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Building resolutions

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Abstract

We consider how a resolution of an abelian group M over \mathbb{Z} could be lifted to a free resolution of the trivial module R over $R[M]$, where R is the field of the rationals. The extended resolution is defined in terms of the exterior and divided powers algebras. Furthermore if the resolution of M is in fact a free resolution over $\mathbb{Z}[G]$ for some group G then the extended resolution will provide a free resolution of the augmentation ideal of $R[M]$ over $R[M \rtimes G]$. Furthermore if R is a subring of the rationals containing \mathbb{Z} and all $j \leq i$ are invertible in R then the extended complex can be defined up to dimension $(i + 1)$ and is exact up to dimension i .

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1. Introduction

The main purpose of this paper is to show how a resolution of a module M over an integral group algebra $\mathbb{Z}[G]$ could be lifted to a resolution \mathcal{Q} of the trivial module R over $R[M]$ that is endowed with G -action, where R is the ring of the rational numbers \mathbb{Q} . In fact all terms of the lifted resolution except the zero term will be free $R[M \rtimes G]$ -modules. If we want to construct only the first $(i + 1)$ -terms of a ‘lifted’ resolution it is sufficient to require that R is a subring of \mathbb{Q} that contains \mathbb{Z} and every $1 \leq j \leq i$ is invertible in R . Thus the lifted resolution will be exact up to dimension i .

The idea of lifting a resolution of M was suggested in [3], where the case G finitely generated abelian, $R = \mathbb{Z}$ is considered. There a topological approach is used to show that

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in small dimensions the complex given by Theorem A is exact. This turns out to be helpful in a low dimensional case of the FP_m -Conjecture for metabelian groups [2].

Our interest in lifting resolutions was originally motivated by attempts to extend the results from [3] to higher dimensional cases. Still we believe that the construction of lifted resolutions deserves its own attention, separately from the FP_m -Conjecture. The complex \mathcal{Q} can be viewed as a generalisation of the complex $D(F)$ considered in [9].

The main results of this paper, Corollaries 1 and 2, will be formulated in Section 2 and proved in Section 4. The proof of Theorem 1 is based on spectral sequence arguments. In Section 6 we restrict to the case $G = 1$ and work with coefficients \mathbb{Z} . This will give a new approach to a non-functorial description of the homology of abelian groups. More about the homology of abelian groups can be found in [7, Theorem C], [1,4,6].

2. Preliminaries

2.1. Divided powers

In this section we consider a free abelian group V . Throughout the paper if not otherwise stated all tensor and exterior powers are over the ring of integers \mathbb{Z} . By definition the i th divided power $\tilde{S}^i(V)$ of V is $\{\lambda \in \otimes^i V \mid \sigma(\lambda) = \lambda \text{ for all } \sigma \in S_i\}$, note in general it is not isomorphic to the i th symmetric power of V . The divided powers algebra $\Gamma(V) = \bigoplus_{i \geq 0} \tilde{S}^i(V)$ is equipped with symmetric multiplication $*$ that is defined on the whole tensor algebra $T(V) = \bigoplus_{i \geq 0} (\otimes^i V)$ by

$$(v_1 \otimes \cdots \otimes v_k) * (v_{k+1} \otimes \cdots \otimes v_{k+s}) = \sum_{(k,s)\text{-shuffle } \sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k+s)}.$$

A (k, s) -shuffle σ is an element of the symmetric group S_{k+s} such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+s)$. $\Gamma(V)$ satisfies the axioms of a divided powers algebra with only even degrees: it is a commutative graded \mathbb{Z} -algebra where the elements of $\tilde{S}^j(V)$ have degree sj for some fixed even number s and such that for every x of degree i there is an element $x^{(k)}$ of degree ki and

$$x^{(0)} = 1, \quad x^{(1)} = x, \quad x^{(k)} * x^{(j)} = (k, j)x^{(k+j)} \quad \text{where } (k, j) = \frac{(k+j)!}{k!j!};$$

$$(x + y)^{(k)} = \sum_{i+j=k} x^{(i)} * y^{(j)} \quad (\text{Leibniz formula});$$

$$(x^{(j)})^{(k)} = (j, j-1)(2j, j-1) \cdots ((k-1)j, j-1)x^{(kj)};$$

$$(x * y)^{(k)} = x^k * y^{(k)} \quad \text{for } y \neq 1.$$

By [1, Expose 8, Proposition 4], $\Gamma(V)$ (denoted $S(V)$ there) is a divided powers algebra with

$$x^{(k)} = \underbrace{x \otimes \cdots \otimes x}_{k \text{ times}} \quad \text{for } x \in V.$$

2.2. Koszul complexes and homology groups

Suppose B is a free abelian group with a linearly ordered basis X . Consider the exact Koszul complex

$$\cdots \rightarrow \mathbb{Z}[B] \otimes \wedge^i B \rightarrow \mathbb{Z}[B] \otimes \wedge^{i-1} B \rightarrow \cdots \rightarrow \mathbb{Z}[B] \otimes B \rightarrow \mathbb{Z}[B] \rightarrow \mathbb{Z} \rightarrow 0$$

with differential

$$\partial(x_1 \wedge \cdots \wedge x_i) = \sum_{1 \leq j \leq i} (-1)^{j-1} (x_j - 1) \otimes (x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_i)$$

for $x_1 < \cdots < x_i$ in X . This complex can be viewed as free resolution over $\mathbb{Z}[B]$ of the trivial module \mathbb{Z} and we use it to calculate the homologies $H_i(B, \mathbb{Z})$. The isomorphism

$$H_i(B, \mathbb{Z}) \simeq \wedge^i B = \wedge^i H_1(B, \mathbb{Z})$$

given by the Koszul resolution is natural (see [5, Chapter 5, Theorem 6.4i]) and the product in $H_*(B, \mathbb{Z})$ induced by the exterior product in the Koszul resolution coincides with the Pontryagin product (see [5, Chapter 5.5]).

3. Definition of the lifted complex

Let M be an abelian group, i.e., \mathbb{Z} -module. We assume now that

$$\cdots \rightarrow \mathbb{Z}\mathcal{A}_i \xrightarrow{d_i} \mathbb{Z}\mathcal{A}_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_2} \mathbb{Z}\mathcal{A}_1 \xrightarrow{d_1} M \rightarrow 0$$

is a resolution of M over \mathbb{Z} , where $\mathbb{Z}\mathcal{A}_j$ is the free \mathbb{Z} -module with basis \mathcal{A}_j . We aim to define a complex

$$\mathcal{Q}: \cdots \rightarrow \mathcal{Q}_m \rightarrow \mathcal{Q}_{m-1} \rightarrow \cdots \rightarrow \mathcal{Q}_0 = R[M] \rightarrow R \rightarrow 0$$

over $R[M]$ and a filtration $\{\mathcal{Q}^{(j)}\}_{j \geq 1}$ of subcomplexes of \mathcal{Q} , i.e.,

$$\mathcal{Q}^{(1)} \subseteq \cdots \subseteq \mathcal{Q}^{(j)} \subseteq \mathcal{Q}^{(j+1)} \subseteq \cdots, \quad \mathcal{Q} = \bigcup_{j \geq 1} \mathcal{Q}^{(j)}.$$

First we define the underlying module structure of $\mathcal{Q}^{(j)}$. If j is odd

$$\mathcal{Q}_m^{(j)} = \bigoplus_{\sum_{i \leq j} t_i = m; i_i \geq 0} (R[M] \otimes \wedge^{i_1}(\mathbb{Z}\mathcal{A}_1) \otimes \tilde{S}^{i_2}(\mathbb{Z}\mathcal{A}_2) \otimes \wedge^{i_3}(\mathbb{Z}\mathcal{A}_3) \otimes \dots \otimes \wedge^{i_j}(\mathbb{Z}\mathcal{A}_j)).$$

If $j \geq 2$ is even

$$\mathcal{Q}_m^{(j)} = \bigoplus_{\sum_{i \leq j} t_i = m, i_i \geq 0} (R[M] \otimes \wedge^{i_1}(\mathbb{Z}\mathcal{A}_1) \otimes \tilde{S}^{i_2}(\mathbb{Z}\mathcal{A}_2) \otimes \wedge^{i_3}(\mathbb{Z}\mathcal{A}_3) \otimes \dots \otimes \tilde{S}^{i_j}(\mathbb{Z}\mathcal{A}_j)).$$

