MINIMAL COCKCROFT SUBGROUPS
by JENS HARLANDER
(Received 20 July, 1992)

Statement of results. Consider any group G. A \([G, 2]\)-complex is a connected 2-dimensional CW-complex with fundamental group G. If X is a \([G, 2]\)-complex and L is a subgroup of G, let \(X_L\) denote the covering complex of X corresponding to the subgroup L. We say that a \([G, 2]\)-complex is \(L\)-Cockcroft if the Hurewicz map

\[ h_L: \pi_2(X) \to H_2(X_L) \]

is trivial. In case \(L = G\) we call X Cockcroft. There are interesting classes of 2-complexes that have the Cockcroft property. A \([G, 2]\)-complex X is aspherical if \(\pi_2(X) = 0\). It was observed in [4] that a subcomplex of an aspherical 2-complex is Cockcroft. The Cockcroft property is of interest to group theorists as well. Let X be a \([G, 2]\)-complex modelled on a presentation \(\langle S; R \rangle\) of the group G. If it can be shown that X is Cockcroft, then it follows from Hopf’s theorem (see [2, p. 31]) that \(H_2(G)\) is isomorphic to \(H_2(X)\). In particular \(H_2(G)\) is free abelian. For a survey on the Cockcroft property see Dyer [5]. A collection \(\{G_\alpha: \alpha \in \Omega\}\) of subgroups of a group G that is totally ordered by inclusion is called a chain of subgroups of G. Defining \(\beta \leq \alpha\) if and only if \(G_\beta \leq G_\alpha\) makes \(\Omega\) into a totally ordered set. The main result of this paper is the following theorem.

**Theorem 1.** Let \(\{G_\alpha: \alpha \in \Omega\}\) be a chain of subgroups of a group G. A \([G, 2]\)-complex X that is \(G_\alpha\)-Cockcroft for all \(\alpha \in \Omega\) is also \(\bigcap_{\alpha \in \Omega} G_\alpha\)-Cockcroft.

Theorem 1 together with Zorn’s lemma give the next result.

**Corollary 1.** Let X be a Cockcroft \([G, 2]\)-complex. Then G contains a minimal subgroup L such that X is \(L\)-Cockcroft.

It is a longstanding open question raised by J. H. C. Whitehead [9] whether a subcomplex of an aspherical complex is aspherical. Suppose X is a subcomplex of an aspherical 2-complex Y and denote by K the kernel of the map \(\pi_1(Y) \to \pi_1(Y)\) induced by inclusion. J. F. Adams [1] showed that if X is not aspherical then K contains a nontrivial perfect subgroup. He studied a certain system of coverings \(\{X_{K_\alpha}\}_{\alpha \in \Omega}\) of X, where \(\{K_\alpha\}_{\alpha \in \Omega}\) is the set of characteristic subgroups of K such that the quotients \(K/K_\alpha\) are \(C\)-conservative for any abelian group C. A group G is \(C\)-conservative if the functor \(C \otimes_{CG} \cdot\) detects monomorphisms between projective \(CG\)-modules; i.e. if \(\Psi: P \to Q\) is a homomorphism between projective \(CG\)-modules and \(C \otimes_{CG} \Psi: C \otimes_{CG} P \to C \otimes_{CG} Q\) is injective, then \(\Psi\) is injective (see also Howie [8]). Adams observed that N, the intersection of all groups \(K_\alpha\), is perfect and that \(H_2(X_N) = 0\). If one assumes X to be non-aspherical; then the second homology of the universal covering of X is non-trivial. Thus \(X_N\) is different from the universal covering and therefore N is non-trivial (see also Howie [6] and [7]).

The proof of Theorem 1 relies on a lemma that deals with arbitrary systems of coverings \(\{X_{G_\alpha}\}_{\alpha \in \Omega}\) of a \([G, 2]\)-complex X. We show that \(H_2(X_N)\) embeds in \(\lim \{H_2(X_{G_\alpha})\}\), where N is the intersection of all the \(G_\alpha\). We use this result also to characterize non-asphericity of a 2-complex X with \(H_2(X) = 0\) by the existence of a certain minimal subgroup of \(\pi_1(X)\).

THEOREM 2. Let $X$ be a $[G, 2]$-complex with $H_2(X) = 0$. The following statements are equivalent:

(i) $X$ is non-aspherical;
(ii) there exists a non-trivial minimal subgroup $L$ of $G$ such that $H_2(X_L) = 0$.

Furthermore, if $X$ is non-aspherical, then no group $L$ as in (ii) can have a nontrivial $\mathbb{Z}$-conservative quotient; in particular $L_{nh}$ is torsion.

