Some Aspects of Efficiency

*Jens Harlander**

1. Definitions and motivation

Roughly speaking, a group is said to be efficient if it can be presented in terms of generators and relations in a very economical way: the difference between the number of relators and generators is (at least in the case of a finite group) the rank of the Schur multiplier. Efficiency is an important concept in computational and geometric/combinatorial group theory. In this paper I will focus on the latter topic stressing a more topological point of view. I will discuss the relevance of efficiency questions to well known open problems in geometric group theory and low dimensional topology, such as the Eilenberg–Ganea problem and Wall's finiteness problem in dimension 2. I will also mention efficiency notions for infinite presentations (so called Cockcroft-properties) that provide delicate approximations for asphericity and are important in the context of the Whitehead problem. None of the results presented here are new, but some of them have never been published.

For a finitely presented group G there exists a finite connected 2-complex K with fundamental group G. We define $\chi_2(G)$ to be the minimal Euler-characteristic of such a complex. A lower bound for this number in terms of the homology of G is easily obtained. We have

$$\begin{aligned}\chi(K) &= 1 - r(H_1(K)) + r(H_2(K)) \\ &\geq 1 - r(H_1(G)) + d(H_2(G)).\end{aligned}$$

Here $r(-)$ denote the torsion-free rank and $d(-)$ the minimal number of generators. The last inequality holds because $H_1(K) = H_1(G)$, $H_2(K)$ is torsion-free (so $r(H_2(K)) = d(H_2(K))$) and $H_2(G)$ is a quotient of $H_2(K)$ by Hopf's theorem (see Brown [10]). So we have

$$\chi_2(G) \geq 1 - r(H_1(G)) + d(H_2(G)).$$

*Supported by the Deutsche Forschungsgemeinschaft (DFG) and the Korean Science and Engineering Foundation (KOSEF).

The group G is called *efficient* if this lower bound can be topologically realized, that is, if there exists a finite connected 2-complex K with fundamental group G such that

$$\chi_2(G) = \chi(K) = 1 - r(H_1(G)) + d(H_2(G)).$$

The inequality, although in slightly different form, goes back to P. Hall. The first examples of non-efficient groups were provided by R. Swan [46] (we will discuss his construction in more detail in the next section). His examples supplied a negative answer to a question of B. H. Neumann [41], who wondered whether finite groups with trivial Schur-multiplier have balanced presentations (and thus are efficient).

Aside from group theory, efficiency also plays an important role in 2-dimensional topology, in particular for questions involving the topological realization of partial resolutions, such as the Eilenberg–Ganea conjecture or Wall's finiteness problem in dimension 2 (see Section 3 for details). Since aspherical groups are clearly efficient, the notion of efficiency can be thought of as a crude approximation of asphericity. I will discuss this connection briefly in Section 4.

There is another, more algebraic way to look at efficiency. A finite presentation F/N of G (i.e. F is a finitely generated free group and N is a normal subgroup, normally generated by finitely many elements such that the quotient F/N is isomorphic to G) is called *efficient* if

$$d_F(N) = d(N/[F, N]),$$

where $d_F(N)$ denotes the minimal number of normal generators for N. We had better convince ourselves that a group G is efficient if and only it has an efficient presentation. Choose a finite connected 2-complex K with fundamental group G and $\chi(K) = \chi_2(G)$. We can assume that K has a single vertex, n one-cells and m two-cells. Let N be the kernel of the epimorphism

$$F = \pi_1(K^{(1)}) \to \pi_1(K) = G.$$

We clearly have $d_F(N) = m$. Now G is efficient if and only if

$$\chi(K) = 1 - n + m = 1 - r(H_1(G)) + d(H_2(G))$$

and this is the case if and only if

$$m = n - r(H_1(G)) + d(H_2(G)).$$

Since

$$d(N/[F, N]) = n - r(H_1(G)) + d(H_2(G))$$

(see the first section in [30] for example) we have shown that G is efficient if and only if F/N is efficient.

The conjugation action of F on the normal subgroup N induces a G-module structure on the quotient $N/[N, N]$. This module is called the *relation module* of the presentation F/N, and $d_G(N/[N, N])$ denotes its rank (the minimal number of module generators). Consider the chain of inequalities

$$d_F(N) \geq d_G(N/[N,N]) \geq d(N/[F,N]).$$

If the presentation F/N is not efficient, then either the first or the second (or both) inequalities are strict. The first difference $d_F(N) - d_G(N/[N,N])$ is called *the relation gap* (a term created by Karl Gruenberg; later we will also hear about his "generation gap") and, so far, no finite presentation with positive relation gap is known. All known failure of efficiency occurs in the transition from the relation module $N/[N,N]$ to $N/[F,N]$.

We should remark here that infinite relation gaps are known to exist. In [5], M. Bestvina and N. Brady construct a finitely generated group that has the finiteness property FP_2 but is not finitely presented. So any presentation F/N with finitely generated F has finitely generated relation module, and hence has an infinite relation gap.

2. Examples and tests

The class of efficient groups contains aspherical groups and small cancellation groups (immediate from the definitions), finitely generated abelian groups and fundamental groups of closed 3-manifolds (Epstein [24]), all finite metacyclic groups (Wamsley [48], Beyl [4]) and many finite simple groups (Campbell, Robertson, Williams [12]). Whether all finite simple groups are efficient is still unclear.

