

# Injective Labeled Oriented Trees are Aspherical

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December 3, 2015

## Abstract

A labeled oriented tree is called *injective*, if each vertex occurs at most once as an edge label. We show that injective labeled oriented trees are aspherical. The proof uses a new relative asphericity test based on a lemma of Stallings.

MSC: 57M20, 57M05, 20F05

Keywords: labeled oriented tree, Wirtinger presentation, 2-complex, asphericity

## 1 Introduction

This article is concerned with the Whitehead conjecture, which states that a subcomplex of an aspherical 2-complex is aspherical. See Bogley [1] and Rosebrock [11] for surveys. The conjecture originally arose in the context of knot theory. The Wirtinger presentation of a knot gives rise to a 2-complex that is a subcomplex of a contractible 2-complex. Thus, an affirmative answer to the conjecture implies the asphericity of knot complements in the 3-sphere. Labeled oriented trees give rise to presentations that generalize Wirtinger presentations for knots, and presentations obtained from injective labeled oriented trees generalize Wirtinger presentations of alternating knots. Labeled oriented trees play a central role in understanding the Whitehead conjecture. Howie [6] showed that the finite case of the Whitehead conjecture reduces, up to the Andrews-Curtis conjecture, to the statement that

presentations arising from labeled oriented trees are aspherical.

A *labeled oriented graph* (LOG) is an oriented graph  $\mathcal{G}$  on vertices  $\mathbf{x}$  and edges  $\mathbf{e}$ , where each oriented edge is labeled by a vertex. Associated with it is the *LOG-presentation*  $P(\mathcal{G}) = \langle \mathbf{x} \mid \{r_e\}_{e \in \mathbf{e}} \rangle$ . If  $e$  is an edge that starts at  $x$ , ends at  $y$  and is labeled by  $z$ , then  $r_e = xz(zy)^{-1}$ . We also use the notation  $e = [x, z, y]$ . A *LOG-complex*  $K(\mathcal{G})$  is the standard 2-complex associated with the LOG-presentation  $P(\mathcal{G})$ , and a *LOG-group*  $G(\mathcal{G})$ , is the group defined by the LOG-presentation. We say a labeled oriented graph is *aspherical* if its associated LOG-complex is aspherical. A labeled oriented graph is called *injective* if each vertex occurs at most once as an edge label. A *labeled oriented tree* (LOT) is a labeled oriented graph where the underlying graph is a tree.

The following is the main result of this article.

**Theorem 1.1** *Injective labeled oriented trees are aspherical.*

The theorem does not extend to labeled oriented graphs. The Wirtinger presentation  $P$  read off an alternating knot diagram with  $n$  crossings is the LOG-presentation of an injective labeled oriented circle  $\mathcal{C}$  with  $n$  edges. This labeled oriented circle  $\mathcal{C}$  is not aspherical, because any one relator in  $P$  is a consequence of the other relators.

We will need the following additional terminology on labeled oriented graphs. A *sub-LOG* of a labeled oriented graph  $\mathcal{G}$  is a connected subgraph  $\mathcal{H}$  containing at least one edge such that each edge label of  $\mathcal{H}$  is a vertex of  $\mathcal{H}$ . A sub-LOG  $\mathcal{H}$  is *proper* if it is not all of  $\mathcal{G}$ . A labeled oriented graph is called *compressed* if no edge is labeled with one of its vertices. It is called *boundary reducible* if there is a boundary vertex that does not occur as edge label, and *boundary reduced* otherwise. A labeled oriented graph is called *interior reducible* if there is a vertex with two adjacent edges with the same label that either point away or towards that vertex, and *interior reduced* otherwise. A labeled oriented graph which is boundary reduced, interior reduced and compressed is called *reduced*.

Howie [7] observed that a labeled oriented tree  $\mathcal{G}$  can be transformed into a reduced labeled oriented tree  $\mathcal{G}_{red}$  so that  $K(\mathcal{G})$  and  $K(\mathcal{G}_{red})$  have the same homotopy type. Here are some details on this transformation. If  $\mathcal{G}$  is not

compressed then it contains an edge of the form  $e = [a, a, b]$  or  $e = [b, a, a]$ . We remove the interior of  $e$  and identify the vertices  $a$  and  $b$  to become one vertex  $a$ . Edges labeled with  $b$  we relabel with  $a$ . This transforms  $\mathcal{G}$  to a labeled oriented tree  $\mathcal{G}'$  with fewer vertices. On the presentation level  $P(\mathcal{G})$  is changed into  $P(\mathcal{G}')$  by  $Q^{**}$ -transformations. On the 2-complex level the change amounts to a 3-deformation  $h: K(\mathcal{G}) \rightarrow K(\mathcal{G}')$ . If  $\mathcal{G}$  is not boundary reduced, then it contains an edge  $e = [a, c, b]$  or  $e = [b, c, a]$ , where  $a$  is a boundary vertex and  $a$  does not appear as an edge label in  $\mathcal{G}$ . We remove the interior of  $e$  and the vertex  $a$  from  $\mathcal{G}$  to obtain a labeled oriented tree  $\mathcal{G}'$  with fewer vertices. As before, on the presentation level this amounts to a  $Q^{**}$ -transformation and hence to a 3-deformation  $h: K(\mathcal{G}) \rightarrow K(\mathcal{G}')$  on the 2-complex level. If  $\mathcal{G}$  is interior reducible at a vertex  $b$  then there exist edges  $e_1 = [a, d, b]$  and  $e_2 = [c, d, b]$ , or  $e_1 = [b, d, a]$  and  $e_2 = [b, d, c]$ . Fold the two edges  $e_1$  and  $e_2$  into one edge and label it by  $d$ , matching orientations and identifying the vertices  $a$  and  $c$  into one vertex  $a$ . Change all edge labels  $c$  into edge labels  $a$ . Again, on the presentation level this amounts to a  $Q^{**}$ -transformation and hence to a 3-deformation on the 2-complex level. If  $\mathcal{H}$  is a sub-LOT of  $\mathcal{G}$  then the reductions just described transform  $\mathcal{H}$  into a sub-LOT  $\mathcal{H}_0$  of  $\mathcal{G}_{red}$  and the homotopy equivalence  $h: K(\mathcal{G}) \rightarrow K(\mathcal{G}_{red})$  restricts to a homotopy equivalence  $h: K(\mathcal{H}) \rightarrow K(\mathcal{H}_0)$ .

In summary we have the following result that will be used later in this article.

**Lemma 1.2** *Let  $\mathcal{G}$  be a labeled oriented tree and  $\mathcal{H}$  be a sub-LOT. Transform  $\mathcal{G}$  to  $\mathcal{G}_{red}$  and let  $\mathcal{H}_0$  be the image of  $\mathcal{H}$  in  $\mathcal{G}_{red}$  under that transformation. Then there is a commutative diagram*

$$\begin{array}{ccc} K(\mathcal{G}) & \xrightarrow{h} & K(\mathcal{G}_{red}) \\ i \uparrow & & \uparrow i \\ K(\mathcal{H}) & \xrightarrow{h_0} & K(\mathcal{H}_0) \end{array}$$

where the horizontal maps are homotopy equivalences and the vertical maps are inclusions.

We conclude this section with an outline of the paper. In Section 2 we review some basic concepts from combinatorial topology and introduce the notion of relative vertex asphericity. Section 3 contains a test for relative vertex asphericity based on a result of Stallings. The material in this section

is of independent interest with possible applications not directly connected with the study of labeled oriented trees. Section 4 introduces altered LOT-presentations and contains Theorem 4.1, a result shown by Huck and the second author [9]. There it was used to show that prime injective labeled oriented trees are aspherical (prime means that the labeled oriented tree does not contain sub-LOTs). Corollary 4.2 is a relative version of Theorem 4.1. Section 5 contains the proof of our main result Theorem 1.1. We also show that if  $\mathcal{H}$  is a sub-LOT of an injective labeled oriented tree  $\mathcal{G}$ , then  $G(\mathcal{H})$  is a subgroup of  $G(\mathcal{G})$ . In Section 6 we extend some of our results to a class of non-injective labeled oriented trees, and we close with an application to virtual knots.

