On Distinguishing Virtual Knot Groups from Knot Groups

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Abstract

We use curvature techniques from geometric group theory to produce examples of virtual knot groups that are not classical knot groups.

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1 Introduction

From a combinatorial viewpoint, classical knot theory studies 4-regular plane graphs with over- and under crossing information added at the nodes. Virtual knot theory extends this to the study of arbitrary 4-regular graphs with over- and under crossing information. Typically this graph is projected on an orientable surface. The notion of a virtual knot was introduced by Kauffman [11]. For a more recent survey on virtual knots we refer to [6]. A Wirtinger presentation can be read off the diagram in the usual way. In the classical situation the group defined by the presentation is the fundamental group of the knot-complement in the 3-sphere, which is isomorphic to the fundamental group of the knot-complement in the thickened 2-sphere. The group defined by the Wirtinger presentation of a virtual knot is the fundamental group of the knot complement in the coned off thickened thickened surface (see [8]) which is a 3-manifold only in the classical case when the surface in the 2-sphere. Another way to put classical knot groups and virtual knot groups into an unified context is to consider surface 2-knots in the 4-sphere, that is embeddings $S \hookrightarrow \mathbb{S}^4$, where S is an oriented surface. It can be shown that virtual knot groups are exactly the fundamental groups of ribbon torus 2-knots, see Satoh [17]. A general reference on the topic of surface knots, in particular diagrammatic methods and Wirtinger presentations, is Carter and Saito's book [4].

Wirtinger presentations of virtual knots are most conveniently recorded by *labeled oriented circles*, LOCs for short. A *labeled oriented graph*, LOG, is a finite oriented graph $\mathcal{G}$ on vertices $\{a, b, c, ...\}$ and every oriented edge is labeled by a vertex (see Figure 1 on the left). The associated Wirtinger presentation has generators in one-to-one correspondence with the vertices of $\mathcal{G}$ and, if (a, b) is an oriented edge in $\mathcal{G}$ (starting at the vertex a and ending at the vertex b) labeled by c, then we have a relation $acb^{-1}c^{-1}$. A *LOC* is just a LOG where the underlying graph is a circle. For instance the LOC in Figure 1 corresponds to the Wirtinger presentation

$$\langle a, b, c, d, e \,| ac = cb, ba = ac, ca = ad, da = ae, ac = ce\rangle.$$

A virtual knot group is the group defined by a Wirtinger presentation coming from a virtual knot. A LOC group is the group defined by a presentation coming from a LOC.

In this article we use curvature techniques from geometric group theory to study virtual knots and their groups. We show that there exist virtual knots whose Wirtinger complex, the standard 2-complex associated with the Wirtinger presentation of the virtual knot, is a non-positively curved square complex. This can never happen in the classical knot case.

In [5] virtual knot diagrams are given which are not knot diagrams. Such diagrams can be constructed from our examples in a straightforward manner. See for instance Figure 9. We also show the asphericity of the Wirtinger complexes of these groups.

2 Knot Groups and Virtual Knot Groups

Theorem 2.1 *The class of virtual knot groups agrees with the class of LOC groups.*

Proof: Given a LOC one can draw a virtual knot diagram by drawing a crossing rectangle for each edge of the LOC as in Figure 1. Every crossing needed when connecting the ends that extend from the rectangles should be made to a virtual crossing.

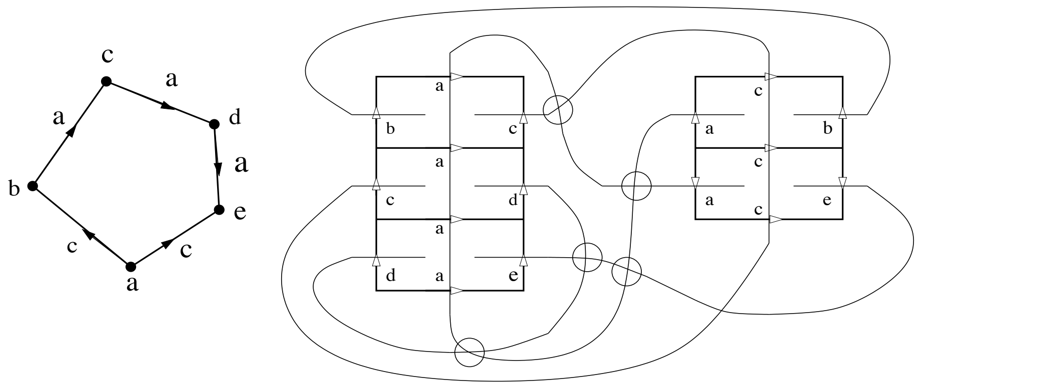

Figure 1: A LOC gives rise to a virtual knot diagram. The circles indicate virtual crossings.

