# ASPHERICAL WORD LABELED ORIENTED GRAPHS AND CYCLICALLY PRESENTED GROUPS

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## Abstract

A *word labeled oriented graph* (WLOG) is an oriented graph $\mathcal{G}$ on vertices $X = \{x_1, \ldots, x_k\}$, where each oriented edge is labeled by a word in $X^{\pm 1}$. WLOGs give rise to presentations which generalize Wirtinger presentations of knots. WLOG presentations, where the underlying graph is a tree are of central importance in view of Whitehead's Asphericity Conjecture. We present a class of aspherical world labeled oriented graphs. This class can be used to produce highly non-injective aspherical labeled oriented trees and also aspherical cyclically presented groups.

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## 1 Introduction

A *word labeled oriented graph* (WLOG) is an oriented graph $\mathcal{G}$ on vertices $X = \{x_1, \ldots, x_k\}$, where each oriented edge is labeled by a non-empty word in $X^{\pm 1}$. Associated with a word labeled oriented graph comes a *WLOG-presentation* $P(\mathcal{G})$ on generators $X$ and relators in one-to-one correspondence with edges in $\mathcal{G}$. For an edge with initial vertex $x_i$, terminal vertex $x_j$, labeled $w$, we have a relation $x_i w = w x_j$. A *WLOG-complex* $K(\mathcal{G})$ is the standard 2-complex associated with the WLOG-presentation $P(\mathcal{G})$, and a *WLOG-group* $G(\mathcal{G})$ is the group defined by the WLOG-presentation. We say a word labeled oriented graph is *aspherical* if its associated WLOG-complex is aspherical. A word labeled oriented graph is called *injective* if no edge label or its inverse is a subword of any other edge label. A word labeled oriented tree (WLOT) is a word labeled oriented graph where the underlying graph is a tree.

If every edge label of a word labeled oriented graph consists of a single letter from $X$, we simply speak of a labeled oriented graph (LOG). LOG presentations generalize Wiritnger presentations of knots. Labeled oriented trees (LOTs) are of central importance in view of Whitehead's Asphericity Conjecture which states that a subcomplex of an aspherical 2-complex is aspherical. LOT presentations which are

Wiritinger presentations of knots are known to be aspherical by a classical result of Papakyriakopoulos [8]. See Bogley [1] and Rosebrock [9] for surveys on the Whitehead Conjecture. Recently the authors have shown that injective labeled oriented trees are aspherical [4]. Here we show that certain injective word labeled oriented graphs are aspherical (in fact, diagrammatically reducible, a strong combinatorial version of asphericity). We should note that a word labeled oriented graph $\mathcal{G}$ can be subdivided to produce a labeled oriented graph $\mathcal{G}'$, and $K(\mathcal{G}')$ is a subdivision of $K(\mathcal{G})$. See Example 2.2 and [5] for details. In particular, if $\mathcal{G}$ is aspherical then so is $\mathcal{G}'$. The labeled oriented graph $\mathcal{G}'$ is typically highly non-injective, even if one starts with an injective WLOG $\mathcal{G}$. Thus our result also gives access to wide classes of non-injective aspherical labeled oriented graphs and trees.
Word labeled oriented trees have appeared before in work of Howie [5], where they were called weakly labeled oriented trees.

Another motivation for this paper comes from the interest in cyclically presented groups. Let $F$ be the free group on generators $\{x_1, \ldots, x_n\}$. Let $\phi$ be the automorphism on $F$ defined by $\phi(x_i) = x_{i+1}$ in case $1 \leq i \leq n-1$, and $\phi(x_n) = x_1$. According to Howie and Williams [6] for $w \in F$

$$P(n,w) = \langle x_1, \ldots, x_n \,|\, w, \phi(w), \ldots, \phi^{n-1}(w)\rangle$$

is called a *cyclic presentation* and the group $G(n,w)$ it defines is called a *cyclically presented group.* The automorphism $\phi$ induces a *shift automorphism* of $G(n,w)$. There is a connection between asphericity and the dynamics of the shift automorphism, see Bogley [2]. We present several classes of cyclically presented groups which arise from word labeled oriented graphs and are aspherical by our main result Theorem 2.1.