## 2 Relative vertex asphericity

In the following we work in the category of combinatorial 2-complexes and combinatorial maps. Recall that a map  $f: X \rightarrow Y$  between CW complexes is said to be *combinatorial* if the restriction of  $f$  to each open cell of  $X$  is a homeomorphism onto its image. Most of the 2-complexes considered in this article will be standard 2-complexes built from group presentations. Such 2-complexes have a single 0-cell.

**Definition 2.1** *A spherical diagram over a 2-complex  $K$  is a combinatorial map  $f: C \rightarrow K$ , where  $C$  is a cell decomposition of the 2-sphere.*

If the 2-complex  $K$  is the standard 2-complex  $K(P)$  associated with a group presentation  $P = \langle \mathbf{x} \mid \mathbf{r} \rangle$  one can define spherical diagrams in graph theoretic terms. Note that  $K(P)$  has a single 0-cell and its oriented 1-cells are in one-to-one correspondence with elements from  $\mathbf{x}$ . Thus there is a one-to-one correspondence between edge paths in  $K(P)$  and words in  $\mathbf{x}^{\pm 1}$ . Suppose  $f: C \rightarrow K(P)$  is a spherical diagram. If  $e$  is an edge in  $C$  that gets mapped to the edge  $x \in \mathbf{x}$  in  $K(P)$ , then we orient  $e$  to make  $f$  orientation preserving on  $e$  and label it by  $x$ . In this way the spherical diagram gives rise to a planar connected oriented graph  $C^{(1)}$  such that

- oriented edges in  $C^{(1)}$  are labeled by elements from  $\mathbf{x}$ ;
- the word read off the boundary path of an inner or the outer region of  $C^{(1)}$  is a cyclic permutation of a word  $r^\epsilon$ , where  $\epsilon = \pm 1$  and  $r \in \mathbf{r}$ .

We can also start with a planar connected oriented graph  $C^{(1)}$ , labeled in the fashion just described, and produce a spherical diagram  $f: C \rightarrow K(P)$ . This provides an alternate definition for spherical diagrams over standard 2-complexes  $K(P)$  in terms of planar connected oriented graphs with labeled edges. This purely combinatorial point of view is taken by Bogley and Pride [2]. They work with pictures over group presentations which are obtained by dualizing spherical diagrams. Another standard reference for diagrams is Gersten [3].

**Definition 2.2** *Let  $\Gamma$  be a graph and  $\Gamma_0$  be a subgraph (which could be empty).*

1. *An edge cycle  $c = e_1 \dots e_q$  in  $\Gamma$  is called homology reduced if it does not contain a pair of edges  $e_i$  and  $e_j$  so that  $e_j = \bar{e}_i$ , where  $\bar{e}_i$  is the edge  $e_i$  with opposite orientation.*
2. *An edge cycle  $c = e_1 \dots e_q$  is said to be homology reduced relative to  $\Gamma_0$  if it does not contain a pair of edges  $e_i$  and  $e_j$  of  $\Gamma - \Gamma_0$  so that  $e_j = \bar{e}_i$ .*

If  $v$  is a vertex of a 2-complex  $K$  then the link  $Lk(K, v)$  is the boundary of a regular neighborhood of  $v$  in  $K$  equipped with the induced cell decomposition. So  $Lk(K, v)$  is a graph. If  $K$  has a single vertex we denote by  $Lk(K)$  the link of that vertex. We refer to the edges in  $Lk(K)$  as *corners*, since they can be thought of as the corners of the 2-cells of  $K$ .

Let  $K$  be a 2-complex with a single vertex and  $K_0$  a subcomplex (which could be empty). Note that  $Lk(K_0)$  is a subgraph of  $Lk(K)$ . Let  $f: C \rightarrow K$  be a spherical diagram and  $v$  be a vertex of the 2-sphere  $C$ . The map  $f$  induces a combinatorial map  $f_L: Lk(C, v) \rightarrow Lk(K)$ . Note that  $Lk(C, v)$  is a circle and the image of that circle, oriented clockwise, is a cycle of corners  $\alpha(v) = \alpha_1 \dots \alpha_q$ , that is a closed edge path, in  $Lk(K)$ . We say that the diagram  $f: C \rightarrow K$  is *vertex reduced at  $v$  relative to  $K_0$*  if the cycle  $\alpha(v)$  in  $Lk(K)$  is homology reduced relative to  $Lk(K_0)$ . We say that the diagram is *vertex reduced relative to  $K_0$*  if it is vertex reduced relative to  $K_0$  at all its vertices. The 2-complex  $K$  is called *vertex aspherical relative to  $K_0$* , VA relative to  $K_0$  for short, if for every spherical diagram  $f: C \rightarrow K$  that is vertex reduced relative to  $K_0$  we have  $f(C) \subseteq K_0$ .

When we leave off “relative to” we implicitly imply relative to the empty set  $\emptyset$ , even if a subcomplex is present. For example if we say a spherical diagram is vertex reduced we mean vertex reduced relative to  $\emptyset$ .

**Theorem 2.3** *If  $K$  is VA relative to  $K_0$ , then  $\pi_2(K)$  is generated, as  $\pi_1(K)$ -module, by the image of  $\pi_2(K_0)$  under the map induced by inclusion. In particular, if  $K_0$  is aspherical, then so is  $K$ .*

**Proof:** This follows from the fact that  $\pi_2(K)$  is generated by spherical diagrams (see [2] Theorem 1.3, p. 162 and the literature cited there) and the fact that each spherical diagram  $f: C \rightarrow K$  is homotopically equivalent to a spherical diagram  $f': C' \rightarrow K$  that is vertex reduced.  $\square$

Vertex asphericity in case  $K_0 = \emptyset$  was considered in Huck, Rosebrock [8]. It presents a generalization of *diagrammatic reducibility*, DR for short. See Gersten [3] for a definition. We note that diagrammatic reducibility implies vertex asphericity, and vertex asphericity implies asphericity (see [8]).

### 3 A test for relative vertex asphericity

A graph is called a *forest* if its connected components are trees.

**Definition 3.1** *Let  $\Gamma$  be a graph and  $\Gamma_0$  be a subgraph.*

- *The graph  $\Gamma$  is called a forest relative to  $\Gamma_0$  if every homology reduced cycle is contained in  $\Gamma_0$ .*
- *The graph  $\Gamma$  is called a tree relative to  $\Gamma_0$  if  $\Gamma$  is connected and every homology reduced cycle is contained in  $\Gamma_0$ .*

**Lemma 3.2** *Let  $\Gamma$  be a graph,  $\Gamma_0$  a subgraph with connected components  $\Gamma_1, \dots, \Gamma_n$ . Let  $\Gamma'$  be the graph obtained by collapsing each component  $\Gamma_i$  to a vertex  $g_i \in \Gamma_i$ . Then  $\Gamma$  is a forest relative to  $\Gamma_0$  if and only if  $\Gamma'$  is a forest.*

**Proof:** If  $\Gamma$  is a forest relative to  $\Gamma_0$  then each homology reduced cycle is contained in some  $\Gamma_i$ . After collapsing each  $\Gamma_i$  to a point there will be no more non-trivial homology reduced cycles and so  $\Gamma'$  is a forest.

For the converse let  $Y$  be the closure of  $\Gamma - \Gamma_0$  in  $\Gamma$ . The intersection  $\Gamma_0 \cap Y$  is a set of vertices. Observe that since  $\Gamma'$  is a forest, if an edge  $e$  of  $Y$  appears in a cycle in  $\Gamma$ , then so must its inverse  $\bar{e}$ . In particular, a homology reduced

cycle in  $\Gamma$  has to be contained in  $\Gamma_0$  and hence in one of the connected components  $\Gamma_i$ .  $\square$

We next discuss a graph theoretic result. Let  $C$  be a cell decomposition of the 2-sphere with oriented edges. A *sink* is a vertex in  $C$  with all adjacent edges pointing towards it, a *source* is a vertex in  $C$  with all adjacent edges pointing away from it. A 2-cell is *consistently oriented* if all its boundary edges are oriented clockwise or all are oriented anti-clockwise. We say a 2-cell has *exponent sum zero* if, when reading around its boundary in clockwise direction, one encounters the same number of clockwise oriented edges as anti-clockwise oriented edges. The lemma below is due to Stallings [12], Lemma 1.2.