Note that in the definition of $\mathcal{Q}_m^{(j)}$ the exterior and divided powers alternate and the elements of $\tilde{S}^{i_{2k}}(\mathbb{Z}\mathcal{A}_{2k})$ and $\wedge^{i_{2k-1}}(\mathbb{Z}\mathcal{A}_{2k-1})$ have degrees $i_{2k}2k$ and $i_{2k-1}(2k - 1)$, respectively. Furthermore \mathcal{Q} is equipped with a strictly anticommutative product whose restriction on the exterior powers is the wedge product and the restriction on the divided powers is the symmetric product $*$. More precisely for j odd and

$$\begin{aligned} & f \otimes \lambda_1 \otimes \lambda_2 \otimes \dots \otimes \lambda_j \\ & \in R[M] \otimes \wedge^{i_1}(\mathbb{Z}\mathcal{A}_1) \otimes \tilde{S}^{i_2}(\mathbb{Z}\mathcal{A}_2) \otimes \wedge^{i_3}(\mathbb{Z}\mathcal{A}_3) \otimes \dots \otimes \wedge^{i_j}(\mathbb{Z}\mathcal{A}_j), \\ & g \otimes \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_j \\ & \in R[M] \otimes \wedge^{k_1}(\mathbb{Z}\mathcal{A}_1) \otimes \tilde{S}^{k_2}(\mathbb{Z}\mathcal{A}_2) \otimes \wedge^{k_3}(\mathbb{Z}\mathcal{A}_3) \otimes \dots \otimes \wedge^{k_j}(\mathbb{Z}\mathcal{A}_j), \end{aligned}$$

we define

$$\begin{aligned} & (f \otimes \lambda_1 \otimes \lambda_2 \otimes \dots \otimes \lambda_j)(g \otimes \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_j) \\ & = (fg)(-1)^\varepsilon \otimes (\lambda_1 \wedge \mu_1) \otimes (\lambda_2 * \mu_2) \otimes \dots \otimes (\lambda_j \wedge \mu_j), \end{aligned}$$

where $\varepsilon = \sum_{1 \leq t < r \leq (j+1)/2} k_{2t-1}i_{2r-1}$. As the elements of the divided powers are central for $\lambda_{j+1} \in \tilde{S}^{i_{j+1}}(\mathbb{Z}\mathcal{A}_{j+1})$, $\mu_{j+1} \in \tilde{S}^{k_{j+1}}(\mathbb{Z}\mathcal{A}_{j+1})$ we have

$$\begin{aligned} & (f \otimes \lambda_1 \otimes \lambda_2 \otimes \dots \otimes \lambda_{j+1})(g \otimes \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_{j+1}) \\ & = ((f \otimes \lambda_1 \otimes \lambda_2 \otimes \dots \otimes \lambda_j)(g \otimes \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_j)) \otimes (\lambda_{j+1} * \mu_{j+1}) \\ & = (fg)(-1)^\varepsilon \otimes (\lambda_1 \wedge \mu_1) \otimes (\lambda_2 * \mu_2) \otimes \dots \otimes (\lambda_j \wedge \mu_j) \otimes (\lambda_{j+1} * \mu_{j+1}). \end{aligned}$$

The multiplicative structure of $\mathcal{Q}^{(j)}$ induces a multiplicative structure on \mathcal{Q} .

We want to construct the differential of the complex $\mathcal{Q} = \bigcup_j \mathcal{Q}^{(j)}$ inductively on j such that for $\lambda_1 \in \mathcal{Q}_{m_1}$, $\lambda_2 \in \mathcal{Q}_{m_2}$

$$\partial(\lambda_1.\lambda_2) = \partial(\lambda_1).\lambda_2 + (-1)^{\deg(\lambda_1)}\lambda_1.\partial(\lambda_2), \tag{1}$$

where $\deg(\lambda_1) = m_1$ is the degree of λ_1 . A complex satisfying (1) is called a *DG* (differential graded) ring. First we define the differential of $\mathcal{Q}^{(1)}$ by

$$\partial(a_1 \wedge \cdots \wedge a_i) = \sum_{1 \leq k \leq i} (-1)^{k-1} (d_1(a_k) - 1) \otimes a_1 \wedge \cdots \wedge \hat{a}_k \wedge \cdots \wedge a_i$$

for $a_1 < \cdots < a_i \in \mathcal{A}_1$ for some fixed linear order $<$ in \mathcal{A}_1 . Note that by construction for $j \geq 1$

$$\mathcal{Q}_m^{(j+1)} = \begin{cases} \bigoplus_{m_1+(j+1).m_2=m} \mathcal{Q}_{m_1}^{(j)} \otimes \wedge^{m_2}(\mathbb{Z}\mathcal{A}_{j+1}) & \text{if } j+1 \text{ is odd,} \\ \bigoplus_{m_1+(j+1).m_2=m} \mathcal{Q}_{m_1}^{(j)} \otimes \tilde{\mathcal{S}}^{m_2}(\mathbb{Z}\mathcal{A}_{j+1}) & \text{if } j+1 \text{ is even.} \end{cases}$$

In addition to (1) we want the differential of \mathcal{Q} to have the following properties

$$\partial(\mathbb{Z}\mathcal{A}_{j+1}) \subseteq \mathcal{Q}_j^{(j)} \cap \text{Ker } \partial, \quad \partial|_{\mathbb{Z}\mathcal{A}_{j+1}} \equiv d_{j+1} \quad \text{modulo } \partial(\mathcal{Q}^{(j)}). \quad (2)$$

We note that the way we construct the differential by induction on j it is not unique but the induced differential on $\mathcal{Q}^{(j+1)}/\mathcal{Q}^{(j)}$ is unique. Secondly the existence of a differential with property (2) should be justified. To do so we assume we have constructed the differential with the above properties on all $\mathcal{Q}_t^{(j)}$ for $t \leq m-1$, where $(m-1)!$ is invertible in R . Furthermore we assume that j is sufficiently small, $j \leq m-1$. The following theorem calculates the homologies of $\mathcal{Q}^{(j)}$ up to dimension $m-1$. As a corollary of Theorem 1 we obtain

$$H_j(\mathcal{Q}^{(j)}) \simeq \text{Ker } d_j \otimes R \simeq \text{Im } d_{j+1} \otimes R.$$

Then using (2) to extend the differential from $\mathcal{Q}^{(j)}$ to $\mathcal{Q}^{(j+1)}$ we need to define only the differential on $\mathbb{Z}\mathcal{A}_{j+1}$. We do it in such a way that the composition

$$\mathbb{Z}\mathcal{A}_{j+1} \xrightarrow{\partial} \mathcal{Q}_j^{(j)} \cap \text{Ker } \partial \rightarrow H_j(\mathcal{Q}^{(j)})$$

is the map

$$d_{j+1} : \mathbb{Z}\mathcal{A}_{j+1} \rightarrow \text{Im } d_{j+1} \subseteq \text{Im } d_{j+1} \otimes R \simeq H_j(\mathcal{Q}^{(j)}).$$

Theorem 1. *Let j, m be fixed positive integers for which the differential of $\mathcal{Q}^{(j)}$ is defined, satisfies (1) and (2) and $j \leq m-1$. Suppose further that R is a subring of \mathbb{Q} containing the integers \mathbb{Z} such that every element $1 \leq i \leq m-1$ is invertible in R . Then for $0 \leq t \leq m-1$ we have*

(1) *If j is odd,*

$$H_t(\mathcal{Q}^{(j)}) = \begin{cases} \wedge^{\frac{t}{j}}(\text{Ker } d_j) \otimes R, & \frac{t}{j} \in \mathbb{Z}, 0 < t \leq m-1, \\ 0, & t = 0 \text{ or } \frac{t}{j} \notin \mathbb{Z}, t \leq m-1. \end{cases}$$

(2) If $j \geq 2$ is even,

$$H_t(Q^{(j)}) = \begin{cases} \tilde{S}^{\frac{t}{j}}(\text{Ker } d_j) \otimes R, & \frac{t}{j} \in \mathbb{Z}, 0 < t \leq m - 1, \\ 0, & t = 0 \text{ or } \frac{t}{j} \notin \mathbb{Z}, t \leq m - 1. \end{cases}$$

The multiplicative structure of $Q^{(j)}$ induces a multiplicative structure on the homologies of $Q^{(j)}$ that coincides with the exterior (resp. symmetric) product in the exterior algebra of $\text{Ker } d_j$ (resp. in the divided powers algebra of $\text{Ker } d_j$) for j odd (resp. j even).

Theorem 1 is the principal result in this paper together with the following two corollaries. The proof of Theorem 1 is rather long and is completed in Section 5 after developing some preliminary results in Section 4.

Corollary 1. Suppose R is a subring of \mathbb{Q} containing the integers \mathbb{Z} and such that every element $1 \leq i \leq m - 1$ is invertible in R then $Q^{(m)}$ is well-defined and is exact in all dimensions $i \leq m - 1$.

