Journal of Knot Theory and Its Ramifications
Vol. 12, No. 7 (2003) 947–962

GENERALIZED KNOT COMPLEMENTS AND SOME ASPHERICAL RIBBON DISC COMPLEMENTS

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Accepted 10 March 2003

ABSTRACT

We generalize some aspects of standard knot-theory to all ribbon-disc complements. We study asphericity of the complement of properly embedded links in certain contractible singular 3-manifolds that should be thought off as replacements of the 3-ball in the classical setting. We apply our results to show asphericity of 2-complexes modelled on labelled oriented graphs that correspond to alternating prime projections on some surface.

1. Introduction

The Whitehead conjecture states that any subcomplex of an aspherical 2-complex is itself aspherical. An affirmative answer implies the asphericity of knot complements, a fact that was not known in the early 40's when Whitehead stated his conjecture and might have motivated it. The asphericity of alternating knot complements was shown by Aumann in 1956 using combinatorial techniques (small cancellation theory) and in 1957 Papakyriakopoulos proved asphericity for general knot-complements using 3-manifold techniques rather than 2-complexes. The Whitehead conjecture however remains open up to this day despite considerable expense of effort (see Bogley [1] for a good survey on Whitehead's asphericity question).

LOT-presentations (LOT stands for labeled oriented tree), certain generalizations of Wirtinger-presentations of knots, play a central role in view of Whitehead's conjecture. A labeled oriented graph (LOG) is an oriented graph on vertices $\{1, \ldots, n\}$, say, where each oriented edge is labeled by a vertex. Associated with it comes a presentation on generators $x_1, \ldots, x_n$ in one-to-one correspondence with the vertices. For an edge with initial vertex i, terminal vertex j and label k we add a relation $x_i x_k = x_k x_j$. We refer to such a presentation as a LOG-presentation and to the standard 2-complex associated with it as a LOG-complex. A LOT is a LOG, where the underlying graph is a tree. It was shown by Howie [6] that if K is a finite 2-complex that 3-deforms to a point and $e \in K$ is an open 2-cell, then $K - e$ 3-deforms to some LOT-complex. He shows in particular that, if the Andrews-Curtis conjecture is true (i.e. if each finite contractible 2-complex 3-deforms to a point) then the asphericity of LOT-complexes implies the Whitehead conjecture in the finite case.

The main purpose of this paper is to generalize some aspects of standard knot theory in order to study the asphericity of LOT-complexes. By standard knot theory we mean the study of knot or link complements in the 3-sphere, or the study of arc complements in the 3-ball.

If one removes a properly embedded arc from a 3-ball then the resulting 3-manifold collapses to one of its Wirtinger-spines. From this spine one can read off a LOT-presentation. Of course, not every LOT-presentation arises in this way. All LOT-complexes however are known to be spines of ribbon-disc complements, that is complements of discs in the 4-ball, embedded in a particular fashion (see [7]). We construct 3-dimensional arc complements that have LOT-complex spines. We show that if one removes a properly embedded arc from a coned off oriented surface (in the standard setting this surface is a disc), then the resulting singular 3-manifold collapses to a LOT-complex; and in fact every LOT-complex occurs as a spine of such a singular 3-manifold. For us a compact singular 3-manifold X is a finite 3-complex with vertex links being oriented surfaces. Note that every point in X that is not a vertex is a manifold point, that is it has a 2-sphere or a 2-disc as its link. A *properly embedded arc* A in the singular 3-manifold X is the image of an embedding $i : I \to X$ of the unit interval $I = [0, 1]$ such that $i(0)$ and $i(1)$ lie on the boundary of X and $A = i(I)$ does not contain points that are not manifold points. The arc-complement $X - N(A)$ is again a compact singular 3-manifold, where $N(A)$ is an open tubular neighborhood of A.

In particular we show the following:

Theorem 1. *Let X be a contractible singular 3-manifold with boundary and A a properly embedded arc in X. If $X - A$ is always aspherical, then LOT-complexes are aspherical.*

Any tame arc projection $\bar{L}$ on a disk gives rise to a LOT $\mathcal{G}$ and a LOT-complex $W(\bar{L})$ (the Wirtinger complex). This is illustrated in Fig. 1. As mentioned above, if $\bar{L}$ is alternating, asphericity of the complex $W(\bar{L})$ can be established by combinatorial techniques (see Wise [11] for a short and elegant proof of this fact). Strong 3-manifold tools such as Dehn's Lemma (that are not available in the singular setting) are not needed.