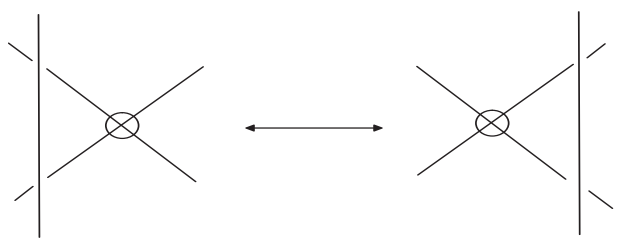

Figure 2: The "forbidden move" which leaves the LOC unchanged

Conversely given a virtual knot diagram one can assign generators to each arc between two consecutive undercrossings and read off Wirtinger relations at each non-virtual crossing. The Wirtinger presentation thus obtained defines a LOC whose group is the virtual knot group. •

Observe that different virtual knots may give rise to the same LOC. If a is the label of two different edges of a LOC then the LOC gives no information which of the two corresponding crossings in the virtual knot should come first when running along the component labeled a. So the move of Figure 2 in a virtual knot diagram, usually known as *forbidden move*, does not change the corresponding LOC (and hence the virtual knot group). Conversely, two different LOCs may produce isotopic virtual knots but different virtual knot diagrams.

The following result is well known [3]. We provide a proof for the convenience of the reader.

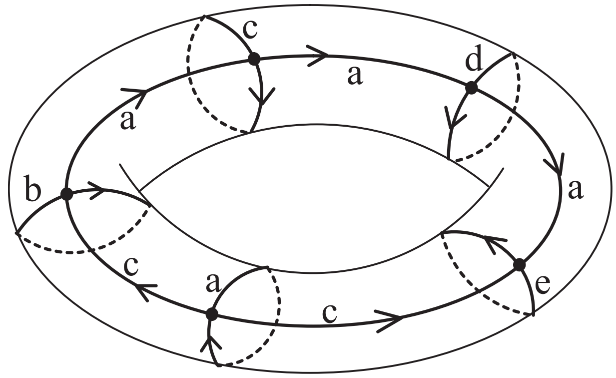

Figure 3: A non-trivial element of $H_2(X)$

Lemma 2.2 *If G is a knot group then $H_2(G) = 0$.*

Proof: Let $M = S^3 - k$ be the knot complement in the 3-sphere of a given knot k and G the fundamental group of M. By the Sphere Theorem M is aspherical. Let m be a meridian in the boundary of M. Let $\hat{M}$ be M together with a 2-cell d attached to M along m. Since M and $\hat{M}$ have the same 3-cells $H_3(\hat{M}, M) = 0$, and since $\hat{M}$ is homotopic to a 3-ball, $H_2(\hat{M}) = 0$. It follows from the long exact homology sequence associated with the pair $(\hat{M}, M)$ that $H_2(M) = 0$. Since M is aspherical it is a $K(G, 1)$-complex and $H_2(M) = H_2(G)$ which implies $H_2(G) = 0$. •

Theorem 2.3 *Let G be a LOC group and assume the 2-complex constructed from the corresponding LOC presentation is aspherical. Then G is a virtual knot group but not a knot group.*

Proof: G is a virtual knot group because of Theorem 2.1. We show that G is not a knot group.

Let X be the 2-complex constructed from the LOC presentation. Then $G = \pi_1(X)$. Since we assume X to be aspherical it is a $K(G, 1)$-complex and $H_2(G) = H_2(X)$.

Now $z = \sum_{i=1}^{n} d_i$ is a cycle, where the summands d_i $(i = 1, ..., n)$ are the 2-cells of X. This can be seen by the following argument: Take the individual 2-cells of X and first identify edges with the same label in each 2-cell.

Then place the resulting cylinders together side by side as the corresponding edges appear in the LOC. This builds a torus. The torus corresponding to the LOC of Figure 1 is depicted in Figure 3. This torus is a non-trivial element of $H_2(X)$. Hence $H_2(G) = H_2(X) \neq 0$. It follows from Lemma 2.2 that G is not a knot group. •

3 Ashperical Examples

A *non-positively curved square-complex* is a combinatorial 2-complex where each face has exactly four sides and the link of a vertex does not contain cycles of length less than four. It is well known that such a complex can be metrized as a piecewise Euclidian complex of non-positive curvature by giving each face the metric of the unit square in the plane. The universal covering of such a metrized complex is a unique geodesic metric space and hence contractible. In particular a non-positively curved square-complex is aspherical. For details on these matters we refer the reader to [1]. The 2-dimensional setting is addressed on page 215, connections with small cancellation conditions is the content of Proposition 5.25 on page 216. Applications to knot theory can be found on page 220. Theorem 5.35 on page 220 is close in spirit to the results obtained in this paper.