**Lemma 3.3** *Given a cell decomposition  $C$  of the 2-sphere with oriented edges. If  $C$  does not contain a consistently oriented 2-cell, then it contains a sink or a source.*

Let  $P = \langle \mathbf{x} \mid \mathbf{r} \rangle$  be a presentation. The link of the single vertex in the associated standard 2-complex is also referred to as the *Whitehead graph*  $W(P)$ . It can be defined directly from the presentation without reference to topological notions. See for example [2], page 170. The Whitehead graph  $W(P)$  is a non-oriented graph on vertices  $\{x^+, x^- \mid x \in \mathbf{x}\}$ , where  $x^+$  is a point of the oriented edge  $x$  of  $K(P)$  close to the beginning of that edge, and  $x^-$  is a point close to the ending of that edge. Vertices  $x^\epsilon$  and  $y^\delta$ , ( $x, y \in \mathbf{x}$ ,  $\epsilon, \delta \in \{\pm\}$ ), are connected by an edge in  $W(P)$  if there is a 2-cell in  $K(P)$  with a corner connecting the two points. For that reason we refer to the edges of  $W(P)$  also as *corners*. The *positive graph*  $W^+(P) \subset W(P)$  is the full subgraph on the vertex set  $\{x^+ \mid x \in \mathbf{x}\}$ , the *negative graph*  $W^-(P) \subset W(P)$  is the full subgraph on the vertex set  $\{x^- \mid x \in \mathbf{x}\}$ .

**Definition 3.4** *A presentation  $P$  is said to satisfy the Stallings test if the following conditions hold:*

- *Relator conditions:*
  - (a) *Relators of  $P$  are cyclically reduced.*
  - (b) *Relators of  $P$  are not positive or negative words.*

- *Forest condition:*

(c)  $W^+(P)$  and  $W^-(P)$  are forests.

The following application of Lemma 3.3 is well known. It first appeared in Gersten [3], Proposition 4.12, in the context of Adian presentations.

**Theorem 3.5** *Let  $P$  be a presentation that satisfies the Stallings test. Then  $K(P)$  is VA (even DR) and hence aspherical.*

The main result of this section is a relative version of Theorem 3.5. Before we can state it we need some notation. If  $P_1 = \langle \mathbf{x}_1 \mid \mathbf{r}_1 \rangle$  and  $P_2 = \langle \mathbf{x}_2 \mid \mathbf{r}_2 \rangle$  are presentations then  $P_1 \cup P_2 = \langle \mathbf{x}_1 \cup \mathbf{x}_2 \mid \mathbf{r}_1 \cup \mathbf{r}_2 \rangle$ . Let  $P = \langle \mathbf{x} \mid \mathbf{r} \rangle$  be a presentation and let  $\{T_1, \dots, T_n\}$  be a set of disjoint *full* sub-presentations of  $P$ . Full means that if  $r$  is a relator in  $P$  that only involves generators from  $T_i$ , then  $r$  is already a relator in  $T_i$ . Disjoint means that the generating sets of  $T_i$  and  $T_j$  are disjoint subsets of  $\mathbf{x}$  in case  $i \neq j$ . Let  $T = T_1 \cup \dots \cup T_n$ . The complex  $K(T) = K(T_1) \vee \dots \vee K(T_n)$  is a sub-complex of  $K(P)$ .

Let  $T_i = \langle \mathbf{t}_i \mid \mathbf{s}_i \rangle$  and let  $\mathbf{u}_i$  be the set of words with letters in  $\mathbf{t}_i^{\pm 1}$  of exponent sum zero. Note that we do not assume that the words in  $\mathbf{u}_i$  are freely or cyclically reduced. Let  $\mathbf{T}_i = \langle \mathbf{t}_i \mid \mathbf{s}_i \cup \mathbf{u}_i \rangle$ . Let  $\mathbf{T} = \mathbf{T}_1 \cup \dots \cup \mathbf{T}_n$  and note that  $P \cup \mathbf{T} = \langle \mathbf{x} \mid \mathbf{r} \cup \mathbf{u}_1 \cup \dots \cup \mathbf{u}_n \rangle$  since  $\mathbf{t}_i \subseteq \mathbf{x}$  and  $\mathbf{s}_i \subseteq \mathbf{r}$  for every  $i$ . The presentation  $P \cup \mathbf{T}$  is infinite and in the group  $G(P \cup \mathbf{T})$  the generators of each  $T_i$  are identified, since we have the relator  $t^{-1}t'$  in  $P \cup \mathbf{T}$  for every pair  $t, t'$  of generators in  $T_i$ . Note that  $\mathbf{T}_i$  is a sub-presentation of  $P \cup \mathbf{T}$  and the subgraph  $W(\mathbf{T}_i)$  of the Whitehead graph  $W(P \cup \mathbf{T})$ , which is spanned by the vertices  $t^\pm$ ,  $t \in \mathbf{t}_i$ , contains the complete graph on these vertices. In fact, every pair of vertices in  $W(\mathbf{T}_i)$  is connected by infinitely many edges, and at every vertex in  $W(\mathbf{T}_i)$  there are attached infinitely many loops.

**Definition 3.6** *Let  $P$  be a presentation, and let  $\{T_1, \dots, T_n\}$  be a set of disjoint full sub-presentations. Let  $T = T_1 \cup \dots \cup T_n$ . Then  $P$  is said to satisfy the Stallings test relative to  $T$  if the following conditions hold:*

- *Relator conditions:*

(a) *Relators of  $P - T$  are cyclically reduced.*

(b) *Relators of  $P - T$  are not positive or negative words.*

(c) *Relators of  $T$  have exponent sum zero.*

(d) Any word  $w$  in the generators of some  $T_i$  that represents the trivial element of the group defined by  $P \cup \mathbf{T}$  has exponent sum zero.

• *Forest condition:*

(e)  $W^+(P \cup \mathbf{T})$  is a forest relative to  $W^+(\mathbf{T})$  and  $W^-(P \cup \mathbf{T})$  is a forest relative to  $W^-(\mathbf{T})$ .

Here is the main result of this section:

**Theorem 3.7** *If  $P$  is a presentation that satisfies the Stallings test relative to  $T$ , then  $K(P)$  is VA relative to  $K(T)$ . Furthermore, the inclusion induced homomorphism  $\pi_1(K(T_i)) \rightarrow \pi_1(K(P))$  is injective for every  $i = 1, \dots, n$ .*

For the proof of this theorem we need the following lemma:

**Lemma 3.8** *Let  $P$  be a presentation and let  $\{T_1, \dots, T_n\}$  be a set of disjoint full sub-presentations,  $T = T_1 \cup \dots \cup T_n$ . Assume that the relators of  $P$  satisfy the relator conditions (a)-(d) of the relative Stallings test (Definition 3.6). If  $K(P)$  is not VA relative to  $K(T)$ , then there is a spherical diagram  $f: C \rightarrow K(P \cup \mathbf{T})$  such that*

1.  $f: C \rightarrow K(P \cup \mathbf{T})$  is vertex reduced relative to  $K(\mathbf{T})$ ,
2.  $f(C)$  is not contained in  $K(\mathbf{T})$ , and
3. If  $v$  is a vertex in  $C$  then the corner cycle  $\alpha(v)$  has length at least two and does not contain two distinct corners  $\alpha_p$  and  $\alpha_q$  that both come from relators of one subpresentation  $\mathbf{T}_i$  of  $P \cup \mathbf{T}$ ,  $i \in \{1, \dots, n\}$ .