Now any tame arc projection $\bar{L}$ on an oriented surface allows us to read off a LOT in the same fashion. In fact every LOT-complex is homotopy equivalent to

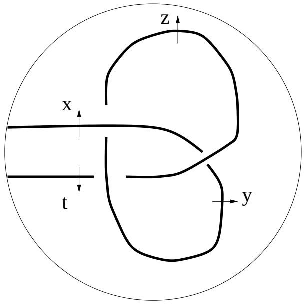

Fig. 1. An arc-projection on a disc with associated LOG.

one of the form $W(\bar{L})$ where $\bar{L}$ is an arc-projection on some orientable surface as we illustrate in the following paragraph. Here we show the following generalization of the classical situation mentioned above:

Theorem 2. *If $\bar{L}$ is an alternating prime arc-projection on a punctured orientable surface then the associated LOT-complex $W(\bar{L})$ is aspherical.*

Our construction will be more general in the sense that we do not just look at arc complements, but at complements of a finite collection of properly embedded arcs and circles in contractible singular 3-manifolds of a very special nature.

2. Generalized Wirtinger-Complexes

In the following we will use the notion of internally collapsing cells in 2-complexes. Suppose e is a 1-cell with two distinct vertices u and v. To internally collapse e onto the vertex u means to identify all of e with u.

Suppose d is a 2-cell with boundary edge-path ew where e is an edge and w an edge-path not containing e in the 2-complex K. Internally collapsing d across e means removing the interior of d and e and whenever a 2-cell d' of K different from d contains e in its boundary replace e by w^{-1} in the boundary of d'. Observe that internal collapses do not change the homotopy type of the 2-complex (see also Lemma 2.2 of [5]).

We start with an orientable surface F with or without boundary. We require F to be connected (otherwise we can apply the following process to every component). We form a singular 3-manifold X by first fattening F to $F \times [0,1]$ and then coning off the top surface $F \times \{1\}$. So $X = F \times [0,2]/F \times \{2\}$. Now X is a compact contractible singular 3-manifold that contains only one singular point (the cone point T). Note that in case F is a disc, X is the 3-ball.

We remove a link L from X consisting of properly embedded arcs and circles contained in $F \times [0,1]$. We refer to the link-complement $X - L$ as a *surface link-complement.* Now project $F \times [0,1]$ to $F \times \{0\}$ to get a link projection $\bar{L}$ on F by assuming L to be in general position. We want our F to be minimal in the following sense: A proper embedding of L in $F \times [0,1] \subseteq X$ is *minimal*, if $\bar{L}$ defines a cell decomposition of F. In order to make an embedding minimal, remove unused handles from F and add discs to unused boundary components (i.e. boundary components d with no arc of L incident with d). The dual cell decomposition of the one given by the projection $\bar{L}$ will be referred to as F'. Every crossing of $\bar{L}$ defines a 2-cell in F'. An orientation of L (or of $\bar{L}$) induces an orientation of the 1-cells of F' since F is orientable (see Fig. 2). In the following we will *always assume that the link-projection is minimal.*

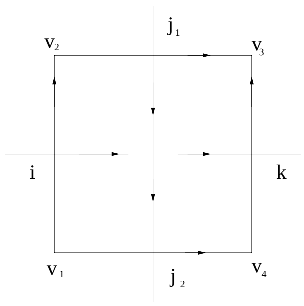

Fig. 2. A relator and the dual link.

Next we collapse $X - L$ to a 2-complex $W^*(\bar{L})$. The cell structure on X is the one induced from F'. We first collapse $F \times [0,1] - L \subseteq X$. Start the collapsing procedure on $\partial(F \times [0,1] - L) - \partial(F \times [0,1])$ and collapse towards the bottom- and top-surface $F'_B = F' \times \{0\}$ and $F'_T = F' \times \{1\}$. We obtain a complex that contains F'_B and F'_T as subcomplexes, and the only 3-cells left are the ones coming from coning off the top surface. We collapse the remaining 3-cells by pushing in the 2-cells of F'_T. Next internally collapse 1-cells of the form $v \times [0,1]$, where v is a vertex of F'. This identifies the vertices in F'_B and F'_T. We denote the resulting 2-complex by $W^*(\bar{L})$.

Figure 3 shows a typical piece of $X - L$. We see a cube $e^2 \times [0,1]$. e^2 is a square 2-cell of F', with two tubes removed coming from the link. On top of the cube sits a 3-cell obtained from coning off the top. The collapsing procedure can be easily visualized in Fig. 3. We first widen the tubes until the top of the upper tube meets the top of the cube, the bottom of the lower tube meets the bottom of the cube and top of the lower tube meets the bottom of the upper tube. So the cube with the tubes removed contributes three 2-cells, the top square, the bottom square and a 2-cell coming from the material in between the tubes, shown in Fig. 4. We now push in the top square to collapse the remaining 3-cell and then (internally) collapse the vertical edges of the cube.

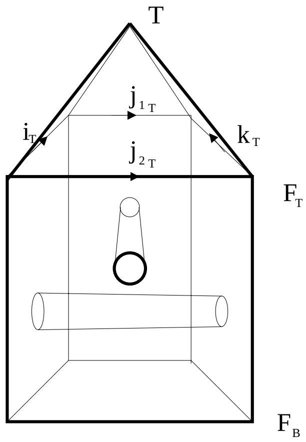

Fig. 3. A part of $X - L$.