## 2 Main Theorem

Let $K$ be the standard 2-complex associated with a finite group presentation. The complex $K$ is called *diagrammatically reducible* (DR) if there does not exist a reduced spherical diagram $f\colon C \to K$. See Gersten [3] for definitions and details. The property DR implies asphericity.
Combinatorial curvature plays an important role in the study of asphericity of 2-complexes. Given a closed surface $S$ with a cell structure, one can assign real numbers $\omega(c)$ to the corners $c$ of the 2-cells of $S$, thought of as angles. The curvature $\kappa(v)$ at a vertex is defined as $2 - \sum \omega(w)$, where the sum is taken over the corners at $v$. The curvature $\kappa(d)$ of a 2-cell $d$ is defined as $\sum \omega(c) - (|\partial d| - 2)$, where the sum is taken over all the corners of $d$ and $|\partial d|$ denotes the number of edges in the boundary of $d$. The combinatorial Gauss-Bonnet Theorem asserts that summing up the curvature at all the vertices and 2-cells yields twice the Euler characteristic of the surface $S$. The combinatorial Gauss-Bonnet theorem is the basis for asphericity tests such as Gersten's weight test (see [3]). More details on combinatorial curvature can be found in Section 14 of McCammond's survey [7].
The remaining of this section is devoted to the proof of our main result.

**Theorem 2.1** *Let $P(\mathcal{G}) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle$ be a WLOG-presentation coming from an injective word labeled oriented graph $\mathcal{G}$. Then each relation $r_i$ is of the form $x_{\alpha(i)} w_i = w_i x_{\beta(i)}$ where $w_i$ is a word $w_i = t_{i,1} \ldots t_{i,s_i}$ with $t_{i,j} \in \{x_1, \ldots, x_n\}^{\pm 1}$. We assume $s_i \geq 2$ for $i = 1, \ldots, k$. We further assume that*

1. *Each relator is cyclically reduced.*
2. *For each relator $r_i$ the words $x_{\alpha(i)} t_{i,1}$, $t_{i,1}^{-1} x_{\alpha(i)}$, $x_{\beta(i)} t_{i,s_i}^{-1}$, $t_{i,s_i} x_{\beta(i)}$ are not pieces (i.e. these words and their inverses are not subwords in another relator or in the same relator at another place).*
3. *No word $w_i$ has the form $x_j^{\pm m}$ for $m \geq 2$.*

*Then $K(\mathcal{G})$ is DR.*

**Proof:** Let $K = K(\mathcal{G})$. Assume there exists a reduced spherical diagram $f \colon C \to K$. We will assign weights to the corners of the cell decomposition of $C$ such that the curvature at each vertex and at each 2-cell is non-positive. Thus the total curvature of $C$ is non-positive, contradicting the fact that $C$ is a sphere with total curvature 4, according to Gauss-Bonnet.
Assume $d$ is a 2-cell in $C$ with boundary word $x_i w_q x_j^{-1} w_q^{-1}$. The *long sides of* $d$ are the sides labeled by $w_q$, the remaining two sides labeled $x_i$ and $x_j$ are called *short sides of* $d$. The first and last vertices of the long sides of $d$ are called *extremal vertices* of $d$. Note that every 2-cell in $C$ has exactly four extremal vertices. The other vertices are referred to as *non-extremal.* If $d_1$ and $d_2$ are 2-cells in $C$ that share a long side, that is the intersection $d_1 \cap d_2$ is a long side for both $d_1$ and $d_2$, then we call $d_1 \cup d_2$ a *stacked pair*, and the long side in the intersection a *stack line.* Note that because $\mathcal{G}$ is injective and $f \colon C \to K$ is reduced, if $d_1 \cup d_2$ is a stacked pair, then $d_1$ and $d_2$ are mapped to the same 2-cell in $K$ under $f$, with the same orientations. A stacked pair is shown in Figure 1.

Consider a long side $S$ of a 2-cell $d$ in $C$. We will assign weights to the corners of $d$ along $S$ so that their sum is $\leq |\partial d|/2 - 1$.

**Case I:** All non-extremal vertices on $S$ have valency 2.

In this case the long side $S$ is a stack line. Assign to the first and the last corner of $S$ weight 1/2 and assign weight 1 to all other corners (see Figure 1). Note that

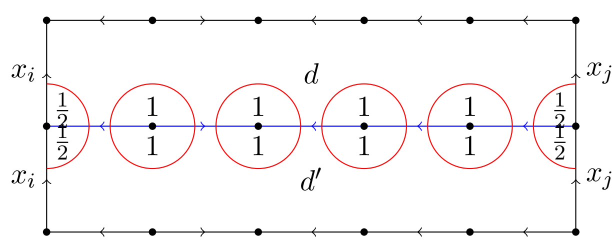


Figure 1. Weights in case I. The stack line is shown in blue.

these weights sum up to $|\partial d|/2 - 1$.