**Proof:** Since  $K(P)$  is not VA relative to  $K(T)$  there exists a spherical diagram  $C^* \rightarrow K(P)$  that is vertex reduced relative to  $K(T)$  but does not map entirely into  $K(T)$ . Since  $K(P)$  is a subcomplex of  $K(P \cup \mathbf{T})$  this diagram can be viewed as a diagram  $C^* \rightarrow K(P \cup \mathbf{T})$  which is vertex reduced relative to  $K(\mathbf{T})$  but does not map entirely into  $K(\mathbf{T})$ . Let  $\Omega$  be the collection of all spherical diagrams that have that feature. Consider the subset  $\Omega_0 \subseteq \Omega$  of those for which the 2-sphere  $C$  contains the smallest number of 2-cells. From  $\Omega_0$  choose a spherical diagram  $f: C \rightarrow K(P \cup \mathbf{T})$  for which  $C$  has the smallest number of edges.

This spherical diagram does not contain a vertex of valency one. If  $v$  were a vertex of valency one in  $C$ , then it would be a vertex in the boundary of a cell  $E$  that maps to some  $K(\mathbf{T}_i)$  since we assumed the relators in  $P$  outside  $T$  to be cyclically reduced. Let  $e$  be the edge in  $C$  that contains  $v$ . See Figure 1. We can remove  $v$  and the interior of  $e$  and transform  $E$  into  $E'$ . Note that the boundary words of  $E$  and  $E'$  are the same up to free or cyclic reduction, hence removing  $v$  and the interior of  $e$  produces a spherical diagram  $f': C' \rightarrow K(P \cup \mathbf{T})$  with fewer edges but the same number of 2-cells contradicting the choice of  $f$ .

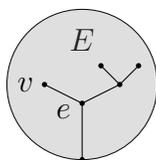


Figure 1: Vertices of valency one in  $C$ .

The spherical diagram under consideration has the first and the second property by choice. Let us look at the third property. Let  $v \in C$  be a vertex and  $\alpha(v)$  its corner cycle. The length of  $\alpha(v)$  is at least two because  $C$  does not contain vertices of valency one. Suppose  $E_p, E_q$  are 2-cells in  $C$  that both get mapped to 2-cells of  $K(\mathbf{T}_i)$  and give rise to the corners  $\alpha_p, \alpha_q \in \alpha(v)$ , respectively. Let us assume first that  $E_p$  and  $E_q$  are distinct (we do not rule out that  $E_p$  and  $E_q$  share boundary edges). We can split  $v$  in  $C$  into two vertices, and fuse the cells  $E_p$  and  $E_q$  into one cell  $E$ . See Figure 2. This creates

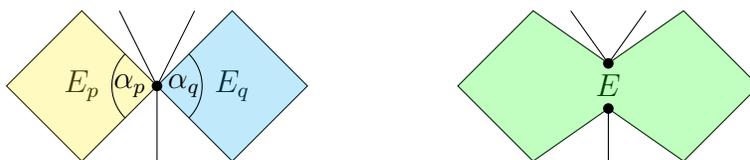


Figure 2: Splitting at a vertex.

a new cell division  $C'$  with one fewer 2-cell than  $C$ . Note that the boundary word of  $E$  is a word obtained by reading around the boundary of the union  $E_p \cup E_q$ , thus it is a word in  $\mathbf{t}_i^{\pm 1}$ , where  $\mathbf{t}_i$  is the generating set of  $T_i$ , of exponent sum zero. So the boundary word of  $E$  is a relator of  $\mathbf{T}_i$ . The splitting process has created a new spherical diagram  $f': C' \rightarrow K(P \cup \mathbf{T})$ . It is vertex reduced relative to  $K(\mathbf{T})$  and  $f'(C')$  is not contained in  $K(\mathbf{T})$ . Since  $C'$

contains fewer 2-cells than  $C$  this contradicts the choice of  $f: C \rightarrow K(P \cup \mathbf{T})$ .

We next assume that there is one 2-cell  $E_r$  at  $v$  in  $C$  that maps to a 2-cell in  $\mathbf{T}_i$  that gives rise to both corners  $\alpha_p$  and  $\alpha_q$ . See Figure 3. We can split

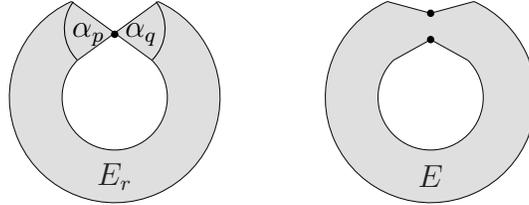


Figure 3: Splitting at a vertex.

at  $v$  as before, but note that now  $C'$  is not a cell division of the 2-sphere because the 1-skeleton  $C''^{(1)}$  is not connected. Let  $C_1''^{(1)}$  and  $C_2''^{(1)}$  be the connected components of  $C''^{(1)}$ . The boundary word for the outer region of each planar connected graph  $C_j''^{(1)}$ ,  $j = 1, 2$ , is a word in  $\mathbf{t}_i^{\pm 1}$  which represents the trivial element of the group defined by  $P \cup \mathbf{T}$  and hence (relator condition (d) of Definition 3.6) has exponent sum zero. Thus the boundary word is a relator in  $\mathbf{T}_i$ . It follows that each  $C_j''^{(1)}$  determines a spherical diagram  $f_j: C_j' \rightarrow K(P \cup \mathbf{T})$  that is vertex reduced relative to  $K(\mathbf{T})$ . Since  $f(C)$  is not in  $K(\mathbf{T})$ , the image  $f_j(C_j')$  is not in  $K(\mathbf{T})$  for one of the  $j$ 's. But  $C_j'$  has fewer 2-cells than  $C$ . This contradicts the choice of  $f: C \rightarrow K(P \cup \mathbf{T})$ .

This shows that the spherical diagram  $f: C \rightarrow K(P \cup \mathbf{T})$  has the three properties stated above.  $\square$

**Proof of Theorem 3.7:** Suppose  $K(P)$  is not VA relative to  $K(T)$ . Then there exists a spherical diagram  $f: C \rightarrow K(P \cup \mathbf{T})$  that satisfies the conditions 1, 2, and 3 stated in Lemma 3.8. The relator conditions (b) and (c) in Definition 3.6 (relative Stallings test) imply that  $C$  does not contain cells with consistently oriented boundary, hence  $C$  contains a sink or a source by Lemma 3.3. Let us assume without loss of generality that  $C$  contains a source, say at the vertex  $v \in C$ . The cycle  $\alpha(v) = \alpha_1 \dots \alpha_l$  satisfies  $l \geq 2$ , is contained in  $W^+(P \cup \mathbf{T})$  and is homology reduced relative to  $W^+(\mathbf{T})$  because  $f: C \rightarrow K(P \cup \mathbf{T})$  is vertex reduced relative to  $K(\mathbf{T})$ . Since  $W^+(P \cup \mathbf{T})$  is a forest relative to  $W^+(\mathbf{T})$  we know that  $\alpha(v)$  is entirely contained in a connected

component of  $W^+(\mathbf{T})$ , and hence in some  $W^+(\mathbf{T}_i)$ , because  $W^+(\mathbf{T})$  is a disjoint union of the  $W^+(\mathbf{T}_i)$ ,  $i = 1, \dots, n$ . Thus, if  $\alpha(v) = \alpha_1 \dots \alpha_l$ , then all corners  $\alpha_j$ ,  $j = 1, \dots, l$  are in  $W^+(\mathbf{T}_i)$ . This contradicts condition 3 of Lemma 3.8. We have reached a contradiction. Thus  $K(P)$  is VA relative to  $K(T)$ .