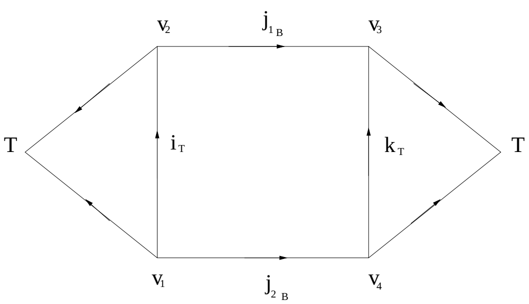

Fig. 4. The additional relator.

Let us describe $W^*(\bar{L})$ in more detail. The vertices are the vertices in F' together with the cone point T. The 1-cells are the 1-cells of F'_B , the 1-cells of F'_T together with one 1-cell for each vertex v of F', connecting v to the vertex T. If i is an edge in F' then we denote the corresponding edges in F'_B, F'_T by i_B, i_T, respectively. The 2-cells are the 2-cells of F'_B, together with one 2-cell for each edge of F' (coned off 1-cells), together with an additional 2-cell for every 2-cell of F'. Indeed, if we have a 2-cell e^2 in F' as shown in Fig. 2 then we have a 2-cell f^2 in $W^*(\bar{L})$ with boundary $i_T j_{1B}(k_T)^{-1}(j_{2B})^{-1}$ as shown in Fig. 4.

We claim that $W^*(\bar{L})$ can be collapsed to the LOG-complex $W(\bar{L})$ corresponding to a LOG $\mathcal{G}$ that can be read off $\bar{L}$ (as indicated in Fig. 1). Collapse the cone on the 1-skeleton of F'_T (this is a contractible subcomplex of $W^*(\bar{L})$) to a point. The resulting 2-complex has one vertex T and edges in correspondence with the edges of F'. The 2-cell f^2 of $W^*(\bar{L})$ with boundary $i_T j_{1B}(k_T)^{-1}(j_{2B})^{-1}$ now becomes a 2-cell with boundary $j_1 j_2^{-1}$. We can internally collapse such 2-cells by identifying their boundary edges. So the 2-cells left are the ones coming from F'_B as shown in Fig. 2. Such a cell has the boundary $ijk^{-1}j^{-1}$ and hence is a LOG-cell.

Note also that the LOG for this LOG-complex can be directly read off the projection $\bar{L}$. Orient the projection and label subarcs running from undercrossing to undercrossing by $1, 2, 3, \ldots$. This forms the vertex set of the LOG. For every crossing add an edge as shown in Fig. 5 (see also Fig. 1 for a worked out example).

Fig. 5. Reading off the LOG from the projection.

We have just seen that every tame proper link projection $\bar{L}$ on an orientable surface F gives rise to a LOG $\mathcal{G}$ with underlying graph consisting of disjoint arcs and circles. On the other hand, we could have started with such a LOG $\mathcal{G}$, drawn the corresponding link projection $\bar{L}$ onto an orientable surface F, such that the boundary of $\bar{L}$ is in the boundary of F. Then thicken F, remove L and cone off $F \times \{1\}$ to end up with a link complement that collapses to the LOG-complex $W(\bar{L})$ by the above process.

Let us illustrate this with an example. Consider the LOT $\mathcal{T}$ of Fi. 6. Figure 7 shows the 2-cells of the corresponding LOT-complex. Figure 8 shows the orientable surface F'. Sides of squares with the same label have to be identified (4_1 with 4_1 and so on). F' has a square cell structure. The arc-projection (thick lines) is dual to this cell-structure.

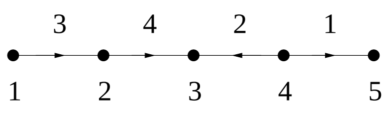

Fig. 6. A simple example.

Note that the indicated process of building a link-projection on a surface F' from a given LOG $\mathcal{G}$ (consisting of arcs and circles) is unique only in case $\mathcal{G}$ is *injective*, i.e. each generator of $\mathcal{G}$ occurs at most once as a conjugator in a relator. This is satisfied in our example.

A simple Euler-characteristic count shows that the surface in our example is a once punctured torus. Note that we have cycles of length 2 in the 1-skeleton of F'

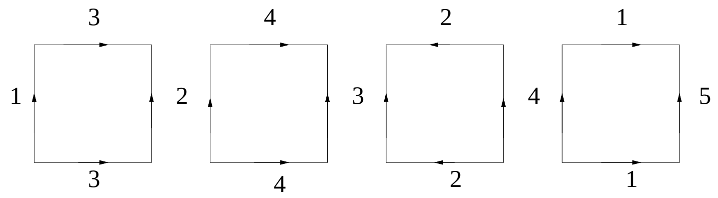

Fig. 7. The 2-cells of the Wirtinger-complex.

