EMBEDDINGS INTO EFFICIENT GROUPS

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A finite presentation \( F/N \) of a group \( G \) is called efficient if \( d_\rho(N) = d(H_1(G)) + d(F) - r(H_1(G)) \). A finitely presented group is called efficient if it admits an efficient presentation. We show that a finitely presented group embeds into an efficient group.

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1. Background

If \( A \) is a \( G \)-group, then \( d_\rho(A) \) denotes the minimal number of \( G \)-group generators of \( A \). For example the normal subgroup \( N \) of a group \( F \) is an \( F \)-group via conjugation and \( d_\rho(N) \) is the minimal number of elements that generate \( N \) as a normal subgroup. If \( G \) acts trivially on \( A \) we omit the subscript and simply write \( d(A) \) for the minimal number of generators for the group \( A \).

Given a finite presentation \( \mathcal{P} = \langle X \mid R \rangle \) of a group \( G \), let \( F \) be the free group on \( X \) and \( N = N(R) \) be the normal closure of \( R \) in \( F \). Then \( F/N = G \). We also refer to \( F/N \) as a presentation for \( G \). Now

\[(*) \quad N/[F, N] = H_2(G) \oplus \mathbb{Z}^{d(F) - r(H_1(G))},\]

where \( r(H_1(G)) \) is the torsion free rank of the finitely generated abelian group \( H_1(G) \). To see this, consider the exact 5-term sequence

\[H_2(F) \rightarrow H_2(G) \rightarrow N/[F, N] \rightarrow F/[F, F] \rightarrow G/[G, G] \rightarrow 0\]

associated with the extension \( N \rightarrow F \rightarrow G \) (see Brown [9, page 47]). Since \( F \) is free, \( H_2(F) = 0 \) and the result follows. In particular we have

\[d(N/[F, N]) = d(H_2(G)) + d(F) - r(H_1(G)).\]

For more details and additional references see Beyl, Tappe [5, page 18]. The presentation \( \mathcal{P} = \langle X \mid R \rangle \) is called efficient if

\[|R| = d_\rho(N) = d(N/[F, N]).\]
The group $G$ is called efficient if it admits an efficient presentation. Examples of efficient groups are finitely generated abelian groups (Epstein [13]), fundamental groups of closed 3-manifolds [13] and also finite groups with balanced presentations. Such finite groups have trivial Schur-multiplier. Whether finite groups with trivial Schur-multiplier are efficient (i.e., admit balanced presentations in this case) was answered negatively by Swan [26]. He gave examples of non-efficient metabelian groups with trivial $H_2$. Finite metacyclic groups are efficient. This was shown by Wamsley [27] and Beyl [4]. Infinite metacyclic groups however need not be efficient, a result due to Baik and Pride [2] (see also Baik [1]). The first examples of torsion-free non-efficient groups were found by Lustig [21]. For more references on the subject of efficiency see Baik, Pride [3], Beyl, Rosenberger [6], Campbell, Robertson, Williams [10, 11], Johnson, Robertson [18], Kenne [20] and Robertson, Thomas, Wotherspoon [24].

Suppose $(X|R)$ is a finite presentation for a group $H$. Assume that $u$ and $w$ are words in $X^{\pm 1}$ and let $G$ be the quotient of $H$ presented by $(X|R, w)$. Suppose the following conditions are satisfied:

1. $[u, w]$ represents the trivial element of $H$;
2. $u$ represents an element of infinite order of $G$;
3. The presentation $(X|R, w)$ is efficient.

The group $G$ can be used to embed a given group into an efficient group by an iterated amalgamated product. Before we state our main result we introduce more notation. Let $S(K, G, l)$ be the fundamental group of a graph of groups supported by a graph with vertices $v_1, v_2, \ldots, v_l$ and oriented edges $e_1, \ldots, e_l$, where $e_i$ starts at $v_{i-1}$ and ends at $v_i$. The group at $v$ is $K$, all other vertex groups are $G$ (as above) and the edge groups are infinite cyclic. Edge maps are given by choosing elements of infinite order in $K$ and the other vertex groups.