The first examples of non-efficient groups, certain finite metabelian groups, were given by Swan [46] in 1965. M. Lustig [42] constructed the first torsion-free non-efficient groups. We will take a closer look at both Swan's and Lustig's examples later on in this section. Baik and Pride [2] showed that Wamsley's and Beyl's results on finite metacyclic groups do not extend to the infinite setting: there do exist non-efficient infinite metacyclic groups. Robertson, Thomas and Wotherspoon [44] gave examples of non-efficient unitary reflection groups and Baik, Pride [3] found examples of non-efficient Coxeter groups (whether all symmetric groups are efficient is still unknown). Examples of non-efficient finite perfect groups can be found in the article of Kovács [39].

To not leave the reader with the wrong impression, I believe that non-efficiency, however hard to detect at the time, is common among finitely presented groups, whereas efficiency should be viewed as a rare exception occuring only in very special situations. Agreeing with Kovács (see [39]) I also would not know where to look for a "new" class of efficient finite groups.

A group G is called *proficient* if it has a presentation F/N such that

$$d_G(N/[N,N]) = d(N/[F,N]).$$

The term "proficient" is due Kovács. In the forthcoming paper of Gruenberg and Kovács [28] it is shown that a finite group is proficient if and only if it is efficient as profinite group. In [30] I used "pseudo-efficient" instead of proficient, but in the light of the mentioned connection with profiniteness, the word proficient seems to be more appropriate. Proficiency is clearly a weaker notion than efficiency (in fact it avoids the relation gap problem) and there are classes of proficient groups whose efficiency is unknown, most prominently perhaps the class of finite nilpotent groups (see Wamsley [49], and also Gruenberg [26], page 50). In fact, any finite non-efficient presentation of a finite nilpotent group will have a relation gap. I don't have any examples of infinite nilpotent groups that are not proficient (but I'm sure such exist).

Most remarkable is the free product

$$G = (\mathbb{Z}_p \times \mathbb{Z}) * (\mathbb{Z}_q \times \mathbb{Z}),$$

where p and q are relatively prime integers. Some 35 odd years ago D. Epstein asked (see [24]) if this group is efficient, and this is still unanswered. A standard presentation is given by

$$\langle a, b, c, d \mid a^p, [a, b], c^q, [c, d] \rangle.$$

If we denote by F the free group on the given four generators and by N the normal closure of the given four relators in F, then, since $H_2(G) = \mathbb{Z}_{pq}$, this presentation is efficient if and only if N can be generated by three elements. Somewhat surprisingly, K. Gruenberg and P. Linnell have shown that the relation module $N/[N, N]$ can be generated by three elements (private correspondence, 1994).

Theorem 2.1 (Gruenberg, Linnell). *The above free product G is proficient. Indeed the relation module of the standard presentation given above can be generated by three elements.*

I will next discuss two quite different methods for determining non-efficiency of a given group, one due to Swan, requiring an understanding of finite index subgroups, the other, much more recent, due to M. Lustig, based on an analysis of the second Fox-ideal.

Proposition 2.2 (Swan [46]). *If G is a finitely presented group and H is a subgroup of finite index, then*

$$\chi_2(G) \geq \chi_2(H)/[G : H].$$

Proof. Let K be a finite 2-complex for G such that $\chi_2(G) = \chi(K)$ and let K_H be the covering of K corresponding to the subgroup H. Then

$$\chi_2(H) \leq \chi(K_H) = [G : H]\chi(K) = [G : H]\chi_2(G).$$

The result follows. ◇

Swan uses this observation to construct examples of non-efficient finite metabelian groups. Let a generator of $\mathbb{Z}_3$ act on $\mathbb{Z}_7^n$ by squaring all elements and let G be the semi-direct product $\mathbb{Z}_7^n \rtimes \mathbb{Z}_3$. Then $H = \mathbb{Z}_7^n$ is a subgroup of index 3 in G. The rank of the well known second homology group of H tells us that

$$\langle x_1, \dots, x_n \mid x_1^7, \dots, x_n^7, [x_i, x_j](1 \le i < j \le n)\rangle$$

is an efficient presentation for H, and so

$$\chi_2(H) = 1 - n + (n + (n-1)n/2).$$

We deduce from Proposition 2.2 that $\chi_2(G) \ge 1/3 + n(n-1)/6$. Since $H_2(G) = 0$ (this takes a bit of work), the homological lower bound is $1 - r(H_1(G)) + d(H_2(G)) = 1$, hence independent of n. So G is non-efficient for $n \ge 3$.

I will now turn to Lustig's minimality test. Let K be a finite connected 2-complex with fundamental group G. Consider the cellular chain complex $(C_*(\tilde{K}), \partial_*)$ for the universal covering $\tilde{K}$ of K. If we choose a basis $c_1, \dots, c_m$ for the free $\mathbb{Z}G$-module $C_2(\tilde{K})$ then every element can be expressed as a unique linear combination $\alpha_1 c_1 + \cdots + \alpha_m c_m$, $\alpha_i \in \mathbb{Z}G$, and the *second Fox ideal* $I_2(K)$ is the 2-sided ideal in $\mathbb{Z}G$ generated by those $\alpha \in \mathbb{Z}G$ that occur as coefficients of elements from the kernel of ∂_2.