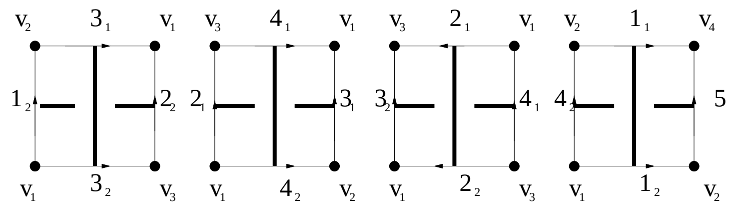

Fig. 8. The surface F'.

other than the boundary cycle, for example $3_1 4_2$.

If the given LOG is not injective, then there is at least one generator x_i which occurs in at least two relators R_j, R_k as conjugator and at most two relators R_t, R_s as initial or terminal vertex. The 2-cells corresponding to R_j, R_k may be identified along their boundaries labeled x_i in any of the two different possible orders. The dual link has two overcrossings in a row and the LOG does not tell which one has to be passed first after the undercrossing coming from R_t. This gives rise to two different surfaces F.

Since any LOT can be transformed into a labeled oriented interval without altering the homotopy type of the associated LOT-complex (see [7]), we proved Theorem 1 stated in the introduction. In fact, we have shown something stronger:

Theorem 3. *Every LOT-complex is homotopy equivalent to a spine of a surface-arc complement.*

3. Generalized Dehn-Complexes

We define the complex $D^*(\bar{L})$. It is obtained from the complex $W^*(\bar{L})$ by coning off the bottom surface to obtain $W^*(\bar{L})/F \times \{0\}$ and pushing in the resulting 3-cells across the faces of F'_B. $D^*(\bar{L})$ is a spine of the singular 3-manifold $Y - L$ obtained from $X - L$ by coning off the bottom surface as well. In order to collapse $Y - L$ to $D^*(\bar{L})$, we first collapse the subcomplex $X - L$ as described in the previous section to arrive at a 3-complex that contains $W^*(\bar{L})$ as a subcomplex. The remaining 3-cells come from coning off the bottom surface and we can collapse them by pushing in the faces of F'_B.

A typical 2-cell of $D^*(\bar{L})$ is shown in Fig. 9 below. Note that the vertices $v_1, \ldots, v_4$ need not all be distinct.

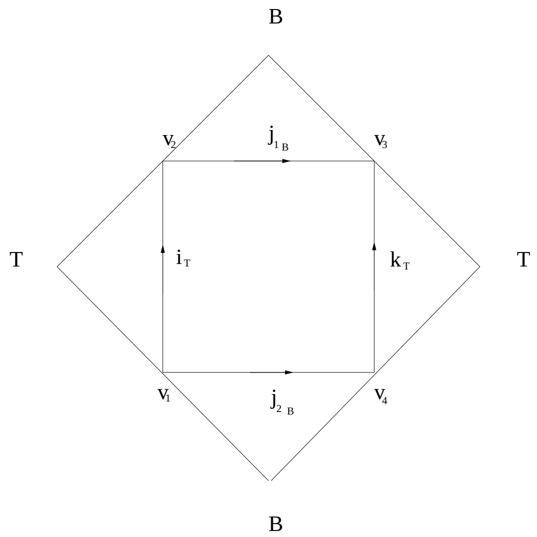

Fig. 9. A 2-cell of $D^*(\bar{L})$.

The natural map $W^*(\bar{L}) \to D^*(\bar{L})$ induces an epimorphism on fundamental groups that will in general have a non-trivial kernel (of course it is an isomorphism in case F is a disc). When one is interested in asphericity, things are not so bad. We will see in a later section that if $D^*(\bar{L})$ satisfies a strong asphericity criterion (diagrammatic reducibility) then $W^*(\bar{L})$ (and hence the Wirtinger complex $W(\bar{L})$) is aspherical.

The Dehn-complex $D(\bar{L})$ is obtained from $D^*(\bar{L})$ by a procedure similar to the one we used to obtain $W(\bar{L})$ from $W^*(\bar{L})$. Internally collapse cone-triangles by pushing in the corresponding edges in F'_T and F'_B. The complex obtained has vertices T, B together with the vertices coming from F'. For every vertex $v \in F'$ we have edges e^v_T, e^v_B connecting v to T and B to v, respectively. The vertices of F' can now be omitted because if a 2-cell of $D(\bar{L})$ contains one of the edges e^v_T or e^v_B in its boundary, it contains the path $e^v_T e^v_B$. The edge obtained from this path after leaving off v we also denote by v (and hope this does not lead to confusion). So $D(\bar{L})$ has two vertices T and B, edges in correspondence with the vertices of F' and 2-cells as shown in Fig. 10.