**Theorem.** Suppose that $K$ is a finitely presented group that admits a generating set consisting of elements of infinite order. Suppose furthermore that, in case both $H_2(K)$ and $H_2(G)$ have torsion, the first torsion-numbers of these abelian groups are not relatively prime. Then there exists an integer $l$ such that $S(K, G, l)$ is efficient.

Note that the condition on the torsion numbers ensures that $d(H_2(K) \oplus H_2(G)) = d(H_2(K)) + d(H_2(G))$.

There is considerable flexibility in choosing $G$. For example we can take $H = \langle a, b | a^n = b^n, u = ab \rangle$ and $w = a^l$. In that case we get $G = \langle a, b | a^n = b^n, a^l = Z \ast Z \ast Z \rangle$, the free product of two cyclic groups of order $n$. Or we could take $H = \langle a, b, c | [a, b], [a, c], [b, c] \rangle$, $u = a$ and $w = [b, c]$. Here we obtain $G = \langle a, b, c | [a, b], [a, c], [b, c] \rangle = Z \oplus Z \oplus Z$. Note that in both cases $H_2(G)$ is torsion-free. Before we prove the Theorem we point out some consequences.

**Corollary 1.** Let $K$ be a finitely presented group and let $d(K) = k$. Let $F_k$ be the free
group of rank $k$. Then there exists an integer $l$ such that $S(K \ast F_k, \mathbb{Z}_n \ast \mathbb{Z}_n, l)$ is efficient. In particular a finitely presented group can be embedded into a finitely presented efficient group.

Proof. Let $y_1, \ldots, y_k$ be a set of generators for $K$ and let $a_1, \ldots, a_k$ be a basis for $F_k$. Then $y_1a_1, \ldots, y_k a_k, a_1, \ldots, a_k$ is a generating set of $K \ast F_k$ consisting of elements of infinite order. Now apply the Theorem.

We remark that the author showed in [15] that a finite group can be embedded into a finite efficient group. In fact, if $K$ is finite, then $K \times \prod_{p < \omega} \mathbb{Z}_p$ is efficient for $l$ big enough and $p$ a prime.

Corollary 2. Let $K$ be a finitely presented group of finite cohomological dimension $k$. If $k \neq 2$, then $K$ can be embedded into an efficient group of cohomological dimension $k$. If $k = 2$, then $K$ can be embedded into an efficient group of virtual cohomological dimension 2.

Proof. If $k = 1$ then, by Stallings' Theorem [25], $K$ is free and thus itself efficient. So suppose $k \geq 2$. Since $K$ is torsion-free, it admits a generating set consisting of elements of infinite order. We can apply the Theorem to see that $\overline{G} = S(K, G, l)$ is efficient for big enough $l$ and an appropriately chosen group $G$. If $k = 2$ take $G = \mathbb{Z}_n \ast \mathbb{Z}_n$. The virtual cohomological dimension of both $\mathbb{Z}$ and $\mathbb{Z}_n \ast \mathbb{Z}_n$ is one and hence $vcd(\overline{G}) = vcd(K) = 2$ (see Bieri [7, page 83]). If $k \geq 3$ take $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Since $cd(G) = 3$ it follows that $cd(\overline{G}) = cd(K) = k$.

Whether a group of cohomological dimension 2 can be embedded into an efficient group of cohomological dimension 2 is related to the question whether a group of cohomological dimension 2 has geometric dimension 2. A discussion of these matters can be found in Section 4 of this paper.