**Case II:** There is a non-extremal vertex of $S$ of valency greater or equal to 3.

Assign weight $2/3$ to the corners at the first and last vertex of $S$ in $d$. Assign weight 1 to corners at vertices of valency 2 on $S$. Let $v$ be a non-extremal vertex on $S$ of valency $\geq 3$. If $v$ is the endpoint of a stack line then assign weight 1 to the corner in $d$ at $v$. If $v$ is not the endpoint of a stack line assign weight $2/3$ to the corner in $d$ at $v$.

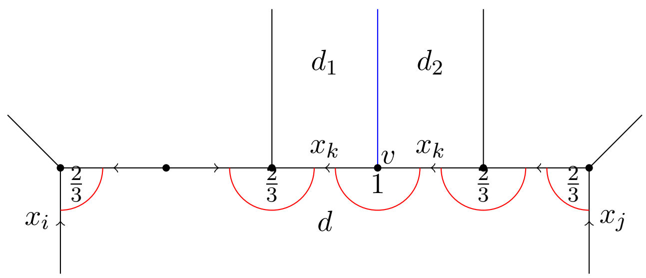


Figure 2. Weights in case II. The stack line is shown in blue.

Note that there are at least 3 corners in $d$ along $S$ of weight $2/3$. There are already two weights $2/3$ at the first and last corner of $d$ along $S$. If there were no other corners with weight $2/3$ then all non-extremal vertices along $S$ would have to be the endpoints of stack lines. But then $S$ would be labeled $x_k^{\pm m}$, for some $x_k$ and some $m \geq 2$, contradicting condition 3 in the theorem. Thus, the sum of the weights along a long side is $\leq |\partial d|/2 - 1$.

Consider a 2-cell $d$ in $C$. Using the above process we have assigned weights at the corners along both long sides of $d$. Since the sum of the weights along a long side of $d$ is $\leq |\partial d|/2 - 1$, the sum of the weights of all corners of $d$ is $\leq |\partial d| - 2$, and hence $\kappa(d) \leq 0$.

It remains to check the curvature at the vertices of $C$. Because of condition 1 all vertices in $C$ have valency at least two. Because of condition 2 an extremal vertex has valency at least three. Since the smallest weight assigned is $1/2$ we have $\kappa(v) \leq 0$ for a vertex of valency greater or equal to four. Thus we only need to worry about vertices of valency two and three. The two corners at a vertex $v$ of valency two both have weight 1 which sums up to 2. So $\kappa(v) = 0$.

Now suppose that $v$ is a vertex of valency three. Let $d_1$, $d_2$ and $d_3$ be the three 2-cells in $C$ that share the vertex $v$ and let $c_i$ be the corner at $v$ in $d_i$.

If $v$ is not the end of a stack line, then the weights assigned to the corners $c_i$ are all $2/3$. This is because weight $1/2$ is only assigned to corners at the end of stack lines and weight 1 is assigned only to corners at the the end of stack lines or at corners at vertices of valency 2. Thus we have $\kappa(v) = 0$.

Next suppose $v$ is the endpoint of a stack line $L$ and assume without loss of generality that $L = d_1 \cap d_2$. Note that $d_1 \cap d_3$ contains a short side of $d_1$ and $d_2 \cap d_3$ contains a short side of $d_2$. Thus $L$ is the only stack line with end vertex $v$. We next argue that $v$ can not be an extremal vertex of $d_3$. Assume this would be the case. The situation is depicted in Figure 3. The stacked 2-cells $d_1$ and $d_2$ have to

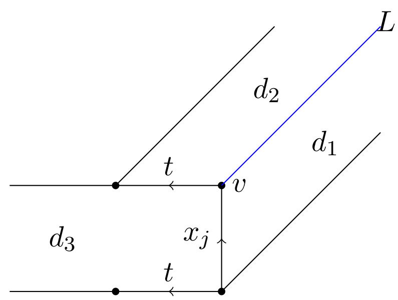


Figure 3. Extremal vertex with stack line (blue) implies that the boundary of $d_3$ is not reduced.

have the same labels on their short sides, so $t = x_j$ and the boundary word of $d$ is not reduced, contradicting the fact that each relator in $P(\mathcal{G})$ is cyclically reduced.