Assume now that $X$ is a subcomplex of an aspherical 2-complex $Y$. As before let $K$ denote the kernel of the homomorphism $\pi_1(X) \to \pi_1(Y)$ induced by the inclusion map. The covering complex $X_K$ of $X$ can be viewed as a subcomplex of the universal covering complex $\hat{Y}$ of $Y$. Since $X_K$ and $\hat{Y}$ are 2-complexes, the map $H_2(X_K) \to H_2(\hat{Y})$ induced by inclusion is injective. Since $H_2(\hat{Y}) = \pi_2(\hat{Y}) = 0$ it follows that $H_2(X_K) = 0$. Theorem 2 applied to the complex $X_K$ together with the fact that $X$ is non-aspherical if and only if $X_K$ is non-aspherical, yield the following result.

COROLLARY 2. Let $X$ be a $[G, 2]$-complex that is a subcomplex of an aspherical 2-complex $Y$. Let $K$ be the kernel of the homomorphism $\pi_1(X) \to \pi_1(Y)$ induced by inclusion. The following statements are equivalent:

(i) $X$ is non-aspherical;
(ii) there exists a non-trivial minimal subgroup $L$ of $K$ such that $H_2(X_L) = 0$.

Furthermore, if $X$ is non-aspherical, then no group $L$ as in (ii) can have a non-trivial $\mathbb{Z}$-conservative quotient; in particular $L_{nh}$ is torsion.

I am grateful to Mike Dyer for many helpful suggestions.

Proof of results. Let $X$ be a $[G, 2]$-complex and let $\{G_\alpha : \alpha \in \Omega\}$ be a chain of subgroups of $G$. Denote by $\hat{X}$ the universal covering complex of $X$ and by $p$ the covering projection

$$p : \hat{X} \to X.$$

The preimage $p^{-1}(c)$ of each open cell $c$ in $X$ consists of open cells $\hat{c}_g$, $g \in G$, such that

$$p|_{\hat{c}_g} : \hat{c}_g \to c$$

is a homeomorphism. For each $G_\alpha$, the orbit complex $\hat{X}/G_\alpha$, denoted by $X_\alpha$, is the covering complex $X_{G_{\alpha}}$ with covering projection

$$p_\alpha : \hat{X} \to X_\alpha.$$

Denote by $N$ the intersection $\bigcap_{\alpha \in \Omega} G_\alpha$ and by $p_N$ the covering projection

$$p_N : \hat{X} \to X_N.$$

Let $p_{\alpha N}$ be the covering projection

$$p_{\alpha N} : X_N \to X_\alpha$$

and let $p_{\beta \alpha}$ be the covering projection

$$p_{\beta \alpha} : X_\alpha \to X_\beta$$

for $\alpha \geq \beta$. The cells in $X_N$ and in $X_\alpha$ are just $N$ and $G_\alpha$ orbits of cells in $\hat{X}$. So if $N \ast \hat{c} = \{n \ast \hat{c} : n \in N, \hat{c} \text{ an open cell of } \hat{X}\}$ is an open cell of $X_N$, then $p_{\alpha N}$ sends this open
cell homeomorphically onto the open cell $G_\alpha * \tilde{c}$ of $X_\sigma$ and $p_{\beta \alpha}$ sends the open cell $G_\alpha * \tilde{c}$ of $X_\sigma$ homeomorphically onto the open cell $G_\beta * \tilde{c}$ of $X_\beta$ for $\alpha \neq \beta$. Now $(C_3(X_\alpha), p_{\alpha \beta})_{\alpha, \beta \in \Omega}$ is an inverse system of Abelian groups with inverse limit

$$\lim_{\alpha \in \Omega} C_3(X_\alpha).$$

**Lemma 1.** $\lim_{\alpha \in \Omega} C_3(X_\alpha) \rightarrow \lim_{\alpha \in \Omega} C_3(X_\alpha)$ is injective and yields an injection from $H_3(X_N)$ to $\lim_{\alpha \in \Omega} H_3(X_\alpha)$ when restricted to $H_3(X_N)$; in particular, if all the $H_3(X_\alpha)$ are trivial, then $H_3(X_N)$ is trivial.

**Proof.** First we show that if $c_1 = N * \tilde{c}_1$ and $c_2 = N * \tilde{c}_2$ are two different open cells in $X_N$, then there exists an element $\beta \in \Omega$ such that $p_{\beta N}(c_1)$ and $p_{\beta N}(c_2)$ are two different open cells in $X_\beta$. Suppose not. Then

$$G_\alpha * \tilde{c}_1 = G_\alpha * \tilde{c}_2$$

for all $\alpha \in \Omega$. So, in particular,

$$\tilde{c}_1 \in G_\alpha * \tilde{c}_2$$

for all $\alpha \in \Omega$. Then for each $\alpha \in \Omega$ there exists a $g_\alpha$ in $G_\alpha$ such that