A LOC is *reduced* if every edge involves three distinct labels, and no two adjacent edges which are both directed towards (sink) or away (source) from the common vertex have the same label.

Theorem 3.1 *Suppose that $\mathcal{C}$ is a reduced LOC and X is the associated Wirtinger complex. Assume that none of the three situations shown in Figure 4 are present in $\mathcal{C}$. Then X is a non-positively curved square-complex and hence is aspherical. In particular the fundamental group of X is a virtual knot group but not a knot group.*

Proof: Figure 4 depicts the three cases that yield length two and three cycles in the vertex link of X. In case C both arrows labeled b could also be reversed. If none of these situations are present in $\mathcal{C}$, then the associated Wirtinger complex X is a non-positively curved square-complex and hence is aspherical (see Rosebrock [15]). The remaining statements follow from Theorem 2.3. •

Example 3.2 *Let $\mathcal{C}(n,k)$, $k \geq 2$, $n \geq 3k$, be the LOCs with vertices $1, 2, ..., n+1$ and edges e_i connecting consecutive vertices i and $i+1$ (modulo $n+1$), oriented arbitrarily. The label on e_i is $k+i$ modulo $n+1$. These LOCs satisfy the conditions of Theorem 3.1 (see Figure 5 in case $n=6$ and $k=2$).*

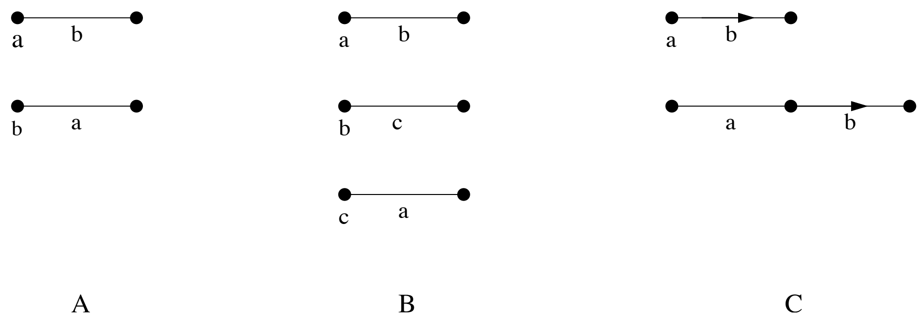

Figure 4: The three cases A, B, C, that lead to short cycles in the vertex link of the associated Wirtinger complex. Unoriented edges can have any orientation. In case C the given edge orientations could both be reversed.

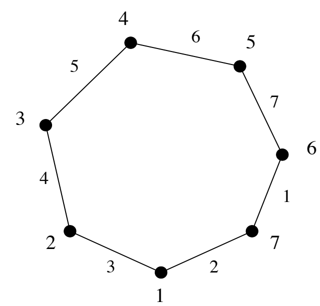

Figure 5: The LOCs $\mathcal{C}(6, 2)$, edges can be oriented arbitrarily.

In [8] and [7] asphericity and geometric properties of LOGs coming from alternating knot projections on surfaces are studied. Note that one obtains an virtual knot projection on the sphere from a knot projection on a surface by projecting the handles of the of the surface onto the sphere. An alternating knot projection on a surface F is termed *prime* if the 1-skeleton of the square cell decomposition dual to the projection does not contain reduced edge loops of length less than four. In the classical case where the projection surface S is a sphere this notation agrees with the usual definition of a prime alternating knot. The following is a special case of the main result of [8] (see also Huck, Rosebrock [10]): Let $\mathcal{C}$ be a LOC coming from a prime alternating knot projection on a surface of genus at least one and let X be the associated Wirtinger complex. Then X is aspherical. This, together with Corollary 2.3 implies the following theorem:

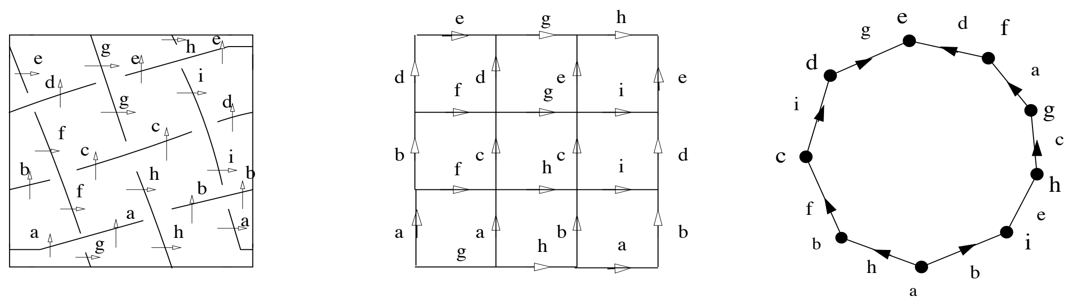

Figure 6: The LOC $\mathcal{T}(3)$ which is not prime.