It follows that the vertex $v$ is a non-extremal vertex of $d_3$ and hence situation is exactly as shown in Figure 2. So the weights at $v$ are 1/2, 1/2, and 1. Thus we have $\kappa(v) = 0$. □

**Example 2.2** The WLOT-presentation

$$P(\mathcal{G}) = \langle a, b, c, d \mid a(da) = (da)b, b(db) = (db)c, c(ac) = (ac)d \rangle$$

satisfies the hypothesis of Theorem 2.1. Hence $K(\mathcal{G})$ is DR. Note that $\mathcal{G}$ is an interval with four vertices, each edge label has length two. Subdividing $\mathcal{G}$ into a labeled oriented interval on seven vertices, introducing a new vertex at every edge midpoint, yields a labeled oriented interval $\mathcal{G}'$ with the presentation

$$P(\mathcal{G}') = \langle a, x, b, y, c, z, d \mid ad = dx, xa = ab, bd = dy, yb = bc, ca = az, zc = cd \rangle.$$

Since $K(\mathcal{G}')$ is obtained from $K(\mathcal{G})$ by subdivision and subdividing preserves DR (see [3], 6.10), the complex $K(\mathcal{G}')$ is also DR. Note that $\mathcal{G}'$ is not injective.

## 3 Cyclically presented groups and further examples

We study the cyclic presentations

$$C(n,w) = P(n, x_1 w x_2^{-1} w^{-1}) =$$

$$= \langle x_1, \ldots, x_n \mid x_1 w = w x_2, x_2\phi(w) = \phi(w)x_3, \ldots, x_n\phi^{n-1}(w) = \phi^{n-1}(w)x_1\rangle$$

where $\phi$ is the shift automorphism. Note that $C(n,w)$ is a WLOG-presentation where the underlying graph is a circle on $n$ vertices.

**Theorem 3.1** *Let $w = x_j^t x_k^m$ where $t \neq 0$ and $m \neq 0$ are integers and $j \neq k$ are integers mod $n$. Assume $j \neq 1, k \neq 2, 2j-1 \neq k, j \neq k-1, j \neq 2k-2, j+k \neq 3$. Then $K(C(n,w))$ is DR.*

**Proof:** We apply Theorem 2.1 to $C(n,w)$. The word labeled oriented circle that defines the WLOG-presentation $C(n,w)$ is injective because all words $w$, $\phi(w)$, $\phi^2(w)$,..., $\phi^{n-1}(w)$ are different and of the same length. Note that the first relator $x_1 x_j^t x_k^m x_2^{-1}(x_j^t x_k^m)^{-1}$ is cyclically reduced because of $j \neq 1, k \neq 2, j \neq k$. Since the other relators are obtained by shifting the subscripts on the first relator, all relators are cyclically reduced and hence condition 1 of the assumptions in Theorem 2.1 holds. Condition 3 follows from the fact that $t \neq 0$, $m \neq 0$, and $j \neq k$. Let $A = \{x_1 x_j^{\epsilon}, x_j^{-\epsilon} x_1, x_2 x_k^{-\tau}, x_k^{\tau} x_2\}$, where $\epsilon = 1$ in case $t > 0$, $\epsilon = -1$ in case $t < 0$, and $\tau = 1$ in case $m > 0$, $\tau = -1$ in case $m < 0$. Note that an element $a \in A$ being a piece would mean $a$ or $a^{-1}$ is contained in a shift of $A$, or $a$ or $a^{-1}$ is equal to a shift of $x_j^{\epsilon} x_k^{\tau}$. The assumption $j \neq k-1$ implies $j-1 \neq k-2$, and $j+k \neq 3$ implies $k-2 \neq 1-j$, which shows that $a$ or $a^{-1}$ is not a shift of an element from $A$. Now $j \neq 2k-2$ implies $j-k \neq k-2$, thus $(x_2 x_k^{-\tau})^{\pm 1}$ or $(x_k^{\tau} x_2)^{\pm 1}$ can not be a shift of $x_j^{\epsilon} x_k^{\tau}$. Furthermore $2j-1 \neq k$ implies $j-1 \neq k-j$, hence $(x_1 x_j^{\epsilon})^{\pm 1}$ or $(x_j^{-\epsilon} x_1)^{\pm 1}$ can not be a shift of $x_j^{\epsilon} x_k^{\tau}$. We have shown that no element from $A$ or its inverse can be a piece. Hence no shift of an element from $A$ or its inverse can be a piece and condition 2 is indeed satisfied. The result follows from Theorem 2.1. □

**Example 3.2** The presentations $C(n, x_3^t x_1^m)$, $n \geq 5$, $m \neq 0$ and $t \neq 0$, satisfy the hypothesis of Theorem 3.1 and are therefore DR. One can show that all presentations $C(n, x_3 x_1), n \geq 5$, do not satisfy the weight test stated in [3]. Thus the most obvious test for asphericity fails for these examples.

Theorem 3.1 is a result about word labeled oriented circles. Because of Whitehead's Asphericity Conjecture we are interested in the asphericity of labeled oriented trees. We can omit one relator of $C(n,w)$ and obtain a word labeled oriented interval presentation that is also DR as a subpresentation of a presentation that is DR.

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