Proof. Note that the remarks before the statement of Theorem 1 show that if $(m - 1)!$ is invertible in R we can construct $Q_{j+1}^{(j+1)}$ once we have constructed $Q_i^{(j)}$ with properties (1) and (2) for $j, t \leq m - 1$. With other words we can construct $Q^{(m)}$ and by Theorem 1 $H_s(Q^{(m)}) = 0$ for $s \leq m - 1$. \square

Corollary 2. If R is the ring of the rationals then the complex Q is well-defined and is exact.

Proof. Note that $Q = \bigcup_j Q^{(j)}$ and hence

$$H_n(Q) = \varinjlim H_n(Q^{(j)}).$$

By Theorem 1, $H_n(Q^{(j)}) = 0$ for $j > n$, hence $H_n(Q) = 0$. \square

Finally we note that if M is a module over $\mathbb{Z}[G]$ we can take \mathcal{A}_j to be free G -sets. The differential of the lifted complex Q commutes with the action of G , where G acts on exterior, divided and tensor products diagonally and on $\mathbb{Z}[M]$ via its action on M .

4. Some exact sequences involving exterior and divided powers

The main result of this section is Proposition 2. We start with some definitions. Suppose V is a free abelian group, we set for $(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \setminus (0, 0)$

$$M_{i,j} = (\wedge^i V) \otimes (\tilde{S}^j(V))$$

and define a \mathbb{Z} -linear map $\theta_{i,j} : M_{i,j} \rightarrow M_{i+1,j-1}$ for $i \geq 0, j \geq 1$ by

$$\begin{aligned} & \theta_{i,j}((v_1 \wedge \cdots \wedge v_i) \otimes (w_1 * \cdots * w_j)) \\ &= \sum_{1 \leq t \leq j} (v_1 \wedge \cdots \wedge v_i \wedge w_t) \otimes (w_1 * \cdots * \hat{w}_t * \cdots * w_j). \end{aligned}$$

Fix a linear well ordered basis $V^{(0)}$ of V over \mathbb{Z} and define $M_{i,j}^{(0)}$ to be the set

$$\begin{aligned} & \{ \lambda(w_1 * \cdots * w_j)(v_1 \wedge \cdots \wedge v_i) \otimes (w_1 * \cdots * w_j) \\ & \mid \text{where all } v_k, w_r \in V^{(0)} \text{ and } v_1 < \cdots < v_i, w_1 \leq \cdots \leq w_j \}, \end{aligned}$$

where $\lambda(w_1 * \cdots * w_j) = \frac{1}{\prod_{v \in V^{(0)}} \alpha_v!}$, α_v is the number of the elements of $\{w_1, \dots, w_j\}$ equal to v . Note that $\lambda(w_1 * \cdots * w_j)w_1 * \cdots * w_j \in \tilde{S}^j(V)$. We write $\lambda(\underline{w})\underline{v} \otimes \underline{w}$ for the element $\lambda(w_1 * \cdots * w_j)(v_1 \wedge \cdots \wedge v_i) \otimes (w_1 * \cdots * w_j)$ from $M_{i,j}^{(0)}$.

Lemma 1. $M_{i,j}^{(0)}$ is a basis of $M_{i,j}$ over \mathbb{Z} .

Proof. It is sufficient to show that $M_{0,j}^{(0)}$ is a basis of $M_{0,j} = \tilde{S}^j(V)$. For $v_1 \otimes \cdots \otimes v_j \in \otimes^j V$ define $S(v_1 \otimes \cdots \otimes v_j)$ as the symmetrisation of $v_1 \otimes \cdots \otimes v_j$, i.e., $S(v_1 \otimes \cdots \otimes v_j) = \sum \pi(v_1 \otimes \cdots \otimes v_j)$, where the sum is over representatives of the left coset classes $S_j/\text{Stab}_{S_j}(v_1 \otimes \cdots \otimes v_j)$. Note that $\tilde{S}^j(V)$ is spanned over \mathbb{Z} by

$$D = \{ S(v_1 \otimes \cdots \otimes v_j) \mid v_1, \dots, v_j \in V^{(0)}, v_1 \leq v_2 \leq \cdots \leq v_j \}$$

and $S(v_1 \otimes \cdots \otimes v_j) = \lambda(v_1 * \cdots * v_j)v_1 * \cdots * v_j$. Finally, we show that there is not a non-trivial \mathbb{Z} -linear dependence between the elements of D . Suppose that

$$\sum z_{\underline{w}} \lambda(\underline{w}) w_1 * \cdots * w_j = 0,$$

where the sum is over all $w_i \in V^{(0)}$ such that $w_1 \leq w_2 \leq \cdots \leq w_j$ and $z_{\underline{w}} \in \mathbb{Z}$. Note that if all $w_i \in V^{(0)}$ and $w_1 \leq w_2 \leq \cdots \leq w_j$ the element $w_1 \otimes \cdots \otimes w_j$ appears in the left hand side of the above equation only in the summand $z_{\underline{w}} \lambda(\underline{w}) w_1 * \cdots * w_j$, hence $z_{\underline{w}} = 0$. \square

Now we order the elements of $M_{0,j}^{(0)}$ in the following way:

$$\lambda(w_1 * \cdots * w_j)(w_1 * \cdots * w_j) < \lambda(w'_1 * \cdots * w'_j)(w'_1 * \cdots * w'_j)$$

if and only if there exists a permutation $\sigma \in S_j$ such that $w_{\sigma(i)} \leq w'_i$ for all $1 \leq i \leq j$ and for at least one i the inequality is strict. An element $\lambda(\underline{w})\underline{v} \otimes \underline{w}$ of $M_{i,j}^{(0)}$, where $i \geq 1$, $j \geq 1$, is said to be good if $v_1 > w_1$ and $M_{i,j}^{(\text{good})}$ is defined to be the \mathbb{Z} -submodule of $M_{i,j}$ spanned by the good elements in $M_{i,j}^{(0)}$. We continue with two easy lemmas. We omit the index of θ when it is clear from the context what it is.

Lemma 2.

- (i) For $i \geq 1, j \geq 1$ we have $\theta(M_{i-1,j+1}) + M_{i,j}^{(\text{good})} = M_{i,j}$;
- (ii) For all $i \geq 1$ the map $\theta : M_{i-1,1} \rightarrow M_{i,0}$ is surjective and the map $\theta : M_{0,i} \rightarrow M_{1,i-1}$ is injective.

Proof. (i) We show that every element $\lambda(\underline{w})\underline{v} \otimes \underline{w}$ in $M_{i,j}^{(0)}$ can be written as a sum of good elements in $M_{i,j}^{(0)}$ modulo the image of $\theta_{i-1,j+1}$. We do this by induction on $\lambda(\underline{w})\underline{w}$ with respect to the order defined on $M_{0,j}^{(0)}$.

If $\lambda(\underline{w})\underline{v} \otimes \underline{w}$ in $M_{i,j}^{(0)}$ is not good then $v_1 \leq w_1$. Then consider

$$\begin{aligned} &\theta(\lambda(v_1 * w_1 * \dots * w_j)(v_1 * w_1 * \dots * w_j)) \\ &= \lambda(w_1 * \dots * w_j)v_1 \otimes (w_1 * \dots * w_j) \\ &\quad + \sum_{k \in I} \lambda(v_1 * w_1 * \dots * \hat{w}_k * \dots * w_j)w_k \otimes (v_1 * w_1 * \dots * \hat{w}_k * \dots * w_j), \end{aligned}$$

where I is a subset of $\{1, \dots, j\}$ such that $\{w_k \mid k \in I\}$ contains all different elements from $\{w_m \mid 1 \leq m \leq j\} \setminus \{v_1\}$. Thus

$$\begin{aligned} &\theta(\lambda(v_1 * w_1 * \dots * w_j)(v_2 \wedge v_3 \wedge \dots \wedge v_i) \otimes (v_1 * w_1 * \dots * w_j)) \\ &= \lambda(w_1 * \dots * w_j)(v_2 \wedge v_3 \wedge \dots \wedge v_i \wedge v_1) \otimes (w_1 * \dots * w_j) \\ &\quad + \sum_{k \in I} \lambda(v_1 * w_1 * \dots * \hat{w}_k * \dots * w_j)(v_2 \wedge v_3 \wedge \dots \wedge v_i \wedge w_k) \\ &\quad \otimes (v_1 * w_1 * \dots * \hat{w}_k * \dots * w_j). \end{aligned}$$

This shows that modulo the image of $\theta_{i-1,j+1}$ the element $\lambda(\underline{w})\underline{v} \otimes \underline{w}$ is congruent to $(-1)^i \sum_{k \in I} (v_2 \wedge \dots \wedge v_i \wedge w_k) \otimes (v_1 * w_1 * \dots * \hat{w}_k * \dots * w_j) \lambda(v_1 * w_1 * \dots * \hat{w}_k * \dots * w_j)$. After reordering the elements from the exterior parts in the above expression and deleting those for which $w_k \in \{v_2, v_3, \dots, v_i\}$ we get a sum of elements $\lambda(\underline{w}')\underline{v}' \otimes \underline{w}' \in M_{i,j}^{(0)}$ with coefficients -1 or 1 and $\lambda(\underline{w}')\underline{w}' < \lambda(\underline{w})\underline{w}$.