Theorem 2.3 (Lustig [42]). *If there is a ring homomorphism ϕ from the group ring $\mathbb{Z}G$ into ring of $k \times k$-matrices ($k \ge 1$) over some commutative ring R with 1, such that $\phi(1) = 1$ and $\phi(I_2(K)) = 0$, then K is minimal: $\chi_2(G) = \chi(K)$.*

As an application of his test Lustig constructs torsion-free non-efficient groups in the following manner. Let H be a torsion-free one-relator group presented by $\langle a, b \mid r\rangle$ with $H_{ab} = \mathbb{Z}$, and let G be the direct product $H \times \mathbb{Z}$. Then G has a presentation

$$\langle a, b, t \mid r, [a, t], [b, t]\rangle.$$

If K is the 2-complex defined by this presentation, then it is not difficult to show (since H is aspherical) that $I_2(K)$ is generated by $1 - t$ and the Fox-derivatives $\frac{\partial r}{\partial a}$ and $\frac{\partial r}{\partial b}$. If $\phi : \mathbb{Z}G \to \mathbb{Z}H_{ab}$ be the ring homomorphism induced by the projection $G \to H_{ab}$, then the image

$$\phi(I_2(K)) = \mathbb{Z}H_{ab}\left\langle \overline{\frac{\partial r}{\partial a}}, \overline{\frac{\partial r}{\partial b}} \right\rangle,$$

(the bar denotes the image in $\mathbb{Z}H_{ab}$) is the so called Alexander ideal of H. So if we take H to have trivial Alexander ideal (for example the figure-8 knot group), then the hypothesis of Theorem 2.3 is satisfied and we deduce

$$\chi_2(G) = \chi(K) = 1 - 3 + 3 = 1.$$

Since $1 - r(H_1(G)) + d(H_2(G)) = 1 - 2 + 1 = 0$, the group G is non-efficient.

3. Some 2-dimensional conjectures

Efficiency and the relation gap problem are at the heart of several unresolved question in group theory involving 2-dimensional topology. I will begin with a discussion of the Eilenberg–Ganea conjecture which is concerned with comparing a geometric dimension notion for a group with an algebraic one. A good reference for this section is [10].

A $K(G,1)$-complex is a CW-complex with fundamental group G and trivial higher homotopy groups. Such a complex exists for every group, and, even more, is unique up to homotopy. The *geometric dimension* of a group G, $\mathrm{gd}(G)$, is the minimal dimension of a $K(G,1)$-complex (which, of course, can be infinite). The *cohomological dimension*, $\mathrm{cd}(G)$, is the minimal length of a projective resolution of (the trivial $\mathbb{Z}G$-module) $\mathbb{Z}$. Since the cellular chain complex of the universal covering of a $K(G,1)$-complex provides one with such a resolution of length $\mathrm{gd}(G)$, it is clear that the cohomological dimension is always less or equal to the geometric dimension, and it is natural to wonder if these two notions can differ at all. It was shown by Eilenberg and Ganea [23] in 1957 that, under the condition that the cohomological dimension of the group in question is different from 1 and 2, the two notions are indeed the same. Roughly ten years later, Stallings [45] could show that groups of cohomological dimension 1 are free, and hence are also of geometric dimension 1. The case $\mathrm{cd}(G) = 2$ remains unsettled and is known as the Eilenberg–Ganea conjecture.

Eilenberg–Ganea Conjecture 3.1. A group G of cohomological dimension 2 is of geometric dimension 2.

The next result is due to J. Hillman (see [35], page 30).

Theorem 3.2 (Hillman [35]). *Let G be a group of cohomological dimension 2. Then the following statements are equivalent:*

1. *There exits a finite 2-dimensional $K(G,1)$-complex (so G has finite geometric dimension 2);*
2. *The group G is efficient.*

Remark. Theorem 3.2 implies of course that an efficient group of cohomological dimension 2 has geometric dimension 2 as well. The other direction however is unknown, because it is unclear (and probably false) that a group G of geometric dimension 2 has a *finite* $K(G,1)$-complex, even if it is finitely presented.

I will end the part on the Eilenberg–Ganea conjecture with some promising potential counter examples constructed by M. Bestvina [6]. Let X be a finite full simplicial complex with vertex set V and edge set E. Let H be the right-angled Coxeter group

defined by

where $c(e)$ denotes t
Bestvina showed that
cocompactly on an a
mological dimension
index, it is a finitely p
freely and cocompac
with free relation mo
geometric dimension
of presentations and
Coxeter groups the r
Bestvina's results se

I will now turn t
particularly nice clas
procedure. It is there
complex, and which
ones. The second q
in case n=2. In ord
the property $D(n)$ i
$H^{n+1}(X, \mathcal{B}) = 0$ fo

Theorem 3.3 (Wall
of a (finite) n-compl

The case $n = 2$ r

***$D(2)$*-Conjecture 3.**
if and only if it satis

In [20] M. Dyer s
implies that this co
viewpoint differs sli
of Dyer's result.

Theorem 3.5 (Dyer
minimal presentatio
gap, then the $D(2)$-

defined by

$$\langle V \mid v^2 (v \in V), c(e)(e \in E) \rangle$$

where $c(e)$ denotes the commutator $[v, w]$ if v and w are the vertices of the edge e. Bestvina showed that, in case X is an acyclic 2-complex, the group H acts properly and cocompactly on an acyclic 2-complex, and so every torsion-free subgroup has cohomological dimension 2. Since the commutator subgroup G is torsion-free and of finite index, it is a finitely presented group of cohomological dimension 2. Even more, it acts freely and cocompactly on an acyclic 2-complex and hence has a finite presentation with free relation module. So G is proficient, and is efficient if and only if it has finite geometric dimension 2 by the above theorem. Is it efficient? On detailed calculations of presentations and Euler-characteristics of finite index subgroups of right angled Coxeter groups the reader should consult Dicks, Leary [17]. For generalizations of Bestvina's results see [33] and [34].

I will now turn to C. T. C . Wall's finiteness conjecture. CW-complexes form a particularly nice class of topological spaces since they can be built up by an inductive procedure. It is therefore natural to ask which spaces have the homotopy type of a CW-complex, and which CW-complexes are homotopically equivalent to n-dimensional ones. The second question was completely answered by C. T. C. Wall [47], except in case n=2. In order to state his result we need a definition. A space X *satisfies the property* $D(n)$ if $H_i(\tilde{X}) = 0$ for $i > n$ (here $\tilde{X}$ is the universal covering) and $H^{n+1}(X, \mathcal{B}) = 0$ for all local coefficient systems $\mathcal{B}$.

