1}\mu - \mu(q)) \ \in C(B),$$

with $\chi(\lambda_q) > 0$ for some $\chi \in Q^*$.

Proposition 6.([BS 81], Theorem 5.2) *The geometric invariant of the cyclic 1-relator module $B = RQ/RQ\mu$ is given by*

$$\Sigma_B = \bigcup_q \{\chi \mid \chi(\lambda_q) > 0\}$$

where q runs through all corners of μ with $\mu(q) \in R$ a unit in R. □

4.2 Conclusion. The conjunction of Lemma 4 and Propositions 5 and 6 establishes our Theorem in the case when the Dedekind domain R has the unique factorization property. This includes, of course, the most important case $R = \mathbb{Z}$.

In general Dedekind domains do not have unique factorization, so that the ideal P of Proposition will not be principal. However, in this situation P will always be of the form $P = RQ \cdot J \cdot \mu$ where $J \subseteq k$ is a fractional ideal in the field of fractions k of R; and Theorem 5.2 of [BS 81] does apply in this generality. (See [BS 81], § 5.3). This establishs our Theorem in full.

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