Let us continue the example we already considered in the previous section. Figure 8 shows a punctured torus with an arc-projection $\bar{L}$. From it we quickly read off the Dehn-complex. Its 2-cells are depicted in Fig. 11.

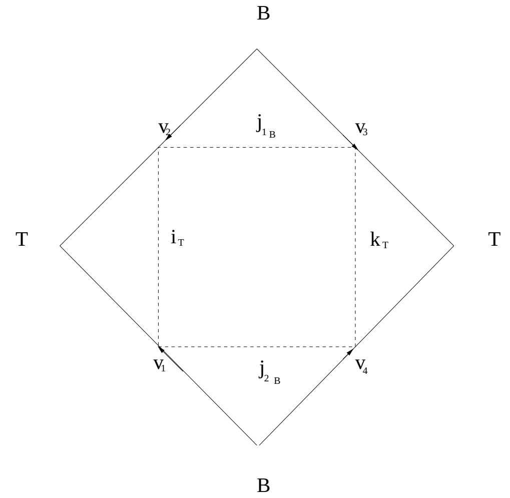

Fig. 10. A 2-cell in $D(\bar{L})$.

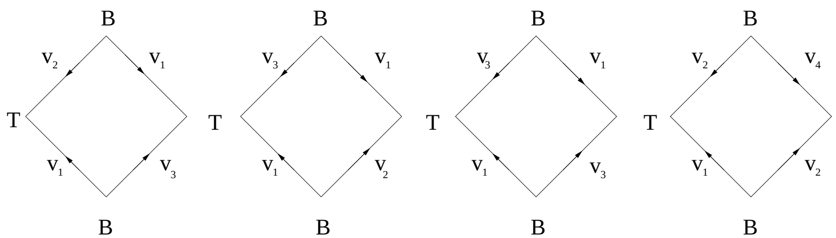

Fig. 11. The Dehn-complex of the simple example.

4. Asphericity Comparison

In this section we show that if the Dehn-complex $D(\bar{L})$ of a link-projection $\bar{L}$ on an aspherical orientable surface F (punctured or not) satisfies a strong combinatorial asphericity condition (see for example [8], [4] or [3] for asphericity conditions as diagrammatically aspherical (DA) and diagrammatically reducible (DR)) then the Wirtinger complex $W(\bar{L})$ is aspherical. Note that every orientable surface except for the sphere is aspherical and every punctured orientable surface is aspherical.

Theorem 4. *Let $\bar{L}$ be a proper link projection on a minimal orientable surface $F \neq S^2$. If the Dehn-complex $D(\bar{L})$ is diagrammatically reducible (DR) then the Wirtinger-complex $W(\bar{L})$ is aspherical.*

Before giving a proof we remark that the above result does not hold if the link projection $\bar{L}$ lives on a sphere. Indeed suppose $\bar{L}$ is a prime alternating knot on S^2. Then it is known that the Dehn-complex $D(\bar{L})$ is DR (even stronger: it is locally

CAT(0), see Wise [11]) but $W(\bar{L})$ is not aspherical being homotopy equivalent to a ball with a knot removed.

Proof. We will first show that if $D(\bar{L})$ is DR then so is $D^*(\bar{L})$. Recall from the construction of these complexes that the first complex is obtained from the second one by internally collapsing 2-cells and then removing unnecessary vertices. The statement will follow from the following. □

Lemma 5. *Suppose the 2-complex N is obtained from the 2-complex K by a series of internal collapses and omitting unnecessary vertices. If N is DR then so is K.*

Proof. Suppose first that N is obtained from K by internally collapsing a 1-cell. A reduced spherical diagram over K would lead to a reduced spherical diagram over N by internal collapses in contradiction to the assumption of N being DR.

Next suppose N is obtained from K by internally collapsing a 2-cell d. Then d contains an edge e which is pushed in across d onto the boundary path w opposite e. Suppose K is not DR. Let S be a reduced spherical diagram over K. We can transform it into a spherical diagram S' over N by replacing each edge in S labeled e with a path labeled w. Note that this will change a face labeled with d into a region with boundary label ww^{-1}. Such a region we simply squeeze off. Clearly the replace move cannot create a folding pair. Neither can squeezing off regions. So the new spherical diagram S' is reduced. This contradicts the fact that N is DR.