(ii) The fact that $\theta : M_{i-1,1} \rightarrow M_{i,0}$ is surjective is obvious. To prove that $\theta : M_{0,i} \rightarrow M_{1,i-1}$ is injective we consider the map $\alpha : M_{1,i-1} \rightarrow M_{0,i}$ sending $v \otimes \underline{w}$ to $v * \underline{w}$. Then $\alpha\theta_{0,i} = i \cdot \text{id}_{M_{0,i}}$ and so $\text{Ker } \theta_{0,i}$ is an abelian subgroup of finite exponent in the free abelian group $M_{0,i}$, hence is trivial. \square

Lemma 3. For $j \geq 1, i \geq 0$ the quotient $M_{i+1,j-1}/\theta(M_{i,j})$ is a free abelian group. Furthermore the image of the good elements in $M_{i+1,j-1}^{(0)}$ is a basis of $M_{i+1,j-1}/\theta(M_{i,j})$ for $j \geq 2$.

Proof. Observe that $\theta^2 = 0$. For $i \geq 1, j \geq 1$, by Lemma 2,

$$\theta(M_{i,j}) = \theta(M_{i,j}^{(\text{good})}) + \theta^2(M_{i-1,j+1}) = \theta(M_{i,j}^{(\text{good})})$$

and applying Lemma 2 again we get

$$M_{i+1,j-1} = \theta(M_{i,j}) + M_{i+1,j-1}^{(\text{good})} = \theta(M_{i,j}^{(\text{good})}) + M_{i+1,j-1}^{(\text{good})} \quad \text{for } j \geq 2, i \geq 1. \quad (3)$$

From now on we aim to prove that the sum in (3) is direct. It is obvious that we can restrict to the case when V is of finite rank. We show that

$$|M_{i+1,j-1}^{(0)}| = \text{rk}(M_{i+1,j-1}) = \mu(i, j) + \mu(i+1, j-1) \quad \text{for } j \geq 2, i \geq 1, \quad (4)$$

where $\mu(i, j)$ is the number of good elements in $M_{i,j}^{(0)}$. An element of $M_{i+1,j-1}^{(0)}$ is either good or not good and there is a bijection between the non-good elements of $M_{i+1,j-1}^{(0)}$ and the good elements in $M_{i,j}^{(0)}$. An element $\lambda(w_1 * \cdots * w_{j-1})(v_1 \wedge \cdots \wedge v_{i+1}) \otimes (w_1 * \cdots * w_{j-1})$ of $M_{i+1,j-1}^{(0)}$ is non-good if $v_1 \leq w_1$ and it corresponds to the good element

$$\lambda(v_1 * w_1 * \cdots * w_{j-1})(v_2 \wedge \cdots \wedge v_{i+1}) \otimes (v_1 * w_1 * \cdots * w_{j-1}).$$

And the good element $\lambda(w'_1 * \cdots * w'_j)(v'_1 \wedge \cdots \wedge v'_i) \otimes (w'_1 * \cdots * w'_j)$ of $M_{i,j}^{(0)}$ corresponds to the non-good element

$$\lambda(w'_2 * \cdots * w'_j)(w'_1 \wedge v'_1 \wedge \cdots \wedge v'_i) \otimes (w'_2 * \cdots * w'_j) \in M_{i+1,j-1}^{(0)}.$$

By (3), (4) and since $\text{rk}(M_{i,j}^{(\text{good})}) \leq \mu(i, j)$ we deduce that both sums in (3) are direct. Hence $M_{i+1,j-1}/\theta(M_{i,j}) \simeq M_{i+1,j-1}^{(\text{good})}$, as required.

If $j \geq 2, i = 0$ by Lemma 1(i) we have $M_{1,j-1} = \theta(M_{0,j}) + M_{1,j-1}^{(\text{good})}$. The sum is direct if $\text{rk}(M_{1,j-1}) = \text{rk}(M_{0,j}) + \text{rk}(M_{1,j-1}^{(\text{good})})$, which is equivalent to the existence of bijection between the non-good elements of $M_{1,j-1}^{(0)}$ and $M_{0,j}^{(0)}$. This can be done exactly as in the first part of the proof. \square

Proposition 1. *Suppose V is a free abelian group.*

(1) *The complex \mathcal{N}*

$$0 \rightarrow \tilde{S}^i(V) \rightarrow V \otimes \tilde{S}^{i-1}(V) \rightarrow \cdots \rightarrow \wedge^{i-2}V \otimes \tilde{S}^2(V) \rightarrow \wedge^{i-1}V \otimes V \rightarrow \wedge^i V \rightarrow 0$$

with the \mathbb{Z} -linear differential that equals the differential θ , i.e.,

$$\begin{aligned} & \partial^{(1)}((v_1 \wedge \cdots \wedge v_k) \otimes (w_1 * \cdots * w_{i-k})) \\ &= \sum_{1 \leq j \leq i-k} (v_1 \wedge \cdots \wedge v_k \wedge w_j) \otimes (w_1 * \cdots * \hat{w}_j * \cdots * w_{i-k}) \end{aligned}$$

is exact.

(2) The complex \mathcal{M}

$$0 \rightarrow \wedge^i(V) \rightarrow V \otimes \wedge^{i-1}(V) \rightarrow \tilde{\mathcal{S}}^2(V) \otimes \wedge^{i-2}(V) \rightarrow \dots \\ \rightarrow \tilde{\mathcal{S}}^{i-1}(V) \otimes V \rightarrow \tilde{\mathcal{S}}^i(V) \rightarrow 0$$

with the \mathbb{Z} -linear differential

$$\partial^{(2)}((w_1 * \dots * w_k) \otimes (v_1 \wedge \dots \wedge v_{i-k})) \\ = \sum_{1 \leq j \leq i-k} (-1)^{j-1} (w_1 * \dots * w_k * v_j) \otimes (v_1 \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_{i-k})$$

has homology groups of finite exponent dividing i . In particular, if R is a subring of \mathbb{Q} containing the ring of integers such that i is invertible in R then the complex $\mathcal{M} \otimes R$ with differential $\partial^{(2)} \otimes \text{id}$ is exact.

Remark. The result in part (2) of Proposition 1 is the best possible as there are examples when the complex \mathcal{M} is not exact.

Proof. We consider the complex

$$\mathcal{T}: 0 \rightarrow \wedge^i(V) \rightarrow \wedge^{i-1}(V) \otimes V \rightarrow \wedge^{i-2}(V) \otimes \tilde{\mathcal{S}}^2(V) \rightarrow \dots \\ \rightarrow V \otimes \tilde{\mathcal{S}}^{i-1}(V) \rightarrow \tilde{\mathcal{S}}^i(V) \rightarrow 0$$

with the \mathbb{Z} -linear differential

$$\partial^{(3)}((v_1 \wedge \dots \wedge v_k) \otimes (w_1 * \dots * w_{i-k})) \\ = \sum_{1 \leq j \leq k} (-1)^{k-j} (v_1 \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_k) \otimes (v_j * w_1 * \dots * w_{i-k}).$$

We claim that

$$\partial^{(1)}\partial^{(3)} + \partial^{(3)}\partial^{(1)} = i \cdot \text{id}.$$

Thus the homology groups of \mathcal{N} and the new complex \mathcal{T} are of finite exponent dividing i . By Lemma 3 the homology groups of \mathcal{N} are trivial or \mathbb{Z} -torsion-free, hence \mathcal{N} is exact. Furthermore using that $\tilde{\mathcal{S}}^t(V) \otimes \wedge^{i-t}(V) \simeq \wedge^{i-t}(V) \otimes \tilde{\mathcal{S}}^t(V)$ via $\underline{w} \otimes \underline{v} \simeq \underline{v} \otimes \underline{w}$ we see that the complex \mathcal{M} is isomorphic to the complex

$$0 \rightarrow \wedge^i(V) \rightarrow \wedge^{i-1}(V) \otimes V \rightarrow \wedge^{i-2}(V) \otimes \tilde{\mathcal{S}}^2(V) \rightarrow \dots \\ \rightarrow V \otimes \tilde{\mathcal{S}}^{i-1}(V) \rightarrow \tilde{\mathcal{S}}^i(V) \rightarrow 0$$

with the \mathbb{Z} -linear differential

$$\begin{aligned} & \partial^{(4)}((v_1 \wedge \cdots \wedge v_k) \otimes (w_1 * \cdots * w_{i-k})) \\ &= \sum_{1 \leq j \leq k} (-1)^{j-1} (v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k) \otimes (v_j * w_1 * \cdots * w_{i-k}). \end{aligned}$$