Theorem 3.3 (Wall [47]). *Let $n \neq 2$. A (finite) CW-complex has the homotopy type of a (finite) n-complex if and only if it satisfies the property $D(n)$.*

The case $n = 2$ remains unsolved. The finite part of this case is refered to as the

***D*(2)-Conjecture 3.4.** A finite 3-complex has the homotopy type of a finite 2-complex if and only if it satisfies $D(2)$.

In [20] M. Dyer shows that the existence of a relation gap (under certain conditions) implies that this conjecture is false. Since [20] has never been published, and my viewpoint differs slightly from the one presented in [20], I have also included a proof of Dyer's result.

Theorem 3.5 (Dyer [20]). *Let G be a group with $H^3(G, \mathbb{Z}G) = 0$ and let F/N be a minimal presentation for G (i.e $\chi_2(G) = 1 - d(F) + d_F(N)$). If F/N has a relation gap, then the $D(2)$-conjecture is false.*

Proof. Let $d_F(N) = m$ and suppose that $d_G(N/[N, N]) < m$. Then we have an exact sequence of $\mathbb{Z}G$-modules

$$0 \to M \to \mathbb{Z}G^{m-1} \to N/[N, N] \to 0.$$

Choose generators $x_1, \dots, x_n$ of F and normal generators $r_1, \dots, r_m$ of N, let K be the standard 2-complex built on the presentation $\langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ and let $(C_*(\tilde{K}), \partial_*)$ be the cellular chain complex of the universal covering of K. Now the kernel of the first boundary map ∂_1 is isomorphic to the relation module $N/[N, N]$, whereas the kernel of the second boundary map ∂_2 is isomorphic to the second homotopy module $\pi_2(K)$. Thus, by Schanuel's Lemma (see [10]) we have

$$\pi_2(K) \oplus \mathbb{Z}G^{m-1} \approx M \oplus C_2(\tilde{K}).$$

Let L be the 2-complex obtained from K by wedging on $m - 1$ 2-spheres. Then

$$\pi_2(L) \approx \pi_2(K) \oplus \mathbb{Z}G^{m-1} \approx M \oplus C_2(\tilde{K}).$$

Now we attach 3-cells to L, using attaching maps that correspond to a basis of the free module $C_2(\tilde{K})$ under the above isomorphism. In this way we obtain a finite 3-complex X with fundamental group G and $\pi_2(X) \approx M$. Notice that X can not have the homotopy-type of a finite 2-complex because

$$\chi(X) = 1 - n + m + (m - 1) - m = 1 - n + (m - 1) < \chi_2(G).$$

It remains to show that X satisfies $D(2)$. Consider the cellular chain complex $(C_*(\tilde{X}), \partial_*)$ of the universal covering $\tilde{X}$ of X. Since, by construction, the boundary map ∂_3 is injective, we have $H_i(\tilde{X}) = 0$ for $i > 2$. In order to show that $H^3(X, \mathcal{B}) = 0$ for all local coefficient systems $\mathcal{B}$ it suffices to show that

$$\mathrm{Hom}_{\mathbb{Z}G}(C_2(\tilde{X}), B) \xrightarrow{\partial_3^*} \mathrm{Hom}_{\mathbb{Z}G}(C_3(\tilde{X}), B)$$

is onto for every $\mathbb{Z}G$-module B. This will follow from the fact that ∂_3 is actually a split-injection. To prove this, we first complete $(C_*(\tilde{X}), \partial_*)$ to a resolution $(C_*, \bar{\partial}_*)$ of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ such that $C_i = C_i(\tilde{X})$ and $\partial_i = \bar{\partial}_i$ for $i = 0, 1, 2$, and $C_3 = C_3(\tilde{X}) \oplus C_3'$ with $\bar{\partial}_3 = \partial_3$ when restricted to the first factor. Furthermore, since $\ker(\partial_2) \approx \pi_2(L)$, we have $\ker(\bar{\partial}_2) = M_1 \oplus M_2$, where $M_1 = \partial_3(C(\tilde{X}))$ is free of rank m, and we can define $\bar{\partial}_3$ so that $\bar{\partial}_3(C_3') = M_2$. Let p be the projection

$$p : C_3 = C_3(\tilde{X}) \oplus C_3' \to C_3(\tilde{X}).$$

For $c \in C_4$ and $\bar{\partial}_4(c) = (a, b) \in C_3(\tilde{X}) \oplus C_3'$ we have

$$0 = \bar{\partial}_3 \circ \bar{\partial}_4(c) = \bar{\partial}_3(a, b) = \partial_3(a) + \bar{\partial}_3(0, b) \in M_1 \oplus M_2,$$

so $\partial_3(a) = \partial_3 \circ p \circ \bar{\partial}_4(c) = 0$ for all $c \in C_4$. Since ∂_3 is injective, we have $p \circ \bar{\partial}_4(c) = 0$ for all $c \in C_4$, thus p presents an element of $H^3(G, C_3(\tilde{X}))$. But this group is zero because $C_3(\tilde{X})$ is free and we assumed $H^3(G, \mathbb{Z}G) = 0$. Consequently we have a map

$$\beta : C_2(\tilde{X}) \to C_3(\tilde{X})$$

such that $\beta \circ \bar{\partial}_3 = p$. If $a \in C_3(\tilde{X})$, we have

$$\beta \circ \bar{\partial}_3(a, 0) = \beta \circ \partial_3(a) = p(a, 0) = a,$$

so β is a splitting for ∂_3 ◇

In [20] Dyer discusses a surprising proficiency result for direct powers of super-perfect finite groups due to K. Gruenberg [27]. A group G is called *super-perfect* if $H_1(G) = H_2(G) = 0$. The smallest such group is the binary icosahedral group of order 120. Note that if F/N is a finite presentation of a finite super-perfect group G, then

$$d_G(N/[N, N]) \geq d(F)$$

(otherwise $H_1(G)$ would have to be infinite) and

$$d(N/[F, N]) = d(F) - r(H_1(G)) + d(H_2(G)) = d(F).$$

This shows that F/N is proficient if and only if $d_G(N/[N, N]) = d(F)$, that is, the group looks balanced from an algebraic viewpoint. In particular the presentation F/N is efficient if and only if it is balanced, i.e. $d_F(N) = d(F)$.