$$\tilde{c}_1 = g_\alpha * \tilde{c}_2.$$

Fix an element $\gamma \in \Omega$; then $g_\alpha * \tilde{c}_2 = \tilde{c}_1 = g_\gamma * \tilde{c}_2$ for all $\alpha \in \Omega$; hence $g_\gamma^{-1} g_\alpha * \tilde{c}_2 = \tilde{c}_2$ for all $\alpha \in \Omega$. Since $G$ acts freely on the set of open cells of $X$ this says that $g_\gamma^{-1} g_\alpha = 1$; thus $g_\gamma = g_\alpha \in G_\alpha$ for all $\alpha \in \Omega$ and therefore $g_\gamma$ is an element of the intersection $N$. Since

$$\tilde{c}_1 = g_\gamma * \tilde{c}_2,$$

we have $c_1 = N * \tilde{c}_1 = N * \tilde{c}_2 = c_2$, which contradicts our assumption that $c_1$ and $c_2$ are different cells. Suppose now that

$$z = \sum_{k=1}^{m} n_k c_k,$$

is a nontrivial element of $C_3(X_N)$, so that the integers $n_k$ are nonzero and the cells $c_k$ are different 2-cells of $X_N$. If $m = 1$, then

$$p_{\alpha N}(z) = n_1 p_{\alpha N}(c_1) \neq 0$$

for all $\alpha \in \Omega$. If $m > 1$ then for every pair $\{i, j\}, i, j \in \{1, \ldots, m\}$, we can find an element $\beta(i, j) \in \Omega$ such that $p_{\beta(i, j) N}(c_i)$ and $p_{\beta(i, j) N}(c_j)$ are two different 2-cells of $X_{\beta(i, j)}$. Let $\beta$ be the largest element among the finitely many $\beta(i, j)$. Then $p_{\beta N}(c_i)$ and $p_{\beta N}(c_j)$ are different cells for any pair $\{i, j\}, i, j \in \{1, \ldots, m\}$, so

$$p_{\beta N}(z) = \sum_{k=1}^{m} n_k p_{\beta N}(c_k) \neq 0.$$

This shows that

$$\lim_{\alpha \in \Omega} p_{\alpha N}(z) \neq 0.$$

**Lemma 2.** $(\lim_{\alpha \in \Omega} p_{\alpha N}) * h_N = \lim_{\alpha \in \Omega} h_\alpha.$
Proof. From the commutative diagram

\[ \begin{array}{ccc}
\pi_2(X) & \xrightarrow{p} & \pi_2(\tilde{X}) \\
& h \downarrow & \downarrow h_N \\
\quad & H_2(\tilde{X}) & C_2(\tilde{X}) \\
& h_N & \downarrow \\
& H_2(X_L) & C_2(X_L)
\end{array} \]

we see that for every \( \alpha \in \Omega \),

\[ p_{\alpha N} \circ h_N = p_{\alpha N} \circ h \circ p_{\alpha}^{-1} = p_{\alpha} \circ h \circ p_{\alpha}^{-1} = h_{\alpha}. \]

Hence \( \lim_{\alpha} p_{\alpha N} \circ h_N = \lim_{\alpha} h_{\alpha} \).

Proof of Theorem 1. Since \( X \) is \( G_\alpha \)-Cockcroft for every \( \alpha \in \Omega \), each \( h_{\alpha} \) is the zero map. Hence \( \lim_{\alpha} h_{\alpha} \) is the zero map. Lemma 2 and the fact that, by Lemma 1, \( \lim_{\alpha} p_{\alpha N} \) is injective show that \( h_N \) is the zero map as well. So \( X \) is \( N \)-Cockcroft.

Proof of Theorem 2. Only the direction (i) \( \Rightarrow \) (ii) requires a proof. If \( \{ G_\alpha : \alpha \in \Omega \} \) is a chain of subgroups of \( G \) such that \( H_2(X_\alpha) = 0 \) for all \( \alpha \), then \( H_2(X_N) = 0 \) by Lemma 1; as before \( X_\alpha \) is the 2-complex \( \tilde{X}_G \), and \( N \) is the intersection of all the \( G_\alpha \). The existence of a minimal subgroup \( L \) such that \( H_2(X_L) = 0 \) now follows from Zorn’s Lemma. If \( L/K \) were a non-trivial \( \mathbb{Z} \)-conservative quotient of \( L \), then \( K \) would be a proper subgroup of \( L \) with \( H_2(X_K) = 0 \) by definition of \( \mathbb{Z} \)-conservative. This contradicts minimality of \( L \).

REFERENCES

2. K. S. Brown, Cohomology of groups (Springer, 1982).

FACHBEREICH MATHEMATIK
JOHANN WOLFGANG GOETHE-UNIVERSITÄT
6000 FRANKFURT/MAIN 11