Note that $\partial^{(3)}|_{\wedge^k(V) \otimes \tilde{\mathcal{S}}^{i-k}(V)} = (-1)^{k-1} \partial^{(4)}|_{\wedge^k(V) \otimes \tilde{\mathcal{S}}^{i-k}(V)}$, i.e., the differentials $\partial^{(3)}$ and $\partial^{(4)}$ are equal up to sign. Then the homologies of \mathcal{M} are isomorphic to the homologies of the new complex \mathcal{T} , which have been proved to be of finite exponent. Finally we prove the claim.

$$\begin{aligned} & (\partial^{(3)}\partial^{(1)} + \partial^{(1)}\partial^{(3)})((v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (w_1 * w_2 * \cdots * w_{i-k})) \\ &= \partial^{(3)}\left(\sum_{1 \leq j \leq i-k} (v_1 \wedge \cdots \wedge v_k \wedge w_j) \otimes (w_1 * \cdots * \hat{w}_j * \cdots * w_{i-k})\right) \\ & \quad + \partial^{(1)}\left(\sum_{1 \leq \alpha \leq k} (-1)^{k-\alpha} (v_1 \wedge \cdots \wedge \hat{v}_\alpha \wedge \cdots \wedge v_k) \otimes (v_\alpha * w_1 * \cdots * w_{i-k})\right) \\ &= \sum_{1 \leq j \leq i-k} \sum_{1 \leq s \leq k} (-1)^{k+1-s} (v_1 \wedge \cdots \wedge \hat{v}_s \wedge \cdots \wedge v_k \wedge w_j) \\ & \quad \otimes (v_s * w_1 * \cdots * \hat{w}_j * \cdots * w_{i-k}) \\ & \quad + \sum_{1 \leq j \leq i-k} (v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (w_1 * w_2 * \cdots * w_{i-k}) \\ & \quad + \sum_{1 \leq \alpha \leq k} (-1)^{k-\alpha} \sum_{1 \leq \beta \leq i-k} (v_1 \wedge \cdots \wedge \hat{v}_\alpha \wedge \cdots \wedge v_k \wedge w_\beta) \\ & \quad \otimes (v_\alpha * w_1 * \cdots * \hat{w}_\beta * \cdots * w_{i-k}) \\ & \quad + \sum_{1 \leq \alpha \leq k} (-1)^{k-\alpha} (v_1 \wedge \cdots \wedge \hat{v}_\alpha \wedge \cdots \wedge v_k \wedge v_\alpha) \otimes (w_1 * w_2 * \cdots * w_{i-k}) \\ &= (i-k)(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (w_1 * w_2 * \cdots * w_{i-k}) + k(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \\ & \quad \otimes (w_1 * w_2 * \cdots * w_{i-k}) \\ &= i(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (w_1 * w_2 * \cdots * w_{i-k}), \end{aligned}$$

as required. This completes the proof of the claim and the proposition. \square

Proposition 2. Suppose $d: V \rightarrow W$ is a homomorphism of free abelian groups.

(1) The following complex denoted by $\mathcal{F}_i(V, d)$

$$\begin{aligned} 0 \rightarrow \tilde{\mathcal{S}}^i(\text{Ker } d) \xrightarrow{\tau} \tilde{\mathcal{S}}^i(V) \rightarrow \text{Im } d \otimes \tilde{\mathcal{S}}^{i-1}(V) \rightarrow \cdots \\ \rightarrow \wedge^{i-1}(\text{Im } d) \otimes V \rightarrow \wedge^i(\text{Im } d) \rightarrow 0 \end{aligned}$$

with the \mathbb{Z} -linear differentials

$$\begin{aligned} & \partial_{\mathcal{F}_i}((v_1 \wedge \cdots \wedge v_k) \otimes (w_1 * \cdots * w_{i-k})) \\ &= \sum_{1 \leq j \leq i-k} (v_1 \wedge \cdots \wedge v_k \wedge d(w_j)) \otimes (w_1 * \cdots * \hat{w}_j * \cdots * w_{i-k}), \end{aligned}$$

where $v_1, \dots, v_k \in \text{Im } d, w_1, \dots, w_{i-k} \in V$ is exact. Here τ is the natural embedding induced by the embedding of $\text{Ker } d$ in V .

(2) Let $\mathcal{R}_i(V, d)$ be the complex

$$\begin{aligned} 0 \rightarrow \wedge^i(\text{Ker } d) \xrightarrow{\tau} \wedge^i(V) \rightarrow (\text{Im } d) \otimes \wedge^{i-1}(V) \rightarrow \tilde{\mathcal{S}}^2(\text{Im } d) \otimes \wedge^{i-2}(V) \rightarrow \cdots \\ \rightarrow \tilde{\mathcal{S}}^{i-1}(\text{Im } d) \otimes V \rightarrow \tilde{\mathcal{S}}^i(\text{Im } d) \rightarrow 0 \end{aligned}$$

with the \mathbb{Z} -linear differential

$$\begin{aligned} & \partial_{\mathcal{R}_i}((w_1 * \cdots * w_k) \otimes (v_1 \wedge \cdots \wedge v_{i-k})) \\ &= \sum_{1 \leq j \leq i-k} (-1)^{j-1} (w_1 * \cdots * w_k * d(v_j)) \otimes (v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{i-k}), \end{aligned}$$

where $w_1, \dots, w_k \in \text{Im } d, v_1, \dots, v_{i-k} \in V$ and τ is the embedding induced by the embedding of $\text{Ker } d$ in V . Then $\mathcal{R}_i(V, d)$ has homology groups of finite exponent not bigger than i . In particular if R is a subring of \mathbb{Q} containing the ring of the integers such that $i!$ is invertible in R then $\mathcal{R}_i(V, d) \otimes R$ is exact.

Proof. If d is injective we identify V with its image $\text{Im } d$ and apply Proposition 1.

If d is not injective we define V_1 to be the quotient $V/\text{Ker } d$ and $\hat{d}: V_1 \rightarrow W$ the map induced by d . As W is free abelian its subgroups are free abelian, in particular $\hat{d}(V_1) \simeq V_1$ is free abelian. Thus $V \simeq V_1 \oplus (\text{Ker } d)$ as abelian groups. Using the fact that

$$\tilde{\mathcal{S}}^k(V) \simeq \tilde{\mathcal{S}}^k(V_1 \oplus (\text{Ker } d)) \simeq \bigoplus_{k_1+k_2=k} (\tilde{\mathcal{S}}^{k_1}(V_1) \otimes \tilde{\mathcal{S}}^{k_2}(\text{Ker } d))$$

(where the tensor product in the right-hand side of the formula corresponds to the symmetric product $*$ on the left-hand side), we see that the complex $\mathcal{F}_i(V, d)$ splits into a direct sum of the complexes

$$\{\mathcal{F}_j(V_1, \hat{d}) \otimes \tilde{\mathcal{S}}^{i-j}(\text{Ker } d)\}_{1 \leq j \leq i} \quad \text{with differentials } \partial_{\mathcal{F}_j} \otimes \text{id}_{\tilde{\mathcal{S}}^{i-j}}$$

and the complex

$$0 \rightarrow \tilde{\mathcal{S}}^i(\text{Ker } d) \xrightarrow{\text{id}} \tilde{\mathcal{S}}^i(\text{Ker } d) \rightarrow 0.$$

By Proposition 1(1), $\mathcal{F}_j(V_1, \hat{d})$ is exact and hence $\mathcal{F}_j(V_1, \hat{d}) \otimes \tilde{\mathcal{S}}^{i-j}(\text{Ker } d)$ and $\mathcal{F}_i(V, d)$ are exact.