Theorem 3.6 (Gruenberg [27]). *Let G be a super-perfect finite group, that is proficient and can be generated by two elements (for example the binary icosahedral group). Then every finite presentation of the direct power G^n, $n \geq 1$, is proficient as well.*

From a naive standpoint it seems rather unlikely that the groups G^n could be balanced (efficient) because of all the additional commutators required in forming direct products. Efficiency questions for direct products are notoriously difficult, the interested reader should consult [14] for additional references. There is one result however, due to Campbell, Robertson and Williams [13], that I would like to explicitely mention. It says that if G is a finite non-perfect group, then there is an integer $k \geq 1$ such that if G^n has a minimal efficient presentation for $1 \leq n \leq k$, then G^n has a minimal efficient presentation for all n (a presentation F/N of G is called *minimal efficient* if it is efficient and $d(F) = d(G)$). This is then used to show that A_4^n is efficient for all n, where A_4 is the alternating group on 4 symbols.

Typically, a presentation of a finite group requires more relators than generators. A group (finite or not) is called *balanced* if it has a finite balanced presentation. In some sense, balanced finite groups seem to be rare. For instance, there are no finite balanced groups known that require more that three generators. I would like to end this section with a result that connects the relation gap problem and the $D(2)$-conjecture to the problem of finding more balanced finite groups.

Theorem 3.7. *One of the following statements (or both) is true:*

1. *There exists a group as in Theorem* 3.5 *(in particular the* $D(2)$*-conjecture is false);*

2. *For each k there exits a finite balanced group requiring more than k generators.*

Proof. Let G be a finite super-perfect group as in Theorem 3.5. Note that $d(G^n)$ tends to infinity as n tends to infinity. If 2 above is false, then there exists a number k_0 so that every finite balanced group can be generated by k_0 many elements. We choose n big enough so that $d(G^n) > k_0$. Since G^n is not balanced, every finite presentation of G^n has a relation gap by Theorem 3.6. Since furthermore $H^3(G^n, \mathbb{Z}G^n) = 0$ (this is indeed true for every finite group), G^n is a group as in Theorem 3.5. $\diamond$

4. Stabilization results

In [18] M. Dunwoody gave generators for the relation module of a certain presentation that can not be lifted to an actual set of relators. Note that a relation gap would be an extreme failure of this lifting: no minimal generating set of the relation module can be lifted to a set of relators. It was realized by W. Metzler [40] that the lifting problem can be bypassed using "stabilization" by direct products. Hog-Angeloni and Metzler have successfully applied this stabilization trick in various situations, for example to distinguish homotopy-type and simple homotopy-type for 2-complexes [40], and also in connection with the Andrews–Curtis conjecture [36]. I used these techniques to "close" the relation gap for a given group [30].

Theorem 4.1. *For a finitely presented group G and a given integer $n > 1$ there exists a positive integer k such that the direct product*

$$G \times \mathbb{Z}_n^k$$

has a presentation with zero relation gap.

I showed in [31] that if one is willing to use amalgamated products rather than direct products one can obtain an efficient presentation for the stabilized group. This shows in particular that every finitely presented group can be embedded into an efficient group.

Theorem 4.2. *Every finitely presented group G can be embedded into an efficient group H. If G has cohomological dimension $n \neq 2$, then H can be chosen to also have cohomological dimension n. If G has cohomological dimension* 2*, then H can be chosen of virtual cohomological dimension* 2.

I should point out that if the word "virtual" could be removed in the last statement above, then the Eilenberg–Ganea conjecture would be true for finitely presented groups (highly unlikely). This is because H, being efficient, would be of geometric dimension 2 by Theorem 3.2, and so every subgroup of H would have geometric dimension 2 as well.

5. Asphericity

A group G is called *aspherical* if it is the fundamental group of a finite aspherical 2-complex. Clearly, aspherical groups are efficient, and so efficiency can be thought of as a first approximation of asphericity. In this section I will discuss more delicate approximations, the so called Cockcroft properties, which can also be viewed as efficiency notions for infinite complexes.

Let K be a connected 2-complex with fundamental group G and let H be a subgroup of G. The complex K is said to *have the Cockcroft property* if the Hurewicz map

$$h : \pi_2(K) \to H_2(K)$$

is the zero map. We say K *has the H-Cockcroft property* if the covering K_H of K corresponding to the subgroup H has the Cockcroft property. Notice that K is aspherical if and only if it has the $\{1\}$-Cockcroft property.

It turns out that in case of a finite complex, the Cockcroft property is essentially efficiency. (A slightly weaker Cockcroft notion, the *p-Cockcroft property*, where p is a certain prime, can be shown to be equivalent to efficiency. We will not persue this here, the interested reader should consult Kilgour, Pride [38].) This follows quickly from Hopf's exact sequence (see [10])

$$\pi_2(K) \xrightarrow{h} H_2(K) \to H_2(G) \to 0.$$

Proposition 5.1. *The finite 2-complex K with fundamental group G has the Cockcroft property if and only if*

$$\chi(K) = 1 - r(H_1(G)) + r(H_2(G)).$$

In particular G is the fundamental group of a 2-complex that has the Cockcroft property if and only if it is efficient and $H_2(G)$ is torsion-free.