Theorem 3.3 *Let $\mathcal{C}$ be a LOC coming from a prime alternating knot projection on a surface of genus at least one and let X be the associated Wirtinger complex. Then X is aspherical. Hence the fundamental group of X is a virtual knot group but not a knot group.*

Example 3.4 *Consider the square $[0,n] \times [0,n]$, n an odd number in $\mathbb{N}$. Draw lines from $(0,j)$ to $(n,j+1)$, $j = 0,...,n-1$ and also from (i,n) to $(i+1,0)$, $i = 0,...,n-1$. Starting at $(0,0)$, travel along these lines from left to right, make the first crossing an over-crossing, the next an under-crossing and so on. After identifying opposite sides we obtain an alternating link projection $A(n)$ on a torus. Note that it is prime for $n \geq 5$. We denote the LOC coming from this projection by $\mathcal{T}(n)$ and the fundamental group of the associated Wirtinger complex by $G(n)$. By Theorem 3.3 $G(n)$, n odd and $n \geq 5$, is a virtual knot group but not a knot group.*

The case $n = 3$ is shown in Figure 6.

We present another class of examples of aspherical LOCs using the cycle-test (see [9]) which is a modification of the weight test (see for example [10]). Both weight and cycle tests are based on combinatorial curvature notions. For applying the weight test one assigns weights (these should be thought as angles) to the corners of the 2-cells of a given 2-complex X. Using the weights one defines local curvature at 2-cells and vertices of X. If this curvature is non-positive everywhere the complex X can be shown to be aspherical. The cycle test can be applied to 2-complexes that can not be non-positively curved as required in the weight test. Rather than assigning curvature to points on X one does so with a given diagram. The cycle test in essence says that if

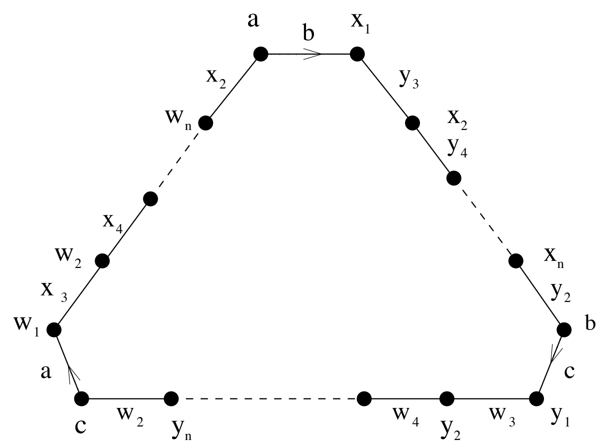

Figure 7: The LOCs $\mathcal{L}_n$

X satisfies certain local conditions then every reduced diagram over X can be non-positively curved and hence can not be a sphere.

Example 3.5 *The Wirtinger complex X_n of the LOCs $\mathcal{L}_n$ shown in Figure 7 with any orientation of the remaining edges are aspherical for all $n \geq 7$. Hence the fundamental group of X_n, $n \geq 7$, is a virtual knot group but not a knot group.*

Proof: Observe that the LOCs shown in Figure 7 are not far from producing non-positively curved square-complexes. Note that the cases A and C of Figure 4 are not present and, for $n \geq 7$, the case B appears exactly once and involves the three edges labeled a, b and c. Indeed, case C does not occur since the LOC is *injective*, that is, each generator appears at most (and hence exactly) once as an edge label. Case A is not present since edges y_i are only bounded by vertices labeled x_j but no vertex y_i bounds an edge labeled x_j (the other cases are similar). Let us look at case B. For $n = 6$ there would be many situations of type B aside from the a, b, c situation, for instance with the generators x_1, y_3, w_5. In case $n \geq 7$ labels different from a, b and c are far enough away from each other and can not produce a case B. Thus for $n \geq 7$, the Wirtinger complex X_n would be a non-positively curved square-complex except for one 3-cycle involving a, b and c. See Figure 8.