Lastly suppose N is obtained from K by dropping unnecessary vertices. In other words K is obtained from N by subdivision of edges. But subdivision of edges preserves the property DR (see the note in Paragraph 6 of [4]). □

We continue the proof of Theorem 4. It remains to show that if $D^*(\bar{L})$ is DR then $W^*(\bar{L})$ is DA (then $W(\bar{L})$, being homotopy equivalent to $W^*(\bar{L})$, is aspherical). The complex $D^*(\bar{L})$ is obtained from $W^*(\bar{L})$ by coning off the bottom surface F'_B and then collapsing the 3-cells by pushing in faces in that surface. Since an orientable surface F'_B (not being a 2-sphere) is DR, Theorem 4 will follow from

Lemma 6. *Suppose K is a 2-complex and K' is a subcomplex that is DR. Suppose the 2-complex N is obtained from K by coning off K' and collapsing 3-cells by pushing in 2-cells of K'. If N is DR then K is DA.*

Proof. Suppose S is a non-empty spherical diagram over K that is reduced and stays reduced when diamond moves are performed. Note that S must contain a face labeled with a 2-cell from $K - K'$ because we assumed K' to be DR. Let $S' \subset S$ be the sub-surface which maps to K' (possibly singular in the sense that boundary points could be pinched together). We may assume that the words read off the boundary of S' are all cyclically reduced words. This can be achieved by performing diamond moves along the boundary.

Take any non-empty component of the complement of S' in S including the one-cells in its boundary. This surface C maps to the closure of $K - K'$ and therefore also to N, since $K - K'$ doesn't differ from the related parts of N. C has a union of 1-spheres as boundary components (which may again be pinched together along vertices). Cone off each of these 1-sphere boundary components. Label the cone points by the cone point of N and the new edges by the appropriate edge labels of N which came from coning off vertices. The resulting 2-sphere can be viewed as a reduced spherical diagram over N in contradiction to the assumption that N is DR. □

5. Main Results

Since every 2-complex which satisfies the weight test is DR (for the Definition of the weight test see [8]) the following is an immediate consequence of Theorem 4.

Corollary 7. *Let $\bar{L}$ be a proper link projection on a minimal orientable surface $F \neq S^2$. If $D(\bar{L})$ satisfies the weight test, then $W(\bar{L})$ is aspherical.*

If the vertex links in the Dehn-complex do not contain cycles of length shorter than four it will satisfy the weight test. Indeed, if we give weight $1/2$ to every edge in the vertex links then the weight test conditions on 2-cells (all being squares) and cycles in the links will be fulfilled. In that case a stronger, metric argument also implies DR. We can metrize the Dehn complex by giving each 2-cell the metric of the unit right-angled square in the plane. This complex satisfies the link condition (i.e. the sum of the angles around an inner vertex is $\geq 2\pi$) if there are no cycles of length shorter than four in the vertex links. In that case $D(\bar{L})$ is locally CAT(0) and hence DR.

We consider link projections with arc and circle components on an orientable surface F. We of course require that the boundary of an arc-component lies on the boundary of F (F closed and all components being circle-components is also allowed). We say a link-projection $\bar{L}$ on a closed oriented surface F is *alternating* if we meet under- and overcrossings on each component in an alternating fashion. We say a link projection on a punctured oriented surface is *alternating* if it is obtained from an alternating link projection on a closed oriented surface by removing disjoint open discs.

We have already mentioned in the introduction that if $\bar{L}$ is an alternating arc-projection on a disc then the associated Dehn-complex $D(\bar{L})$ (and hence the associated Wirtinger-complex, being homotopy equivalent to the Dehn-complex) is aspherical. This simply is not true if we consider projections on higher genus surfaces:

Example 8. There exists an alternating arc-projection $\bar{L}$ on a once punctured torus with non-aspherical Dehn-complex.

Consider the alternating arc-projection $\bar{L}$ shown in Fig. 8 from which one reads off the LOT $\mathcal{T}$ shown in Fig. 6. The associated Dehn complex is shown in Fig. 11.

If we collapse the edge labeled by v_4 in this Dehn-complex we obtain a standard complex built on the presentation

$$\langle v_1, v_2, v_3 \mid v_1 v_2^{-1} v_1 v_3^{-1}, v_1 v_3^{-1} v_1 v_2^{-1}, v_1 v_3^{-1} v_1 v_3^{-1}, v_1 v_2^{-1} v_2^{-1} \rangle$$

which is homotopy equivalent to the standard complex built on $\langle v_2 \mid v_2^8, v_2^2 \rangle$. So the fundamental group is finite (cyclic of order 2) and there is no aspherical 2-complex with non-trivial finite fundamental group. In particular the Dehn-complex is not aspherical.

On the other hand the LOT-complex of Fig. 6 is DR, which is a consequence of Theorem 1.1 of [9]. In fact the LOT-complex is injective and does not contain a reducible (as explained in [9]) sub-LOT.

Note that one of the links $lk(T)$ or $lk(B)$ in $D(\bar{L})$ will certainly contain a cycle of length 2 in case $\bar{L}$ is not alternating. But alternating is not sufficient to rule out cycles of length 2. In fact, in the classical setting where $\bar{L}$ is an alternating knot projection on a sphere F the following statements are equivalent (see [11] and [2] page 220-224):

- $\bar{L}$ comes from a prime knot;
- the vertex links in $D(\bar{L})$ do not contain cycles of length shorter than four;
- the 1-skeleton of F' (the cell decomposition of F dual to the one induced by $\bar{L}$) does not contain cycles of length two.