The Cockcroft property was first studied by Cockcroft [15] in connection with the *Whitehead conjecture* which states that a subcomplex of an aspherical 2-complex is itself aspherical. The more delicate H-Cockcroft notions (and also the unfortunate terminology) were introduced by M. Dyer. In order to illustrate the importance of the Cockcroft properties in the light of the Whitehead conjecture I will state (and prove) a slight generalization of a result of Cockcroft by Brandenburg and Dyer.

Theorem 5.2 (Cockcroft [15], Brandenburg, Dyer [9]). *Suppose L is a subcomplex of an aspherical 2-complex K. Denote by H the kernel of the homomorphism*

$$\pi_1(L) \to \pi_1(K)$$

induced by inclusion. Then L has the H-Cockcroft property.

Proof. Observe that the covering L_H of L corresponding to the subgroup H of $\pi_1(L)$ is a subcomplex of the universal covering $\tilde{K}$ of K. From the long exact homology sequence associated to the pair $(L_H, \tilde{K})$ we see that the map $H_2(L_H) \to H_2(\tilde{K})$, induced by inclusion, is injective. Since $\tilde{K}$ is contractible, $H_2(\tilde{K}) = 0$ and so $H_2(L_H) = 0$ as well, which certainly implies that L_H has the Cockcroft property. $\diamond$

There is a much deeper result of J. F. Adams [1] (see also Bogley [7]) that says that L_P is acyclic, where P is the Adams radical of the kernel H.

Having now talked about both the Whitehead– and the Eilenberg–Ganea conjecture, I should mention that, by a result of N. Brady and M. Bestvina [5], not both conjectures can be true!

I would like to end this section by briefly mentioning the so called Cockcroft-thresholds. Consider the set of subgroups H of $G = \pi_1(K)$ such that K has the H-Cockcroft property, ordered by inclusion. Minimal elements of this ordered set are called *Cockcroft-thresholds*. They do exist (see [32] and Gilbert, Howie [25]), but are not unique in general (see Pride [43]). From a group theoretic angle it would be nice if the set of Cockcroft-thresholds would only be dependent on G and not on the choice of the minimal complex K carrying G.

Question (M. Lustig, 1993). Suppose K_1 and K_2 are two 2-complexes with the same Euler-characteristic and fundamental group G. If H is a subgroup of G such that K_1 has the H-Cockcroft property, does K_2 have the H-Cockcroft property as well?

Bogley, Gilbert and Howie showed in [8] that the answer is positive if H is assumed to be a normal subgroup, but the general case remains open.

More information on the Cockcroft properties and thresholds can be found in Dyer's papers [22], [21]. The reader interested in the Whitehead conjecture should consult Bogley's survey article [7].

6. Lifting the generation gap

In the previous sections we have encountered groups, Epstein's free product of finitely generated abelian groups, finite index subgroups of certain right-angled Coxeter groups, direct products of the binary icosahedral group, that are good candidates for a relation gap. Looking at these examples it becomes apparent how limited the available techniques are to pinpoint the minimal number of relators required to present a group on a given generating set. In this final section I would like to give another, perhaps more flexible approach to the relation gap problem, based on results obtained by Cossey, Gruenberg and Kovács a number of years ago.

The kernel of the augmentation map from the group ring $\mathbb{Z}G$ to the ring of integers $\mathbb{Z}$, that sends every group element g to 1, is known as the augmentation ideal IG. If G is generated by the elements $g_1, \dots, g_n$, the augmentation ideal is generated by $g_1 - 1, \dots, g_n - 1$ as a left $\mathbb{Z}G$-module. So we always have $d(G) \geq d_G(IG)$. The difference

$$d(G) - d_G(IG)$$

is called the *generation gap* of the group G (this terminology is due to K. Gruenberg).

Theorem 6.1 (Cossey, Gruenberg, Kovács [16]). *Let P be a finite perfect group and denote by P^n the n-fold direct product. Then $d_{P^n}(IP^n) = d_P(IP)$.*

Since $d(P^n)$ tends to infinity as n gets large, one can produce groups with arbitrarily large generation gaps in this way. I should point out that, in contrast to the relation gap, infinite generation gaps can not exist since $d(G)$ is finite if and only if $d_G(IG)$ is finite.

Fix a finite perfect group P and let H be a finitely presented group that contains P^n as a subgroup for all n (examples of finitely presented groups that contain all finite groups as subgroups are the Houghton groups contained in Brown [10]). Let $G(n)$ be the group defined by $\langle H, t \mid [p, t], p \in P^n \rangle$. Then $G(n)$ is an HNN-extention $H *_{P^n} t$. If $\langle X \mid R \rangle$ is a finite presentation for H and $w(p)$, $p \in P^n$, is a word in X that represents p as an element of H, then

$$\langle X, t \mid R, [w(p), t], p \in P^n \rangle$$

is a presentation for $G(n)$. Let F be the free group on $X \cup \{t\}$ and let $N(n)$ be the normal closure of $R \cup \{[w(p), t] \mid p \in P^n\}$ in F. From a naive point of view one would think that $d_F(N(n))$ tends to infinity as n gets large because one is inclined to believe that at least $d(P^n)$ relators are required.

Proposition 6.2. *There exists a constant k such that*

$$d_{G(n)}(N(n)/[N(n), N(n)]) \leq k$$

for all n.