Suppose the map  $\pi_1(K(T_i)) \rightarrow \pi_1(K(P))$  is not injective for some  $i = 1, \dots, n$ . Then there exists a cyclically reduced word  $w$  in the generators of  $T_i$  that represents the trivial element of  $\pi_1(K(P))$  but is not trivial in  $\pi_1(K(T))$ . Hence  $w$  is the boundary word of a vertex reduced disc diagram  $g: D \rightarrow K(P)$ . Note that  $D$  has to contain 2-cells that are not mapped to  $K(T)$  because the map  $\pi_1(K(T_i)) \rightarrow \pi_1(K(T)) = \pi_1(K(T_1)) * \dots * \pi_1(K(T_n))$  is injective. By the relator condition (d) of Definition 3.6 (relative Stallings test) the word  $w$  has exponent sum zero and hence is a relator of  $\mathbf{T}_i$ . We can attach a disc  $D'$  to  $D$  and obtain a spherical diagram  $f: C \rightarrow K(P \cup \mathbf{T})$ . Note that this spherical diagram is vertex reduced. If it were not, then there would have to be a vertex on the boundary of  $D$  where the spherical diagram  $f: C \rightarrow K(P \cup \mathbf{T})$  is not vertex reduced. But that would mean that  $D$  contains a 2-cell with boundary word  $w$ . This would imply that  $w$  is a relator in  $P$ . Since we assumed  $T_i$  to be a full sub-presentation, the word  $w$  would have to be a relator in  $T_i$ , which is not the case because  $w$  does not represent the trivial element of  $\pi_1(K(T_i))$ . Thus  $f: C \rightarrow K(P \cup \mathbf{T})$  is indeed vertex reduced. Since  $K(P)$  is VA relative to  $K(T)$  we have that  $f(C) \subseteq K(\mathbf{T})$ , which implies that  $g(D) \subseteq K(P) \cap K(\mathbf{T}) = K(T)$ . We have reached a contradiction.  $\square$

## 4 Altering LOT-presentations and orientations

We say a labeled oriented graph  $\mathcal{Q}$  is a *reorientation* of a labeled oriented graph  $\mathcal{P}$  if  $\mathcal{Q}$  is obtained from  $\mathcal{P}$  by changing the orientation of each edge of a subset of the set of edges of  $\mathcal{P}$ .

The next result is proved in Section 3 of [9].

**Theorem 4.1** *Let  $\mathcal{P}$  be a compressed injective labeled oriented tree that does not contain a boundary reducible sub-LOT. Then there is a reorientation  $\mathcal{Q}$  of  $\mathcal{P}$  such that  $W^+(Q)$  and  $W^-(Q)$  are trees, where  $Q$  is the LOT-presentation associated with  $\mathcal{Q}$ .*

We make the following assumption for the remainder of this section:  $\mathcal{P}$  is a reduced labeled oriented tree and  $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$  is a set of proper maximal sub-LOTS. We assume that the  $\mathcal{T}_i$  are pairwise disjoint, so  $\mathcal{T}_i \cap \mathcal{T}_j = \emptyset$  in case  $i \neq j$ . Let  $\mathcal{T} = \mathcal{T}_1 \cup \dots \cup \mathcal{T}_n$ . Let  $P, T, T_i$  be the LOT-presentations associated with  $\mathcal{P}, \mathcal{T}, \mathcal{T}_i$ , respectively.

From each  $\mathcal{T}_i$  choose a vertex  $t_i$ . If an edge in  $\mathcal{P}$  is labeled with a vertex from  $\mathcal{T}_i$ , replace this edge label with  $t_i$ . Now collapse each  $\mathcal{T}_i$  in  $\mathcal{P}$  to the vertex  $t_i$  to obtain a labeled oriented tree  $\mathcal{P}'$ . We say that  $\mathcal{P}$  is *injective relative to  $\mathcal{T}$*  if  $\mathcal{P}'$  is injective. It is clear that if  $\mathcal{P}$  is injective itself, then  $\mathcal{P}$  is also injective relative to  $\mathcal{T}$ .

**Corollary 4.2** *If  $\mathcal{P}$  is injective relative to  $\mathcal{T}$ , then there exists a reorientation  $\mathcal{Q}$  of  $\mathcal{P}$ , where only certain edges of  $\mathcal{P} - \mathcal{T}$  are reoriented, so that  $Q$ , the LOT-presentation associated with  $\mathcal{Q}$ , satisfies the Stallings test relative to  $T$ .*

**Proof:** The LOT  $\mathcal{P}'$  is compressed. In order to see this assume  $\mathcal{P}'$  is not compressed. Then there exists a  $\mathcal{T}_i$  and an edge  $e$  in  $\mathcal{P}$  not contained in  $\mathcal{T}_i$  but connected to  $\mathcal{T}_i$  that has label  $t_i$ , where  $t_i$  is the vertex chosen from  $\mathcal{T}_i$  for the construction of  $\mathcal{P}'$ . Since we assume  $\mathcal{T}_i$  to be a maximal proper sub-LOT the union  $\mathcal{T}_i \cup e = \mathcal{P}$ , otherwise  $\mathcal{T}_i \cup e$  would be a larger proper sub-LOT. Since  $\mathcal{T}_i$  is a sub-LOT, the vertex of  $e$  that is not in  $\mathcal{T}_i$  does not occur as edge label in  $\mathcal{P}$ . But then  $\mathcal{P}$  is not boundary reduced, contradicting our assumption that  $\mathcal{P}$  is reduced.

Theorem 4.1 implies that there is a reorientation  $\mathcal{Q}'$  of  $\mathcal{P}'$ , such that  $W^+(Q')$  and  $W^-(Q')$  of the LOT-presentation  $Q'$  of  $\mathcal{Q}'$  are trees. Let  $\mathcal{Q}$  be a reorientation of  $\mathcal{P}$  where no edge of  $\mathcal{T}$  is reoriented (so  $\mathcal{T}$  is contained in  $\mathcal{Q}$ ), and collapsing each  $\mathcal{T}_i$  in  $\mathcal{Q}$  to the vertex  $t_i$  results in  $\mathcal{Q}'$ . Let  $P, P', Q, Q', T$ , and  $T_i$  be LOT-presentations associated with  $\mathcal{P}, \mathcal{P}', \mathcal{Q}, \mathcal{Q}', \mathcal{T}$ , and  $\mathcal{T}_i$ , respectively. Note that  $W^+(\mathbf{T}) \subseteq W^+(Q \cup \mathbf{T})$  and that the  $W^+(\mathbf{T}_i)$ ,  $i = 1, \dots, n$  are the connected components of  $W^+(\mathbf{T})$ . Collapsing the components  $W^+(\mathbf{T}_i)$  of  $W^+(\mathbf{T})$  to  $t_i^+$  yields the tree  $W^+(Q')$ . So it follows from Lemma 3.2 that  $W^+(Q \cup \mathbf{T})$  is a forest relative to  $W^+(\mathbf{T})$ . Since  $W^+(Q')$  is

connected, so is  $W^+(Q \cup \mathbf{T})$ . Hence  $W^+(Q \cup \mathbf{T})$  is a tree relative to  $W^+(\mathbf{T})$ . In the same way we can argue that  $W^-(Q \cup \mathbf{T})$  is a tree relative to  $W^-(\mathbf{T})$ .

This shows that the forest condition (e) of Definition 3.6 (the relative Stallings test) holds. The relator conditions (a)-(d) also hold because  $Q$  is a LOT-presentation that comes from the reduced LOT  $\mathcal{Q}$ . So all relators of  $Q$  are cyclically reduced and have exponent sum zero. This implies that all relators of  $Q \cup \mathbf{T}$  have exponent sum zero and hence any word in the generators of  $Q$  that represents the trivial element in the group defined by  $Q \cup \mathbf{T}$  has exponent sum zero.  $\square$

Let  $\mathbf{x}$  be a set and  $w$  be a word in  $\mathbf{x}^{\pm 1}$ . Let  $S$  be a subset of  $\mathbf{x}$ . Define  $w_S$  to be the word obtained from  $w$  by replacing  $x^\epsilon$  in  $w$  by  $x^{-\epsilon}$ ,  $\epsilon = \pm 1$ , if and only if  $x \in S$ . If  $\mathbf{w}$  is a set of words in  $\mathbf{x}^{\pm 1}$ , then let  $\mathbf{w}_S$  be the set of words  $w_S$ ,  $w \in \mathbf{w}$ . If  $P = \langle \mathbf{x} \mid \mathbf{r} \rangle$  is a presentation, denote by  $P_S = \langle \mathbf{x} \mid \mathbf{r}_S \rangle$ .