This motivates the following

Definition 9. An alternating link-projection $\bar{L}$ on an orientable surface F is *prime* if the 1-skeleton of F' does not contain cycles of length shorter than four except for cycles made up of boundary edges.

Theorem 10. *Let $\bar{L}$ be a prime alternating link-projection on the orientable surface F. We assume $F \neq S^2$ and that $\bar{L}$ defines a cell decomposition of F. Then the vertex links of the Dehn complex $D(\bar{L})$ do not contain cycles of length shorter than four. In particular $D(\bar{L})$ is DR and hence $W(\bar{L})$ is aspherical.*

Proof. Let us consider the link of the top-vertex T. By construction of $D(\bar{L})$ we can assign to every edge in $lk(T)$ an edge in F'. It is easy to see that in case of an alternating link-projection this assignment is one-to-one, so we can think of $lk(T)$ as sitting inside the 1-skeleton of F'. So any cycle in $lk(T)$ gives a cycle in the 1-skeleton of F'.

Assume we have a cycle of length less than four. By the definition of prime, this cycle is completely made up of boundary edges. Since F was obtained from a closed surface with a closed link projection by removing disks, every boundary component of F meets $\bar{L}$ in an even number of points. Hence every boundary component of F' is a circle made up of an even number of edges.

So our cycle consists of two edges e_1 and e_2 that make up a boundary component of F'. So we have the situation depicted in Fig. 12 in contradiction to $\bar{L}$ being alternating. □

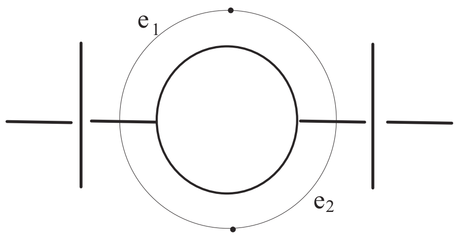

Fig. 12. Surface with boundary of length 2.

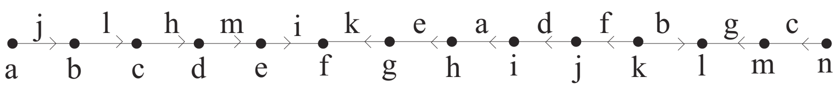

Fig. 13. Injective LOT $\mathcal{T}$.

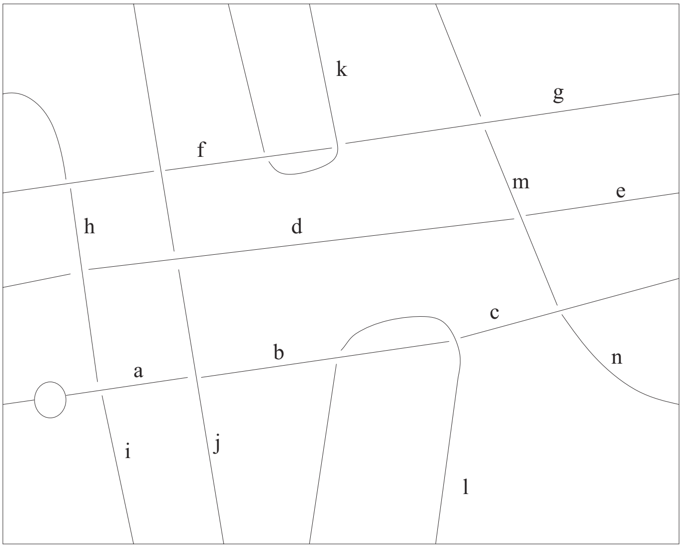

Fig. 14. Torus F together with a link projection $\bar{L}$.

Let us look at an example. Consider the injective LOT $\mathcal{T}$ of Fig. 13. An alternating arc-projection $\bar{L}$ on a torus from which one can read off the LOT $\mathcal{T}$ is shown in Fig. 14 (opposite sides of the square are to be identified). Observe that the

1-skeleton of the dual decomposition F' does not contain cycles of length shorter than four except for the cycle of length 2 that comes from the boundary shown as a circle in the lower left of the figure. Hence we have an example that satisfies the hypothesis of Theorem 10 and we can conclude that the corresponding LOT-complex $W(\bar{L})$ is aspherical. Note that $W(\bar{L})$ is not C(4), T(4) because of cycles of length 2 and 3 in the vertex-link.

We will next address a particular non-prime situation.