Proof. Let F_1 be the free group of X and N_1 be the normal closure of R in F_1. One has a short exact sequence of $\mathbb{Z}G(n)$-modules (see Hannerbauer [29])

$$0 \to \mathbb{Z}G(n) \otimes_H N_1/[N_1, N_1] \to N(n)/[N(n), N(n)] \to \mathbb{Z}G(n) \otimes_{P^n} IP^n \to 0.$$

By Theorem 6.1 we see that

$$d_{G(n)}(N(n)/[N(n), N(n)]) \le d_H(N_1/[N_1, N_1]) + d_P(IP). \qquad \diamond$$

So if one could prove that $d_F(N(n)) \ge d(P^n)$, one could produce arbitrarily large finite relation gaps. Yet another way to produce examples of groups that are likely to have finite relation gaps is described by Howie [37]. He also uses a limiting process (quite different from the one given above) that "approximates" the groups constructed by Brady and Bestvina [5] that are known to have infinite relation gaps.

Acknowledgements. I would like to thank Ann-Chi Kim for organizing Groups – Korea 98 at Pusan National University and his students and local organizers for assisting me during the conference. I thank Young-Gheel Baik for his hospitality, Bill Bogley for numerous helpful discussions and Laci Kovács for useful comments on an earlier version of this paper. I finally gratefully acknowledge financial support from the DFG and KOSEF.

References

[1] J. F. Adams, A new proof of a theorem of W. H. Cockcroft. J. London Math. Soc. 30 (1955), 482–488.

[2] Y. G. Baik, S. J. Pride, Generators of the second homotopy module arising from group constructions. Preprint, University of Glasgow 1993.

[3] Y. G. Baik, S. J. Pride, On the efficiency of Coxeter groups. Bull. London Math. Soc. 29 (1997), 32–36.

[4] F. R. Beyl, The Schur-multiplier of metacyclic groups. Proc. Amer. Math. Soc. 40 (1973), 413–418.

[5] M. Bestvina, N. Brady, Morse theory and finiteness properties of groups. Invent. Math. 129 (1997), 445–470.

[6] M. Bestvina, The virtual cohomological dimension of Coxeter groups. In: Geometric Group Theory, Vol. 1 (G. A. Niblo and M. A. Roller, eds.), London Math. Soc. Lecture Notes Ser. 181, Cambridge University Press, 1993.

[7] W. A. Bogley, J. H. C. Whitehead's asphericity question. In: Two-Dimensional Homotopy and Combinatorial Group Theory (C. Hog-Angeloni, A. Sieradski, W. Metzler, eds.), London Math. Soc. Lecture Notes Ser. 197, Cambridge University Press, 1993.

[8] W. A. Bogley, N. D. Gilbert, J. Howie, Cockcroft properties of Thompson's group. Preprint 1997.

[9] J. Brandenburg, M. N. Dyer, On J. H. C. Whiteheads asphericity question I. Comment. Math. Helv. 56 (1981), 431–446.

[10] K. S. Brown, Cohomology of Groups. Grad. Texts in Math. 87, Springer-Verlag, New York 1982.

[11] K. S. Brown, F

[12] C. M. Campbe
ple groups anc
eds.), Lecture N
65–72.

[13] C. M. Campbel
of imperfect gr

[14] C. M. Campb
PSL(2, p)^3 anc

[15] W. H. Cockcro
(1951), 375–38

[16] J. Cossey, K. W
finite groups. J.

[17] W. Dicks, I. J. L
Group Theory (
Notes Ser. 252,

[18] M. J. Dunwood

[19] M. N. Dyer, L
Math. Helv. 62

[20] M. N. Dyer, On
of Oregon 1983

[21] M. N. Dyer, Cr
motopy and Co
eds.), London M

[22] M. N. Dyer, Co

[23] S. Eilenberg, T.
Ann. of Math. 1

[24] D. B. A. Epstein
Series (2) 12 (19

[25] N. Gilbert, J. Ho
23 (1995), 255–

[26] K. W. Gruenberg
Math. Soc. 25, P

[27] K. W. Gruenberg
Arch. Math. 35 (

[28] K. W. Gruenber
groups. In prepar

[29] T. Hannerbauer,
Glasgow Math. J

[11] K. S. Brown, Finiteness properties of groups. J. Pure Appl. Algebra 44 (1987), 45–75.

[12] C. M. Campbell, E. F. Robertson, P. D. Williams, Efficient presentations for finite simple groups and related groups. In: Groups-Korea 88 (A. C. Kim and B. H. Neumann, eds.), Lecture Notes in Math. 1398, Springer-Verlag, Berlin–Heidelberg–New York 1989, 65–72.

[13] C. M. Campbell, E. F. Robertson, P. D. Williams, Efficient presentations of direct powers of imperfect groups. Algebra Colloq. 4 (1997), 21–27.

[14] C. M. Campbell, I. Miyamoto, E. F. Robertson, P. D. Williams, The efficiency of $\mathrm{PSL}(2, p)^3$ and other direct products of groups. Glasgow Math. J. 39 (1997), 259–268.

[15] W. H. Cockcroft, Note on a theorem of J. H. C. Whitehead. Quart. J. Math. Oxford (2) 2 (1951), 375–384.

[16] J. Cossey, K. W. Gruenberg, L. G. Kovács, The presentation rank of a direct product of finite groups. J. Algebra 28 (1974), 597–603.

[17] W. Dicks, I. J. Leary, On subgroups of Coxeter groups. In: Geometry and Cohomology in Group Theory (P. H. Kropholler, G. A. Niblo, R. Stöhr, eds.), London Math. Soc. Lecture Notes Ser. 252, Cambridge University Press, 1998.

[18] M. J. Dunwoody, Relation modules. Bull. London Math. Soc. 4 (1972), 151–155.