The proof of the second part of Proposition 2 is similar. Using

$$\wedge^j(V_1 \oplus (\text{Ker } d)) \simeq \bigoplus_{j_1+j_2=j} (\wedge^{j_1}(V_1) \otimes \wedge^{j_2}(\text{Ker } d))$$

(where the tensor product in the right hand side of the formula corresponds to the wedge product \wedge on the left-hand side), we see that $\mathcal{R}_j(V, d)$ splits as a direct sum of complexes

$$\{\mathcal{R}_j(V_1, \hat{d}) \otimes \wedge^{i-j}(\text{Ker } d)\}_{1 \leq j \leq i} \quad \text{with the differential } \partial_{\mathcal{R}_j} \otimes \text{id}_{\wedge^{i-j}},$$

and the complex

$$0 \rightarrow \wedge^i(\text{Ker } d) \xrightarrow{\text{id}} \wedge^i(\text{Ker } d) \rightarrow 0.$$

By Proposition 1(2), $\mathcal{R}_j(V_1, \hat{d})$ has homology groups of finite exponent dividing j and as $\wedge^{i-j}(\text{Ker } d)$ is a free \mathbb{Z} -module $H_*(\mathcal{R}_j(V_1, \hat{d}) \otimes \wedge^{i-j}(\text{Ker } d)) = H_*(\mathcal{R}_j(V_1, \hat{d})) \otimes \wedge^{i-j}(\text{Ker } d)$. This completes the proof. \square

5. Proof of Theorem 1

We prove Theorem 1 by induction on j . Consider first the case $j = 1$ and denote by A_1 the free abelian group $\mathbb{Z}A_1$. Let \mathcal{P} be the Koszul resolution of \mathbb{Z} over $\mathbb{Z}[A_1]$ constructed using the basis A_1 , i.e., $P_i = \mathbb{Z}[A_1] \otimes (\wedge^i A_1)$ with differentials

$$\partial(a_1 \wedge \cdots \wedge a_i) = \sum_{1 \leq j \leq i} (-1)^{j-1} (a_j - 1) \otimes a_1 \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_i$$

for $a_1 < \cdots < a_i \in A_1$. We consider \mathcal{P} as a free resolution of the trivial module \mathbb{Z} over $\mathbb{Z}[\text{Ker } d_1]$ and use this resolution to calculate $H_i(\text{Ker } d_1, \mathbb{Z})$. By the description of the homologies of torsion free abelian group as exterior powers given in Section 1.2

$$H_i(\mathbb{Z} \otimes_{\mathbb{Z}(\text{Ker } d_1)} \mathcal{P}) = H_i(\text{Ker } d_1, \mathbb{Z}) \simeq \wedge^i \text{Ker } d_1.$$

Remember that by definition

$$\mathcal{Q}^{(1)} \simeq (\mathbb{Z} \otimes_{\mathbb{Z}(\text{Ker } d_1)} \mathcal{P}) \otimes_{\mathbb{Z}} R.$$

As R is flat over \mathbb{Z}

$$H_i(\mathcal{Q}^{(1)}) = H_i(\mathbb{Z} \otimes_{\mathbb{Z}(\text{Ker } d_1)} \mathcal{P}) \otimes R \simeq (\wedge^i \text{Ker } d_1) \otimes R,$$

thus Theorem 1 holds for $j = 1$. Note that at this stage we have not used the fact that some of the elements of R are invertible. This fact will be used later on when we refer to Proposition 2(2). We remind the reader that by construction for $j \geq 2$

$$Q_m^{(j)} = \begin{cases} \bigoplus_{m_1+j.m_2=m} Q_{m_1}^{(j-1)} \otimes \wedge^{m_2}(\mathbb{Z}A_j) & \text{if } j \text{ is odd,} \\ \bigoplus_{m_1+j.m_2=m} Q_{m_1}^{(j-1)} \otimes \tilde{S}^{m_2}(\mathbb{Z}A_j) & \text{if } j \text{ is even} \end{cases} \quad (5)$$

with differential

$$\partial(\mathbb{Z}A_j) \subseteq Q_{j-1}^{(j-1)} \cap \text{Ker } \partial, \quad \partial|_{\mathbb{Z}A_j} \equiv d_j \quad \text{modulo } \partial(Q^{(j-1)}).$$

Note that by (1) we have the following more explicit formulas for the differentials. If $j \geq 2$ is even

$$\begin{aligned} \partial(\lambda \otimes (w_1 * \dots * w_{m_2})) &= \partial(\lambda) \otimes (w_1 * \dots * w_{m_2}) \\ &+ (-1)^{m_1} \sum_{1 \leq i \leq m_2} \lambda w_1 \dots \partial(w_i) \dots w_{m_2}, \end{aligned} \quad (6)$$

where $\lambda \in Q_{m_1}^{(j-1)}$, $w_1, \dots, w_{m_2} \in \mathbb{Z}A_j$, $\lambda w_1 \dots \partial(w_i) \dots w_{m_2}$ is the product of the corresponding elements in $Q^{(j)}$.

If $j \geq 3$ is odd

$$\begin{aligned} \partial(\lambda \otimes (v_1 \wedge \dots \wedge v_{m_2})) &= \partial(\lambda) \otimes (v_1 \wedge \dots \wedge v_{m_2}) \\ &+ \sum_{1 \leq i \leq m_2} (-1)^{i-1+m_1} \lambda v_1 \dots \partial(v_i) \dots v_{m_2} \end{aligned} \quad (7)$$

for $\lambda \in Q_{m_1}^{(j-1)}$, $v_1, \dots, v_{m_2} \in \mathbb{Z}A_j$.

Now we consider the inductive step for $j \geq 2$. Assume Theorem 1 holds for the complex $Q^{(j-1)}$ and we consider the filtration $\{\mathcal{F}^p\}_{p \geq 1}$ of $Q^{(j)}$ where \mathcal{F}^p contains all direct components of $Q^{(j)}$ in (5) with $m_2 \leq p$. Thus

$$\begin{aligned} (\mathcal{F}^p / \mathcal{F}^{p-1})_s &\simeq Q_{s-pj}^{(j-1)} \otimes \wedge^p(\mathbb{Z}A_j) \quad \text{for } j \text{ odd;} \\ (\mathcal{F}^p / \mathcal{F}^{p-1})_s &\simeq Q_{s-pj}^{(j-1)} \otimes \tilde{S}^p(\mathbb{Z}A_j) \quad \text{for } j \text{ even;} \end{aligned}$$

with differentials $\partial|_{Q^{(j-1)}} \otimes \text{id}$ by (1). The associated spectral sequence as defined in [8, Corollary 11.12] is

$$E_{p,q}^1 = H_{p+q}(\mathcal{F}^p / \mathcal{F}^{p-1})$$

with differential

$$d_{p,q}^1 : E_{p,q}^1 \rightarrow E_{p-1,q}^1,$$

which by [8, Exercise 11.13] is the connecting homomorphism of the exact sequence of complexes $0 \rightarrow \mathcal{F}^{p-1}/\mathcal{F}^{p-2} \rightarrow \mathcal{F}^p/\mathcal{F}^{p-2} \rightarrow \mathcal{F}^p/\mathcal{F}^{p-1} \rightarrow 0$.

(I) Now we assume that $j \geq 2$ is odd. Using the description of the differential of $\mathcal{F}^p/\mathcal{F}^{p-1}$ and assumption that the inductive hypothesis holds for $j - 1$ we get

$$E_{p,q}^1 \simeq \begin{cases} H_{p+q-jp}(\mathcal{Q}^{(j-1)}) \otimes \wedge^p(\mathbb{Z}\mathcal{A}_j) & \text{if } p+q-jp > 0, \\ R \otimes \wedge^p(\mathbb{Z}\mathcal{A}_j) & \text{if } p+q-jp = 0, \\ 0 & \text{if } p+q-jp < 0, \end{cases}$$

$$\simeq \begin{cases} \text{not known groups} & \text{if } p+q-jp \geq m, \\ \tilde{S}^{\frac{p+q-jp}{j-1}}(\text{Ker } d_{j-1}) \otimes \wedge^p(\mathbb{Z}\mathcal{A}_j) \otimes R & \text{if } \frac{q}{j-1} \in \mathbb{Z}, 0 \leq p+q-jp \leq m-1, \\ 0 & \text{if } \frac{q}{j-1} \notin \mathbb{Z}, 0 \leq p+q-jp \leq m-1, \\ 0 & \text{if } p+q-jp < 0. \end{cases}$$

Claim. After substituting $\text{Ker } d_{j-1}$ with $\text{Im } d_j$ in the above formula the differential of $E_{p,q}^1$ for $\frac{q}{j-1} \in \mathbb{Z}, 0 \leq p+q-jp \leq m-1$ is $\tilde{\partial} \otimes \text{id}_R$, where $\tilde{\partial}$ up to sign is the differential defined in Proposition 2(2) for $d = d_j : V = \mathbb{Z}\mathcal{A}_j \rightarrow W = \mathbb{Z}\mathcal{A}_{j-1}$.