Theorem 11. *Let $\bar{L}$ be a alternating link-projection on the orientable surface $F \neq S^2$. We assume that $\bar{L}$ defines a cell decomposition of F. Let γ be a simple closed path in F that separates the surface into two pieces F_1 and F_2 and intersects $\bar{L}$ transversely. If both $\bar{L}_1 = \bar{L} \cap F_1$ and $\bar{L}_2 = \bar{L} \cap F_2$ are prime alternating, then $D(\bar{L})$ is DR and hence $W(\bar{L})$ is aspherical.*

Proof. Let γ_i be the boundary component in F_i, $i = 1, 2$ obtained from cutting F along γ. Note that since we assume both $\bar{L}_1$ and $\bar{L}_2$ are prime alternating $n = |\gamma_i \cap L_i|$ is even. If $n \neq 2$ then $\bar{L}$ will be prime alternating and we are done by Theorem 10. So suppose $n = 2$. The situation is shown in Fig. 15.

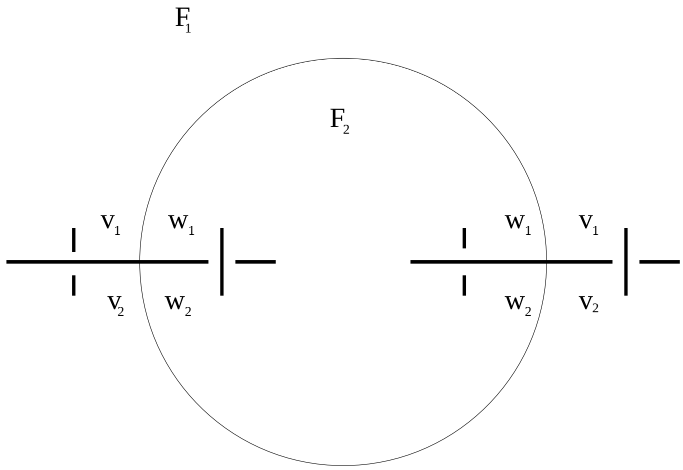

Fig. 15. A part of the link $\bar{L}$ on F.

Now $D(\bar{L})$ is obtained from $D(\bar{L}_1) \cup D(\bar{L}_2)$ by identifying edges $v_1 = w_1$ and $v_2 = w_2$. Choose a square in $D(\bar{L}_1)$ that contains the edge-path $v_2^{-1} v_1$ in its boundary and let d be the diagonal of that square connecting the endpoints of v_1 and v_2. We subdivide so that d is an edge (see Fig. 16). Note that d is a local geodesic in the locally CAT(0)-square complex $D(\bar{L}_1)$ (each square being a right-angled unit square).

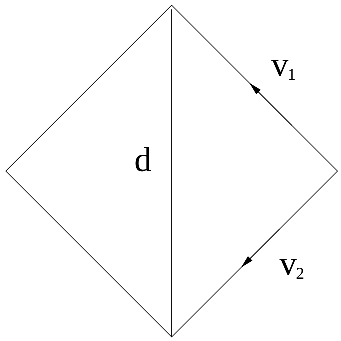

Fig. 16. Subdividing a 2-cell.

Similarly choose a square in $D(\bar{L}_2)$ that contains the edge-path $w_2^{-1}w_1$ in its boundary and let d' be the diagonal of that square connecting the endpoints of w_1 and w_2. Now glue $D(\bar{L}_1)$ to $D(\bar{L}_2)$ by identifying $d = d'$ to obtain a 2-complex K. Since we glued along local geodesics K is locally $CAT(0)$ and hence DR. Now note that $D(\bar{L})$ is obtained from K by internally collapsing two 2-cells. First collapse the triangle with boundary $dv_2^{-1}v_1$ by pushing the edge d onto the edges $v_2^{-1}v_1$. The triangle with boundary $d'w_2^{-1}w_1$ now has become a square with boundary $v_2^{-1}v_1w_2^{-1}w_1$ which we collapse by pushing the two sides $v_2^{-1}v_1$ onto the two sides $w_2^{-1}w_1$. Notice that this last collapse is not an internal collapse in the strict sense as defined in the beginning of Paragraph 2 because the square is collapsed across two edges simultaneously. But note that this process is a composition of a subdivision of a 2-cell and two internal collapses and hence the property DR is preserved (DR is preserved under subdivision, see Paragraph 6 of [4]).

By Lemma 5 $D(\bar{L})$ is DR. □

There is a universal method to construct examples. Take any alternating knot (or link) projection in the plane which is complicated enough. Draw it on a square and cut through some strands which are part of the boundary of the outer region. Connect these arcs with the boundary of the square such that opposite sides of the square have the same number of strands (at least four). After identifying opposite sides there is an example of a link projection on a torus.

Example 12. Take two copies of the torus of Fig. 14 together with their link-projections and identify their boundaries. From the resulting double torus remove a small disk cutting the link-projection twice. The resulting surface F is an example for Theorem 11.

The authors know of no other way to show the asphericity of the corresponding LOT, the methods in [9] fail in this case.

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