[19] M. N. Dyer, Localization of group rings and applications to 2-complexes. Comment. Math. Helv. 62 (1987), 1–17.

[20] M. N. Dyer, On constructing complexes dominated by 2-complexes. Preprint, University of Oregon 1983.

[21] M. N. Dyer, Crossed modules and π_2- homotopy modules. In: Two-Dimensional Homotopy and Combinatorial Group Theory (C. Hog-Angeloni, A. Sieradski, W. Metzler, eds.), London Math. Soc. Lecture Notes Ser. 197, Cambridge University Press 1993.

[22] M. N. Dyer, Cockcroft 2-complexes. Preprint, University of Oregon 1993.

[23] S. Eilenberg, T. Ganea, On the Lusternik–Schnierelmann category of abstract groups. Ann. of Math. 1965 (1957), 517–518.

[24] D. B. A. Epstein, Finite presentations of groups and 3-manifolds. Quart. J. Math. Oxford Series (2) 12 (1961), 205–212.

[25] N. Gilbert, J. Howie, Threshold subgroups for Cockcroft 2-complexes. Comm. Algebra 23 (1995), 255–275.

[26] K. W. Gruenberg, Relation Modules of Finite Groups. Regional Conf. Series of the Amer. Math. Soc. 25, Providence 1976.

[27] K. W. Gruenberg, The partial Euler characteristic of the direct powers of a finite group. Arch. Math. 35 (1980), 267–274.

[28] K. W. Gruenberg, L. G. Kovács, Proficient presentations and direct products of finite groups. In preperation.

[29] T. Hannerbauer, Relation modules of amalgamated free products and HNN-extentions. Glasgow Math. J. 31 (1989), 263–270.

[30] J. Harlander, Closing the relation gap by direct product stabilization. J. Algebra 182 (1996), 511–521.

[31] J. Harlander, Embeddings into efficient groups. Proc. Edinburgh Math. Soc. 40 (1997), 317–324.

[32] J. Harlander, Minimal Cockcroft subgroups. Glasgow Math. J. 36 (1994), 87–90.

[33] J. Harlander, H. Meinert, Higher generation subgroup sets and the virtual cohomological dimension of graph products of finite groups. J. London Math. Soc. (2) 53 (1996), 99–117.

[34] J. Harlander, On the dimension of groups acting on buildings. To appear in the Proceedings of Groups-St Andrews, Bath 1997.

[35] J. A. Hillman, 2-Knots and their Groups. Austral. Math. Soc. Lecture Notes Series 5, Cambridge University Press, 1989.

[36] C. Hog-Angeloni, W. Metzler, Stabilization by free products giving rise to Andrews–Curtis equivalence. Note Math. 10, (1990), 305–314.

[37] J. Howie, Bestvina-Brady groups and the plus construction. Preprint.

[38] C. W. Kilgour, S. J. Pride, Cockcroft presentations. J. Pure Appl. Algebra 106 (1996), 275–295.

[39] L. G. Kovács, Finite groups with trivial multiplicator and large deficiency. In: Groups—Korea '94 (A. C. Kim, D. L. Johnson, eds.), Walter de Gruyter, Berlin–New York 1995, 211–225.

[40] W. Metzler, Die Unterscheidung von Homotopie-Typ und einfachem Homotopie-Typ bei 2-dimensionalen Komplexen. J. Reine Angew. Math. 403 (1990), 201–219.

[41] B. H. Neumann, Some groups with trivial multiplicator. Publ. Math. Debrecen 4 (1955), 190–194.

[42] M. Lustig, Fox ideals, $\mathcal{N}$-torsion and applications to 3-manifolds. In: Two-Dimensional Homotopy and Combinatorial Group Theory (ed. C. Hog-Angeloni, A. Sieradski, W. Metzler, eds.), London Math. Soc. Lecture Notes Ser. 197, Cambridge University Press 1993.

[43] S. J. Pride, Examples of presentations which are minimally Cockcroft in several different ways. J. Pure Appl. Algebra 88 (1993), 199–204.

[44] E. F. Robertson, R. M. Thomas, C. I. Wotherspoon, A class of inefficient groups with symmetric presentation. In: Groups—Korea '94 (A. C. Kim, D. L. Johnson, eds.), Walter de Gruyter, Berlin–New York 1995, 277–284.

[45] J. Stallings, On torsion-free groups with infinitely many ends. Ann. of Math. 88 (1968), 312–334.

[46] R. G. Swan, Minimal resolutions for finite groups. Topology 4 (1965), 193–208.

[47] C. T. C. Wall, Finiteness conditions for CW-complexes. Ann. of Math. 81 (1965), 56-69.

[48] J. W. Wamsley, The deficiency of metacyclic groups. Proc. Amer. Math. Soc. 24 (1970), 724–726.

[49] J. W. Wamsley, The multiplicator of finite nilpotent groups. Bull. Austral. Math. Soc. 3 (1970), 1–8.

On an

Marcel H

Let G denote a gro
trivial group. If n
in short) if for all x

P_1: $\langle x\rangle \trianglelefteq G$;
P_2: $x \notin P_1$, but $\langle x$
P_3: $x \notin P_1 \cup P_2$,
$x^{g_2} \notin \langle x, x^{g_1}\rangle$
⋮
P_n: $x \notin \bigcup_{i=1}^{n-1} P_i$,
satisfying x^{g_1}

It is easy to see
Such groups are ca
Dedekind and Baer

Theorem 1 ([5, Th
abelian or $G = Q$
an elementary abel
order.

In this theorem
The next type i
the authors togethe
p, q different prime

* The first author i
for its hospitalit
C.N.R. grants.