Proof. We use the description of $d_{p,q}^1$ as connecting homomorphism. Let $r = \sum_i \lambda_i \otimes v^{(i)}$ be an element of $\mathcal{Q}_{s-pj}^{(j-1)} \otimes \wedge^p \mathbb{Z}\mathcal{A}_j \subset \mathcal{F}^p$, where $\lambda_i \in \mathcal{Q}_{s-pj}^{(j-1)}, v^{(i)} = v_1^{(i)} \wedge v_2^{(i)} \wedge \dots \wedge v_p^{(i)}$ is an element of a fixed basis of $\wedge^p \mathbb{Z}\mathcal{A}_j$ over \mathbb{Z} . We assume that the image of r in $\mathcal{F}^p/\mathcal{F}^{p-1}$ belongs to $\text{Ker}(\partial_{\mathcal{F}^p/\mathcal{F}^{p-1}})$. Thus $\partial(r) \in \mathcal{F}^{p-1}$. By (7) we have $\partial(r) \in \sum_i \partial(\lambda_i) \otimes v^{(i)} + \mathcal{F}^{p-1}$, hence $\partial(\lambda_i) = 0$ for all i . Then by the description of d^1 as connecting homomorphism we have for the class $[r] \in H_m(\mathcal{F}^p/\mathcal{F}^{p-1})$ of r that

$$d^1([r]) = [\partial(r)] \in H_{m-1}(\mathcal{F}^{p-1}/\mathcal{F}^{p-2}).$$

Since $\partial(\lambda_i) = 0, \text{deg}(\lambda_i) = s - pj$ and by (7)

$$\begin{aligned} \partial(r) &= \sum_i \left(\partial(\lambda_i) \otimes v^{(i)} + \lambda_i \sum_{t \leq p} (-1)^{t-1+\text{deg}(\lambda_i)} v_1^{(i)} \dots v_{t-1}^{(i)} \partial(v_t^{(i)}) v_{t+1}^{(i)} \dots v_p^{(i)} \right) \\ &= \sum_i \left(\lambda_i \sum_{t \leq p} (-1)^{t-1+s-pj} v_1^{(i)} \dots v_{t-1}^{(i)} \partial(v_t^{(i)}) v_{t+1}^{(i)} \dots v_p^{(i)} \right). \end{aligned} \tag{8}$$

Furthermore by (2) there exists $\mu_t^{(i)} \in \mathcal{Q}_j^{(j-1)}$ such that

$$\partial(v_t^{(i)}) - d_j(v_t^{(i)}) = \partial(\mu_t^{(i)}). \tag{9}$$

We aim to prove that for every $t \leq p$

$$\lambda_i v_1^{(i)} \dots v_{t-1}^{(i)} \partial(\mu_t^{(i)}) v_{t+1}^{(i)} \dots v_p^{(i)} \in \mathcal{F}^{p-2} + \partial(\mathcal{F}^{p-1}). \tag{10}$$

Indeed

$$\begin{aligned}
 & \partial(\lambda_i v_1^{(i)} \cdots v_{t-1}^{(i)} \mu_t^{(i)} v_{t+1}^{(i)} \cdots v_p^{(i)}) \\
 &= \partial(\lambda_i) v_1^{(i)} \cdots v_{t-1}^{(i)} \mu_t^{(i)} v_{t+1}^{(i)} \cdots v_p^{(i)} + (-1)^{\deg(\lambda_i)} \lambda_i \partial(v_1^{(i)} \cdots v_{t-1}^{(i)} \mu_t^{(i)} v_{t+1}^{(i)} \cdots v_p^{(i)}) \\
 &= (-1)^{\deg(\lambda_i)} \lambda_i \partial(v_1^{(i)} \cdots v_{t-1}^{(i)} \mu_t^{(i)} v_{t+1}^{(i)} \cdots v_p^{(i)}) \\
 &= (-1)^{\deg(\lambda_i)} \lambda_i \left(\sum_{k < t} (-1)^{k-1} v_1^{(i)} \cdots \partial(v_k^{(i)}) \cdots \mu_t^{(i)} \cdots v_p^{(i)} \right. \\
 &\quad \left. + \sum_{k > t} (-1)^{k-1} v_1^{(i)} \cdots \mu_t^{(i)} \cdots \partial(v_k^{(i)}) \cdots v_p^{(i)} \right. \\
 &\quad \left. + (-1)^{t-1} v_1^{(i)} \cdots \partial(\mu_t^{(i)}) \cdots v_p^{(i)} \right) \\
 &\in (-1)^{\deg(\lambda_i)+t-1} \lambda_i v_1^{(i)} \cdots \partial(\mu_t^{(i)}) \cdots v_p^{(i)} + \mathcal{F}^{p-2}.
 \end{aligned}$$

Note that the above calculation implies (10). Then by (8) and (9)

$$\begin{aligned}
 [\partial(r)] &= \left[\sum_i \left(\lambda_i \sum_{t \leq p} (-1)^{t-1+s-pj} v_1^{(i)} \cdots v_{t-1}^{(i)} (\partial(\mu_t^{(i)}) + d_j(v_t^{(i)})) v_{t+1}^{(i)} \cdots v_p^{(i)} \right) \right] \\
 &= \left[\sum_i \left(\lambda_i \sum_{t \leq p} (-1)^{t-1+s-pj} v_1^{(i)} \cdots v_{t-1}^{(i)} \partial(\mu_t^{(i)}) v_{t+1}^{(i)} \cdots v_p^{(i)} \right) \right] \\
 &\quad + \left[\sum_i \left(\lambda_i \sum_{t \leq p} (-1)^{t-1+s-pj} v_1^{(i)} \cdots v_{t-1}^{(i)} d_j(v_t^{(i)}) v_{t+1}^{(i)} \cdots v_p^{(i)} \right) \right] \\
 &\in H_{m-1}(\mathcal{F}^{p-1} / \mathcal{F}^{p-2}).
 \end{aligned}$$

By (10)

$$\left[\lambda_i \sum_{t \leq p} (-1)^{t-1+s-pj} v_1^{(i)} \cdots v_{t-1}^{(i)} \partial(\mu_t^{(i)}) v_{t+1}^{(i)} \cdots v_p^{(i)} \right] = 0,$$

hence

$$[\partial(r)] = \left[\sum_i \left(\lambda_i \sum_{t \leq p} (-1)^{t-1+s-pj} v_1^{(i)} \cdots v_{t-1}^{(i)} d_j(v_t^{(i)}) v_{t+1}^{(i)} \cdots v_p^{(i)} \right) \right].$$

Finally $d_j(v_t^{(i)}) \in \mathbb{Z}\mathcal{A}_{j-1}$ is a central element in \mathcal{Q} (remember $j - 1$ is even), hence

$$\begin{aligned}
 [\partial(r)] &= (-1)^{s-pj} \left[\sum_i \left(\sum_{t \leq p} (-1)^{t-1} \lambda_i d_j(v_t^{(i)}) v_1^{(i)} \cdots v_{t-1}^{(i)} v_{t+1}^{(i)} \cdots v_p^{(i)} \right) \right] \\
 &= (-1)^{s-pj} \sum_i \sum_{t \leq p} (-1)^{t-1} [\lambda_i d_j(v_t^{(i)})] \otimes v_1^{(i)} \cdots v_{t-1}^{(i)} v_{t+1}^{(i)} \cdots v_p^{(i)}.
 \end{aligned}$$

The latest equality comes from the fact that the differential of $\mathcal{F}^{p-1}/\mathcal{F}^{p-2}$ is $\partial \otimes \text{id}_{\wedge^{p-1}}$. This completes the proof of the claim. \square

Then the bigraded module $\{E_{p,q}^1\}_{p+q-jp \leq m-1}$ splits into chains of complexes:

$$\begin{aligned} 0 \rightarrow \wedge^p(V) \otimes R &\rightarrow (\text{Im } d) \otimes \wedge^{p-1}(V) \otimes R \rightarrow \tilde{S}^2(\text{Im } d) \otimes \wedge^{p-2}(V) \otimes R \rightarrow \dots \\ &\rightarrow \tilde{S}^{p-1}(\text{Im } d) \otimes V \otimes R \rightarrow \tilde{S}^p(\text{Im } d) \otimes R \rightarrow 0 \end{aligned}$$

with differentials up to sign the ones given in Proposition 2(2) for $d = d_j : V = \mathbb{Z}\mathcal{A}_j \rightarrow \mathbb{Z}\mathcal{A}_{j-1}$, the elements of V have degree j and the elements of $\text{Im } d$ degree $j - 1$. According to Proposition 2(2) these complexes are exact in all dimensions except at the very beginning, where the homology is isomorphic to $\wedge^p(\text{Ker } d_j) \otimes R$, i.e.,

$$E_{p,q}^2 = \begin{cases} \text{not known} & \text{if } p + q - jp \geq m, \\ \wedge^p(\text{Ker } d_j) \otimes R & \text{if } p + q - jp = 0, \\ 0 & \text{if } p + q - jp \in (-\infty, m - 1] \setminus \{0\}. \end{cases}$$

If $d^2 : E_{p,q}^2 \rightarrow E_{p-2,q+1}^2$ is non-trivial for some $p + q \leq m - 1$ then $E_{p,q}^2 \neq 0 \neq E_{p-2,q+1}^2$. Since $(p - 2) + (q + 1) - j(p - 2) \leq m - 1$ by the above description of $E_{p,q}^2$ we have $\frac{q}{p} = j - 1 = \frac{q+1}{p-2} > \frac{q}{p}$, a contradiction. Thus $E_{p,q}^2 = E_{p,q}^3$ and the same argument shows $E_{p,q}^3 = E_{p,q}^4 = \dots = E_{p,q}^\infty$ for $p + q \leq m - 1$. As

$$E_{p,q}^2 \implies H_n(\mathcal{Q}^{(j)}),$$

we get for $1 \leq n \leq m - 1$

$$H_n(\mathcal{Q}^{(j)}) = \begin{cases} \wedge^{\frac{n}{j}}(\text{Ker } d_j) \otimes R & \text{if } \frac{n}{j} \in \mathbb{Z}, \\ 0 & \text{if } \frac{n}{j} \notin \mathbb{Z}. \end{cases}$$

This completes the inductive case when j is odd.