The map  $x \rightarrow x^\epsilon$ , where  $x$  is a generator in  $P$  and  $\epsilon = 1$  if  $x$  is not in  $S$  and  $\epsilon = -1$  if  $x$  is in  $S$ , results in a homeomorphism  $\phi: K(P) \rightarrow K(P_S)$  on the corresponding standard 2-complexes. Furthermore, if  $f: C \rightarrow K(P)$  is a vertex reduced spherical diagram, then so is  $\phi \circ f: C \rightarrow K(P_S)$ . In particular both  $P$  and  $P_S$  present the same group. So if  $P$  is a LOT-presentation, then  $P_S$  is also a presentation of a LOT-group.

Let  $\mathcal{P}$  be a reduced labeled oriented tree and let  $P$  be the associated LOT-presentation. Let  $S$  be a subset of the generators of  $P$ . Let  $\mathcal{Q}$  be the reorientation of  $\mathcal{P}$  where exactly those edges are reoriented which have their label in  $S$ . Let  $Q$  be the LOT-presentation associated with  $\mathcal{Q}$ .

The following lemma is essentially Lemma 5.2 of [8]. We include a proof for the convenience of the reader.

**Lemma 4.3** *The Whitehead graphs  $W(P_S)$  and  $W(Q)$  are equal.*

**Proof:** The graphs  $W(P_S)$  and  $W(Q)$  have the same vertices because the presentations  $P_S$  and  $Q$  have the same set of generators. In transforming  $P$  to  $P_S$  an  $x^\epsilon$ ,  $x \in S$ ,  $\epsilon = \pm 1$ , is replaced by  $x^{-\epsilon}$  in all the relators of  $P$ . Whereas when passing from  $P$  to  $Q$  an  $x^\epsilon$ ,  $\epsilon = \pm 1$ , is replaced by  $x^{-\epsilon}$  only in the relators  $r_e$  of  $P$ , where  $e$  is an edge in  $\mathcal{P}$  with edge label  $x$ . Thus we

obtain  $P_S$  from  $Q$  by replacing an  $x^\epsilon$ ,  $x \in S$ ,  $\epsilon = \pm 1$ , by  $x^{-\epsilon}$  only in the relators  $r_e$ , where  $e$  is an edge in  $\mathcal{Q}$  that contains  $x$  as a vertex.

Suppose  $e = [x, z, y]$  is an edge in  $\mathcal{Q}$ , so  $r_e = xz(zy)^{-1} = xzy^{-1}z^{-1}$ . If neither  $x$  nor  $y$  are in  $S$ , then  $r_e = (r_e)_S$  is also a relator in  $P_S$ . If  $x \in S$  and  $y \notin S$ , then  $(r_e)_S = x^{-1}zy^{-1}z^{-1}$  is a relator in  $P_S$ . Note that  $r_e$  and  $(r_e)_S$  contribute the same edges to the Whitehead graph. See Figure 4. The other two cases,  $x \notin S$  but  $y \in S$ , and both  $x$  and  $y$  in  $S$ , lead to relators  $(r_e)_S = xzyz^{-1}$  and  $(r_e)_S = x^{-1}zyz^{-1}$  in  $P_S$  that contribute the same edges to the Whitehead graph as  $r_e$ . This shows that the Whitehead graphs of  $Q$  and  $P_S$  are the same.  $\square$

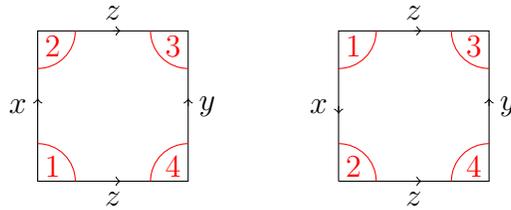


Figure 4: The relators  $r_e = xzy^{-1}z^{-1}$  and  $(r_e)_S = x^{-1}zy^{-1}z^{-1}$  contribute the same edges to the Whitehead graphs.

Let  $\mathcal{P}$  and  $\mathcal{T}$  be as in the assumption for this section.

**Theorem 4.4** *If  $\mathcal{P}$  is injective relative to  $\mathcal{T}$ , then there exists a subset  $S$  of the generators in  $P$  so that  $P_S$  satisfies the Stallings test relative to  $T_S$ .*

**Proof:** By Corollary 4.2 there exists a reorientation  $\mathcal{Q}$  of  $\mathcal{P}$  so that both  $W^\pm(Q \cup \mathbf{T})$  are trees relative to  $W^\pm(\mathbf{T})$ . Only edges in  $\mathcal{P} - \mathcal{T}$  change orientation. Let  $S_0$  be the subset of generators of  $P$  that occur as labels on edges that change orientation when passing from  $\mathcal{P}$  to  $\mathcal{Q}$ . We enlarge  $S_0$  to a set  $S = S_0 \cup \bigcup_{i=1}^n S_i$ , where  $S_i$  is the empty set if no generator of  $T_i$  is contained in  $S_0$ , and  $S_i$  is the set of generators of  $T_i$  if  $S_0$  contains a generator from  $T_i$ . Let  $\mathcal{Q}^*$  be the reorientation of  $\mathcal{P}$  where exactly the edges with labels in  $S$  are reoriented. Note that  $\mathcal{Q}^*$  contains a reorientation  $\mathcal{T}^*$  of  $\mathcal{T}$ , and that  $\mathcal{Q} - \mathcal{T} = \mathcal{Q}^* - \mathcal{T}^*$ . Since  $\mathbf{T} = \mathbf{T}^*$  this implies that  $Q \cup \mathbf{T} = Q^* \cup \mathbf{T}^*$ . Thus  $W(Q \cup \mathbf{T}) = W(Q^* \cup \mathbf{T}^*)$ , and so both  $W^\pm(Q^* \cup \mathbf{T}^*)$  are trees relative to  $W^\pm(\mathbf{T}^*)$ . changed

We first show that the forrest condition (e) of Definition 3.6 holds. Note that

$$P \cup \mathbf{T} = \langle \mathbf{x} \mid \mathbf{r} \cup \bigcup_{i=1}^n \mathbf{u}_i \rangle,$$

where  $P = \langle \mathbf{x} \mid \mathbf{r} \rangle$  and  $\mathbf{u}_i$  is the set of words of exponent sum zero in the generators of  $T_i$ . Thus

$$P_S \cup \mathbf{T}_S = \langle \mathbf{x} \mid \mathbf{r}_S \cup \bigcup_{i=1}^n \mathbf{v}_i \rangle,$$

where  $\mathbf{v}_i$  is the set of words of exponent sum zero in the generators of  $(T_i)_S$ . Since the generating sets of  $T_i$  and  $(T_i)_S$  are the same, we have that  $\mathbf{v}_i = \mathbf{u}_i$  for all  $i$ . So

$$P_S \cup \mathbf{T}_S = \langle \mathbf{x} \mid \mathbf{r}_S \cup \bigcup_{i=1}^n \mathbf{u}_i \rangle.$$

Now

$$Q^* \cup \mathbf{T}^* = \langle \mathbf{x} \mid \mathbf{s} \cup \bigcup_{i=1}^n \mathbf{w}_i \rangle,$$

where  $Q^* = \langle \mathbf{x} \mid \mathbf{s} \rangle$  and  $\mathbf{w}_i$  is the set of words of exponent sum zero in the generators of  $T_i^*$ . Since the generating sets of  $T_i$  and  $T_i^*$  are the same, we have that  $\mathbf{w}_i = \mathbf{u}_i$  for all  $i$ . So

$$Q^* \cup \mathbf{T}^* = \langle \mathbf{x} \mid \mathbf{s} \cup \bigcup_{i=1}^n \mathbf{u}_i \rangle.$$

Since  $W(P_S) = W(Q^*)$  by Lemma 4.3 and the same set of relators  $\mathbf{u} = \bigcup_{i=1}^n \mathbf{u}_i$  is added when enlarging  $P_S$  to  $P_S \cup \mathbf{T}_S$  as when enlarging  $Q^*$  to  $Q^* \cup \mathbf{T}^*$ , it is clear that  $W(P_S \cup \mathbf{T}_S) = W(Q^* \cup \mathbf{T}^*)$ . Since both  $W^\pm(Q^* \cup \mathbf{T}^*)$  are tree relative to  $W^\pm(\mathbf{T}^*)$ , and  $W(\mathbf{T}^*) = W(\mathbf{T}_S)$ , it follows that both  $W^\pm(P_S \cup \mathbf{T}_S)$  are trees relative to  $W^\pm(\mathbf{T}_S)$ . Thus the forrest condition holds.