Suppose that S is a reduced closed surface diagram over X_n. If no vertex

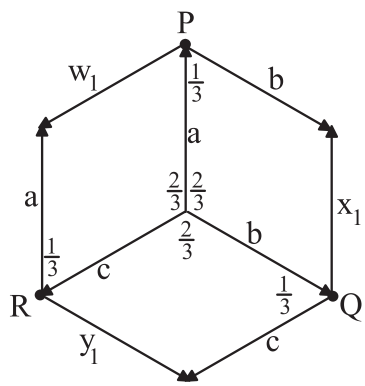

Figure 8: Weights in case of a 3-cycle

in S has valence less that four, we assign the weight $1/2$ to every corner. This makes S non-positively curved, hence S is not a sphere.

Next suppose that S does have a vertex of valence less than four. Since the vertex link in X_n, $n \geq 7$, does not contain 2-cyles and only one 3-cycle we have to see a local situation as shown in Figure 8 in S. We give the weights $2/3$ to the corners around the valence three vertex and weights $1/3$ as assigned in Figure 8. All other weights are $1/2$. The only possible problem might occur at the vertices P, Q or R. For example if the valence at P is four then the weight sum around P is $3/2 + 1/3 < 2$, which gives positive curvature at P. However, a close look at the LOC $\mathcal{L}_n$ and the vertex link of X_n reveals that the valence of P, Q and R must be at least 5. Thus the assigned weights make S into a non-positively curved complex. Hence S is not a 2-sphere. We conclude that X_n, $n \geq 7$, does not admit reduced spherical diagrams and hence is aspherical. •

4 Torsion

We end this article with an example of a virtual knot group that has torsion. Since classical knot groups are torsion free (knot complements in the 3-sphere are aspherical by the sphere theorem, see Papakyriakopoulos [14]), this provides yet another example of a virtual knot group that is not a knot group.

Consider the LOC of Figure 9 and let G be its group. Its Wirtinger

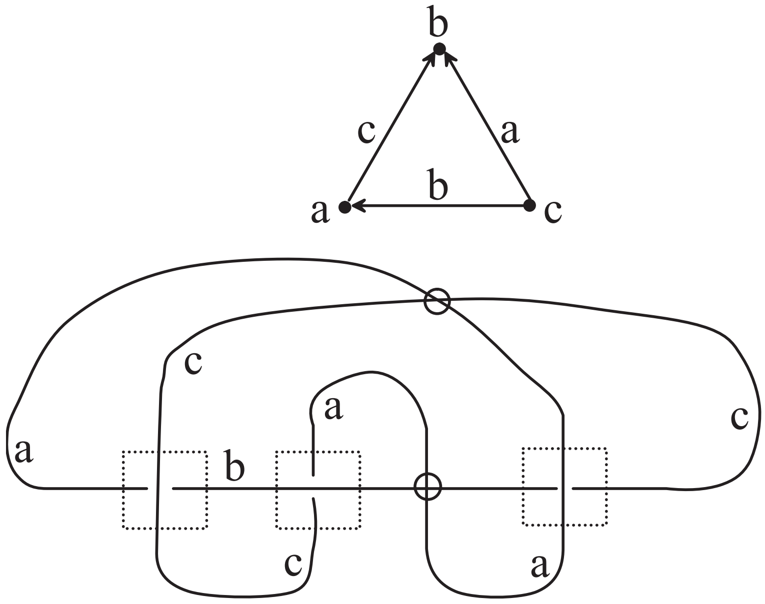

Figure 9: A virtual knot whose group has torsion.

presentation is

$$\langle a, b, c \mid ac = cb, ca = ab, cb = ba \rangle.$$

We have $c = bab^{-1}$ and, after performing a Tietze transformation we obtain a 2-generator presentation

$$\langle a, b \mid abab^{-1} = ba, bab^{-1}a = ab \rangle.$$

The first relation implies that $bab^{-1} = a^{-1}ba$, and the second implies $bab^{-1} = aba^{-1}$, hence $a^{-1}ba = aba^{-1}$, which shows that a^2 is central in G. Note that $G/[G,G] = \mathbb{Z}$, generated by a, so a^2 is a central element of infinite order in G. Since a and b are conjugate in G it follows that $b^2 = 1$ in the quotient $Q = G/\langle a^2 \rangle$. Thus Q is presented by

$$\langle a, b \mid aba = bab, a^2, b^2 \rangle,$$

which shows that $Q = D_3$, the dihedral group of order 6. Since a^2 is in the center we can write G as an extension of the infinite cyclic group generated by a^2 and the finite group Q. Thus G is virtually an infinite cyclic group (i.e. G has a finite index subgroup which is infinite cyclic) that is not cyclic since it maps onto D_3. Thus G has to have torsion, because a torsion free virtually cyclic group is cyclic.

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