(II) Now we assume that j is even, this case is similar to the previous one. Using the inductive hypothesis

$$\begin{aligned} E_{p,q}^1 &\simeq \begin{cases} H_{p+q-jp}(\mathcal{Q}^{(j-1)}) \otimes \tilde{S}^p(\mathbb{Z}\mathcal{A}_j) & \text{if } p + q - jp > 0, \\ \tilde{S}^p(\mathbb{Z}\mathcal{A}_j) \otimes R & \text{if } p + q - jp = 0, \\ 0 & \text{if } p + q - jp < 0, \end{cases} \\ &\simeq \begin{cases} \text{not known groups} & \text{if } p + q - jp \geq m, \\ \wedge^{\frac{p+q-jp}{j-1}}(\text{Ker } d_{j-1}) \otimes \tilde{S}^p(\mathbb{Z}\mathcal{A}_j) \otimes R & \text{if } \frac{q}{j-1} \in \mathbb{Z}, 0 \leq p + q - jp \leq m - 1, \\ 0 & \text{if } \frac{q}{j-1} \notin \mathbb{Z}, 0 \leq p + q - jp \leq m - 1, \\ 0 & \text{if } p + q - jp < 0. \end{cases} \end{aligned}$$

We substitute $\text{Ker } d_{j-1}$ with $\text{Im } d_j$ in the above formula and similarly to the previous case E^1 has differentials $\tilde{\partial} \otimes \text{id}_R$, where $\tilde{\partial}$ up to sign is the differential of $\mathcal{F}_i(V, d)$ given in

Proposition 2(1) for $d = d_j : V = \mathbb{Z}\mathcal{A}_j \rightarrow W = \mathbb{Z}\mathcal{A}_{j-1}$ (this description of the differential follows from (6) in exactly the same way as the proof of the claim in the previous case follows from (7)). Then the bigraded module $\{E_{p,q}^1\}_{p+q-jp \leq m-1}$ splits into chains of complexes:

$$\begin{aligned} 0 \rightarrow \tilde{S}^p(V) \otimes R &\rightarrow \text{Im } d_j \otimes \tilde{S}^{p-1}(V) \otimes R \rightarrow \dots \\ &\rightarrow \wedge^{p-1}(\text{Im } d_j) \otimes V \otimes R \rightarrow \wedge^p(\text{Im } d_j) \otimes R \rightarrow 0, \end{aligned}$$

where $d = d_j : V = \mathbb{Z}\mathcal{A}_j \rightarrow \mathbb{Z}\mathcal{A}_{j-1}$ and with differential up to sign given by Proposition 2(1). According to Proposition 2(1) these complexes are exact in all dimensions except at the very beginning, i.e., for $p + q - jp \leq m - 1$

$$E_{p,q}^2 = \begin{cases} \tilde{S}^p(\text{Ker } d_j) \otimes R & \text{if } p + q - jp = 0, \\ 0 & \text{if } p + q - jp \in (-\infty, m - 1] \setminus \{0\}. \end{cases}$$

As before examining the bidegree of the differential d^r of E^r we obtain that $E_{p,q}^\infty = E_{p,q}^2$ for $p + q \leq m - 1$. This completes the inductive case when j is even as the convergence of the spectral sequence

$$E_{p,q}^2 \implies H_n(\mathcal{Q}^{(j)})$$

implies for $1 \leq n \leq m - 1$

$$H_n(\mathcal{Q}^{(j)}) = \begin{cases} \tilde{S}^{\frac{n}{j}}(\text{Ker } d_j) \otimes R & \text{if } \frac{n}{j} \in \mathbb{Z}, \\ 0 & \text{if } \frac{n}{j} \notin \mathbb{Z}. \end{cases}$$

6. Lifting a resolution of M of length 2

Suppose $\mathcal{A}_i = \emptyset$ for $i \geq 3$ and let us consider the complex \mathcal{Q} defined in Section 2 for $R = \mathbb{Z}$. Then $\mathcal{Q} = \mathcal{Q}^{(2)}$. As shown at the beginning of Section 4

$$H_*(\mathcal{Q}^{(1)}) = \wedge^* \text{Ker } d_1.$$

Using the spectral sequence argument from Section 5 for $R = \mathbb{Z}$ and $j = 2$ we get

$$E_{p,q}^1 = \begin{cases} \wedge^{q-p}(\text{Ker } d_1) \otimes \tilde{S}^p(\mathbb{Z}\mathcal{A}_2) & \text{if } q \geq p, \\ 0 & \text{if } q < p. \end{cases}$$

We substitute $\text{Ker } d_1$ with $\text{Im } d_2$ in the above formula and note that the differentials of E^1 up to a sign are given by the differentials of $\mathcal{F}_i(V, d)$ in Proposition 2(1), for $d = d_2 : V = \mathbb{Z}\mathcal{A}_2 \rightarrow \mathbb{Z}\mathcal{A}_1$. It is important that the first part of Proposition 2 does not require that some of the elements of R are invertible and we could apply it for $R = \mathbb{Z}$.

As $\text{Ker } d_2 = 0$ this gives that (E^1, d^1) splits into exact sequences, the spectral sequence collapses and hence

$$E^2 = 0.$$

In particular, \mathcal{Q} is exact in all dimensions and we obtain the following result.

Theorem 2. *If $\mathcal{A}_i = \emptyset$ for $i \geq 3$ then the complex \mathcal{Q} for $R = \mathbb{Z}$ is exact in all dimensions.*

We can use the resolution \mathcal{Q} to calculate the homology groups $H_*(M, \mathbb{Z})$. The complex $\mathbb{Z} \otimes_{\mathbb{Z}[M]} \mathcal{Q}$ splits into direct sum of the subcomplexes

$$\mathcal{B}_t(d_2) : \cdots \rightarrow \wedge^i(\mathbb{Z}\mathcal{A}_1) \otimes \tilde{\mathcal{S}}^{t-i}(\mathbb{Z}\mathcal{A}_2) \rightarrow \wedge^{i+1}(\mathbb{Z}\mathcal{A}_1) \otimes \tilde{\mathcal{S}}^{t-i-1}(\mathbb{Z}\mathcal{A}_2) \rightarrow \cdots$$

with differential as in Proposition 2(1) and $\wedge^i(\mathbb{Z}\mathcal{A}_1) \otimes \tilde{\mathcal{S}}^{t-i}(\mathbb{Z}\mathcal{A}_2)$ placed in degree $i + 2(t - i)$. Thus

$$H_n(M, \mathbb{Z}) \simeq \bigoplus_{i+2j=n} H_n(\mathcal{B}_{i+j}(d_2))$$

but the isomorphism is not functorial and depends on the choice of exact sequence $0 \rightarrow \mathbb{Z}\mathcal{A}_2 \rightarrow \mathbb{Z}\mathcal{A}_1 \rightarrow M \rightarrow 0$ of \mathbb{Z} -modules. Note that

$$H_n(\mathcal{B}_n(d_2)) \simeq \wedge^n M,$$

which is not a surprise as $\wedge^n M$ naturally embeds in $H_n(M, \mathbb{Z})$ as shown in [5, Theorem 6.4], [2, Proposition 5.1]. We summarise

Corollary 3. *If $0 \rightarrow \mathbb{Z}\mathcal{A}_2 \xrightarrow{d_2} \mathbb{Z}\mathcal{A}_1 \rightarrow M \rightarrow 0$ is an exact sequence of abelian groups then there is a non-functorial isomorphism*

$$H_n(M, \mathbb{Z}) \simeq \bigoplus_{0 \leq j \leq \lfloor \frac{n}{2} \rfloor} H_n(\mathcal{B}_{n-j}(d_2)).$$

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