We have to check the relator conditions (a)-(d).

(a) The relators of  $P_S - T_S$  are cyclically reduced because the relators of  $P - T$  are cyclically reduced.

(b) Note that every relator of  $P - T$  is of the form  $r_e = ab(bc)^{-1}$ , where  $e = [a, b, c]$  is an edge of  $\mathcal{P}$ . So every relator contains some generator and its inverse. Hence so does every relator in  $P_S - T_S$ . Thus no relator of  $P_S - T_S$  is a positive or negative word.

(c) Let  $u$  be a relator of  $T_S$ . Then  $u_S$  is a relator of  $T$ . Since  $T$  is a disjoint union  $T = T_1 \cup \dots \cup T_n$ , it follows that  $u_S$  is a relator in some  $T_i$ . So  $u_S$  is a word in the generators of  $T_i$  of exponent sum zero. By construction, the set  $S$  contains either no generator of  $T_i$ , or all of them. Hence  $u$  is a word in the generators of  $(T_i)_S$  of exponent sum zero.

(d) Let  $w$  be a word in the generators of some  $(T_i)_S$  that represents the trivial element in the group defined by  $P_S \cup \mathbf{T}_S = \langle \mathbf{x} \mid \mathbf{r}_S \cup \bigcup_{i=1}^n \mathbf{u}_i \rangle$ , where  $P = \langle \mathbf{x} \mid \mathbf{r} \rangle$  and  $\mathbf{u}_i$  is the set of words of exponent sum zero in the generators of  $T_i$  (see above). Then  $w_S$  is a word in the generators of  $T_i$  that represents the trivial element in the group defined by  $\langle \mathbf{x} \mid \mathbf{r} \cup \bigcup_{i=1}^n \mathbf{u}_{iS} \rangle$ . Since by construction  $S$  contains either no generator of  $T_i$  or all of them, it follows that  $\mathbf{u}_{iS} = \mathbf{u}_i$  for all  $i$ . So  $w_S$  is a word in the generators of  $T_i$  that represents the trivial element in the group defined by  $P \cup \mathbf{T} = \langle \mathbf{x} \mid \mathbf{r} \cup \bigcup_{i=1}^n \mathbf{u}_i \rangle$ . Since all relators of  $P \cup \mathbf{T}$  have exponent sum zero,  $w_S$  has exponent sum zero. Thus  $w_S \in \mathbf{u}_i$  and hence  $w \in \mathbf{u}_{iS} = \mathbf{u}_i$ . In particular  $w$  has exponent sum zero.  $\square$

## 5 Proof of Theorem 1.1

This section is devoted to proving the following theorem.

**Theorem 5.1** *An injective labeled oriented tree  $\mathcal{P}$  is aspherical. Furthermore, if  $\mathcal{H}$  is a sub-LOT of  $\mathcal{P}$ , then the inclusion induced homomorphism  $\pi_1(K(\mathcal{H})) \rightarrow \pi_1(K(\mathcal{P}))$  is injective.*

**Proof:** We proceed by induction on the number of vertices. If  $\mathcal{P}$  consists of a single vertex the result is true.

If  $\mathcal{P}$  is not reduced we transform it into a reduced injective labeled oriented tree  $\mathcal{P}_{red}$  that contains fewer vertices than  $\mathcal{P}$ . Thus, by induction hypothesis,  $\mathcal{P}_{red}$  is aspherical, and hence so is  $\mathcal{P}$ , because  $K(\mathcal{P})$  is homotopically equivalent to  $K(\mathcal{P}_{red})$  (Lemma 1.2). Let  $\mathcal{H}_0$  be the image of  $\mathcal{H}$  in  $\mathcal{P}_{red}$  under the

transformation. By induction hypothesis we know that the inclusion induced homomorphism  $\pi_1(K(\mathcal{H}_0)) \rightarrow \pi_1(K(\mathcal{P}_{red}))$  is injective. It follows from the commutative diagram in Lemma 1.2 that  $\pi_1(K(\mathcal{H})) \rightarrow \pi_1(K(\mathcal{P}))$  is injective as well.

From now on we assume that  $\mathcal{P}$  is reduced. Let  $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$  be the (possibly empty) set of maximal proper sub-LOTs of  $\mathcal{P}$ . Note that every  $\mathcal{T}_i$  is compressed and injective and contains fewer vertices than  $\mathcal{P}$ . Hence, by induction, each  $\mathcal{T}_i$  is aspherical and  $\pi_1(K(\mathcal{H})) \rightarrow \pi_1(K(\mathcal{T}_i))$  is injective for all sub-LOTs  $\mathcal{H}$  of  $\mathcal{T}_i$ .

**Case 1.** Suppose that for some  $i, j$  we have  $\mathcal{T}_i \cap \mathcal{T}_j \neq \emptyset$ .

As an example<sup>1</sup> consider the injective LOT  $\mathcal{P}$  (with any orientation of its edges) with non-empty intersection of maximal sub-LOTs as depicted in Figure 5. Here let  $\mathcal{T}_i$  be all of  $\mathcal{P}$  without edges and vertices labeled by  $a_i$  and  $b_i$  for  $i = 1, 2, 3$ . Then  $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$  is the set of maximal proper sub-LOTs of  $\mathcal{P}$ .

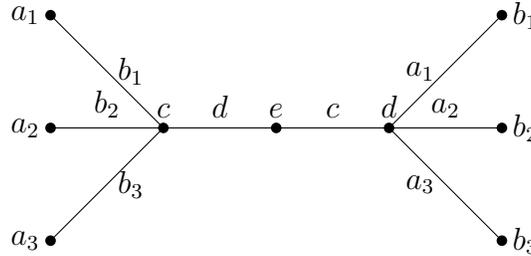


Figure 5: A labeled oriented tree that is the union of three pairwise intersecting maximal sub-LOTs.

We continue with the proof of the theorem in Case 1. We assume without loss of generality that  $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \emptyset$ . Then, by maximality it follows that  $\mathcal{P} = \mathcal{T}_1 \cup \mathcal{T}_2$ . Note that there might be more than two  $\mathcal{T}_i$  (as in the example shown in Figure 5). The intersection  $\mathcal{T}_{12} = \mathcal{T}_1 \cap \mathcal{T}_2$  is a sub-LOT. Indeed, if  $b$  is an edge label in  $\mathcal{T}_{12}$ , then  $b$  has to be a vertex of  $\mathcal{T}_1$ , because  $\mathcal{T}_1$  is a sub-LOT, and  $b$  has to be a vertex of  $\mathcal{T}_2$ , because  $\mathcal{T}_2$  is a sub-LOT. Thus  $b$  is a vertex of  $\mathcal{T}_{12}$ . Since  $\pi_1(K(\mathcal{T}_{12})) \rightarrow \pi_1(K(\mathcal{T}_i))$ ,  $i = 1, 2$ , is injective by induction hypothesis, we see that  $\pi_1(K(\mathcal{P})) = \pi_1(K(\mathcal{T}_1)) *_{\pi_1(K(\mathcal{T}_{12}))} \pi_1(K(\mathcal{T}_2))$  is an

<sup>1</sup>We thank Manuela Ana Cerdeiro for pointing this example out to us.

amalgamated product. Furthermore, since both  $K(\mathcal{T}_i)$  and the intersection  $K(\mathcal{T}_{12})$  are aspherical by induction hypothesis, and the inclusion induced maps  $\pi_1(K(\mathcal{T}_{12})) \rightarrow \pi_1(K(\mathcal{T}_i))$ ,  $i = 1, 2$ , are injective, a theorem of Whitehead [13] (see also Gersten [4], Theorem 5.1) implies that  $K(\mathcal{P})$  is aspherical as well.

Since  $\pi_1(K(\mathcal{P}))$  is an amalgamated product both inclusion induced homomorphisms  $\pi_1(K(\mathcal{T}_i)) \rightarrow \pi_1(K(\mathcal{P}))$ ,  $i = 1, 2$ , are injective. We show that this is true for all proper maximal sub-LOTs and not just for  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . Let  $\mathcal{T}_j$  be a maximal proper sub-LOT. If  $\mathcal{T}_1 \cap \mathcal{T}_j = \emptyset$ , then  $\mathcal{T}_j \subseteq \mathcal{T}_2$  and hence  $\mathcal{T}_j = \mathcal{T}_2$ . If  $\mathcal{T}_1 \cap \mathcal{T}_j \neq \emptyset$  and  $j \neq 1$ , then  $\mathcal{P} = \mathcal{T}_1 \cup \mathcal{T}_j$  and  $\pi_1(K(\mathcal{P})) = \pi_1(K(\mathcal{T}_1)) *_{\pi_1(K(\mathcal{T}_{1j}))} \pi_1(K(\mathcal{T}_j))$ . In particular  $\pi_1(K(\mathcal{T}_j)) \rightarrow \pi_1(K(\mathcal{P}))$  is injective.

Now suppose  $\mathcal{H}$  is a sub-LOT of  $\mathcal{P}$ . If  $\mathcal{H} = \mathcal{P}$  then  $\pi_1(K(\mathcal{H})) \rightarrow \pi_1(K(\mathcal{P}))$  is injective. If  $\mathcal{H}$  is a proper sub-LOT, then  $\mathcal{H}$  is contained in some  $\mathcal{T}_j$ . Since  $\pi_1(K(\mathcal{H})) \rightarrow \pi_1(K(\mathcal{T}_j))$  is injective by induction hypothesis and  $\pi_1(K(\mathcal{T}_i)) \rightarrow \pi_1(K(\mathcal{P}))$  is injective for all  $1 \leq i \leq n$ , we see that  $\pi_1(K(\mathcal{H})) \rightarrow \pi_1(K(\mathcal{P}))$  is injective.

**Case 2.** The  $\mathcal{T}_i$ ,  $i = 1, \dots, n$ , are pairwise disjoint.

Let  $P$ ,  $T_i$ , and  $T$  be the LOT-presentations of  $\mathcal{P}$ ,  $\mathcal{T}_i$ , and  $\mathcal{T}$ , respectively. By Theorem 4.4 there exists a subset  $S$  of the set of generators of  $P$  so that  $P_S$  satisfies the Stallings test relative to  $T_S$ . It follows from Theorem 3.7 that  $K(P_S)$  is VA relative to  $K(T_S)$  and that the inclusion induced maps  $\pi_1(K((T_i)_S)) \rightarrow \pi_1(K(P_S))$  are injective. Using the homeomorphism  $\phi: K(P) \rightarrow K(P_S)$  defined in Section 4 we conclude that  $K(P)$  is VA relative to  $K(T)$  and the inclusion induced maps  $\pi_1(K((T_i))) \rightarrow \pi_1(K(P))$  are injective. By induction hypothesis  $\pi_2(K(T_i)) = 0$  and hence  $\pi_2(K(T)) = 0$ . Now  $\pi_2(K(P)) = 0$  follows from Theorem 2.3. Injectivity of the inclusion induced map  $\pi_1(K(\mathcal{H})) \rightarrow \pi_1(K(\mathcal{P}))$ , where  $\mathcal{H}$  is a sub-LOT of  $\mathcal{P}$ , follows by the argument given in the paragraph before the statement of Case 2.

This completes the proof of Theorem 5.1.  $\square$

## 6 Further consequences and applications

Many of the arguments given in the previous section can be used to prove asphericity of labeled oriented trees that are not injective.

**Theorem 6.1** *Let  $\mathcal{P}$  be a reduced labeled oriented tree and let  $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$  be the set of maximal proper sub-LOTs. Let  $\mathcal{T} = \mathcal{T}_1 \cup \dots \cup \mathcal{T}_n$ . Assume each  $\mathcal{T}_i$  is aspherical.*

1. *If the  $\mathcal{T}_i$  are pairwise disjoint and  $\mathcal{P}$  is injective relative to  $\mathcal{T}$ , then  $\mathcal{P}$  is aspherical.*
2. *If  $\mathcal{T}_i \cap \mathcal{T}_j \neq \emptyset$  for some  $i \neq j$ , then  $\mathcal{P} = \mathcal{T}_i \cup \mathcal{T}_j$ . If both inclusion induced homomorphisms  $\pi_1(K(\mathcal{T}_i \cap \mathcal{T}_j)) \rightarrow \pi_1(K(\mathcal{T}_i))$  and  $\pi_1(K(\mathcal{T}_i \cap \mathcal{T}_j)) \rightarrow \pi_1(K(\mathcal{T}_j))$  are injective, then  $\mathcal{P}$  is aspherical.*

**Proof:** We prove 1: Let  $P$ ,  $T_i$ , and  $T$  be the LOT-presentations of  $\mathcal{P}$ ,  $\mathcal{T}_i$ , and  $\mathcal{T}$ , respectively. By Theorem 4.4 there exists a subset  $S$  of the set of generators of  $P$  so that  $P_S$  satisfies the Stallings test relative to  $T_S$ . It follows from Theorem 3.7 that  $K(P_S)$  is VA relative to  $K(T_S)$ . Using the homeomorphism  $\phi: K(P) \rightarrow K(P_S)$  defined in Section 4 we conclude that  $K(P)$  is VA relative to  $K(T)$ . Since we assumed the  $\mathcal{T}_i$  to be aspherical it follows that  $\pi_2(K(T)) = 0$ . Now  $\pi_2(K(P)) = 0$  follows from Theorem 2.3. The second part of the theorem follows from a theorem of Whitehead [13] concerning the asphericity of unions of aspherical spaces. We used Whitehead's theorem already when we considered Case 1 in the proof of Theorem 5.1.  $\square$

The  $\mathcal{T}_i$  may be aspherical for a variety of reasons without being injective. For instance they could satisfy small-cancellation conditions.

We conclude this article with an application to long virtual knots. See Kauffman [10] for an overview of virtual knot theory. A virtual link diagram is a planar 4-regular graph with under- and over crossing information at some nodes. A virtual knot diagram is a virtual link diagram with only one link component. A long virtual knot diagram  $k$  is obtained by cutting a virtual knot diagram at a point on an edge, thus producing a graph that has exactly two nodes of valency one. A Wirtinger presentation  $P(k)$  can be read off

in the usual way. It is easy to see that  $P(k) = P(\mathcal{P})$ , where  $\mathcal{P}$  is a labeled oriented interval. More details on the connection between labeled oriented intervals and long virtual knots can be found in [5]. We say a long virtual knot diagram is *aspherical*, if the standard 2-complex associated with the Wirtinger presentation is aspherical. A virtual knot diagram is *alternating* if one encounters over- and under-crossings in an alternating fashion when traveling along the diagram. A long alternating virtual knot diagram is obtained when cutting an alternating virtual knot diagram.

**Corollary 6.2** *A long alternating virtual knot diagram  $k$  is aspherical.*

**Proof:** The labeled oriented interval that records the Wirtinger presentation of  $k$  is injective. The result follows from Theorem 5.1.  $\square$

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