2. The main lemma

The proof of the main theorem in this article is based on an idea of Wolfgang Metzler. He realized (see [23]) that wedging on standard 2-complexes of $\mathbb{Z}_2 \times \mathbb{Z}_4$ to a given 2-complex allows one to bypass the commutator question, a serious obstruction encountered when attempting to generalize results from higher dimensions into dimension 2. Hog-Angeloni and Metzler have successfully applied this trick to various situations (see Metzler [23] and also Metzler, Hog-Angeloni [16]). We will present a generalized $\mathbb{Z}_2 \times \mathbb{Z}_4$ trick, which is tailored to our situation. Suppose $P_H = \langle X | R \rangle$ is a finite presentation for the group $H$ and $u$ and $w$ are words in $X^{\pm 1}$ so that the commutator $[u, w]$ represents the trivial element of $H$. Let $G$ be the quotient of $H$ represented by $P_G = \langle X | R, w \rangle$. Let $n$ be the order of the element of $G$ represented by $u$ (the order can be infinite). Next assume that $K$ is another group admitting a finite
presentation $\mathcal{P}_K = (Y|S, [f, t])$, where $t$ is a consequence of $S, [f, t]$ in $\mathcal{P}_K$, that is $t \in N(S, [f, t])$, and $f$ represents an element of order $n$ in $K$. We can form the free product with amalgamation $\hat{G} = K \ast_{Z_n} G$ with presentation $\mathcal{P} = (X, Y|R, w, S, [f, t], u = f)$. The key observation here is that the normal closure of the relations in $\mathcal{P}$ is generated by $|R| + 1 + |S| + 1$ elements, which is one less than expected. Indeed $\{R, S, w = t, u = f\}$ is a generating set for that normal closure. Just observe that $[f, t] = [u, w] = 1$

modulo the relations $f = u, t = w$ and $R$. Since we assumed that $t = 1$ modulo $S$ and $[f, t]$, this shows that $w = 1$ modulo $R, S, f = u$ and $t = w$. If we iterate the above process we obtain the following.

**Lemma.** Suppose $K$ is a group admitting a presentation $\mathcal{P}_K = (Y|S, [f, t]), 1 \leq i \leq l$, where each $t_i$ is contained in $N(S, [f_i, t_i]), \ldots, [f_i, t_i])$, and each $f_i$ represents an element of infinite order. Let $\mathcal{P} = (X, Y|S, [f, t], R_i, w_i, u_i = f_i)$, $1 \leq i \leq l$, $(X, R_i, w_i)$ presenting $G, u_i$ representing an element of infinite order in $G$, be the standard presentation for the amalgamated product $S(K, G, l)$. Then the normal closure of the relations in $\mathcal{P}$ is generated by $|S| + (|R| + 2)l$ elements.

A free product version of the above Proposition with $Z_2 \times Z_4$ factors is implicit in [23], dealing with commutators of relators, that is with elements of $[N, N]$ rather than $[F, N]$.

3. Proof of the theorem

Let $F/N$ be a finite presentation for the group $K$, where $F$ is a free group with basis $Y$ and each element $y$ of $Y$ represents an element of infinite order in $K$. Let $m = d(N/[F, N])$. We can find elements $s_1, \ldots, s_m$ of $N$ so that $s_1[F, N], \ldots, s_m[F, N]$ generates $N/[F, N]$. Since $N$ is the normal closure of finitely many elements, we can find elements $f_i \in F, t_i \in N, 1 \leq i \leq l$, so that $N = N(s_1, \ldots, s_m, [f_i, t_i])$. Thus $\mathcal{P}_K = (Y|s_1, \ldots, s_m, [f_i, t_i])$, $1 \leq i \leq l$, presents $K$. Note that because $\{y^{x^t}, r \mid y \in Y, r \in N\}$ generates $[F, N]$ we may assume that each $f_i$ is equal to some $y^{x^t}$, in particular that each $f_i$ has infinite order in $K$. Let $\mathcal{P}_G = (X|R, w)$ be an efficient presentation of a group $G$ as in the previous section. Then we have a word $u$ in $X^{x^t}$ representing an element of infinite order in $G$ and $[u, w] = 1$ modulo $R$. Let
$P = (X_i, Y|S, [f_i, t_i], R_i, w_i, f_i = u_i),$

$1 \leq i \leq l$, be the standard presentation for the amalgamated product $S(K, G, l)$ as in the Lemma. Let $\tilde{F}$ be the free group on the generators in $P$ and let $\tilde{N}$ be the normal closure of the relations in $P$. Furthermore let $F_i$ be the free group on $X_i$ and let $N_i$ be the normal closure of $R_i$ and $w_i$ in $F(X_i)$. So $F_i/N_i$ presents the vertex group $G$ at $v_i$ in the above amalgamated product. We know from the Lemma that $d_\rho(N) \leq m + (|R| + 2)l$. We claim that $d(\tilde{N}/[\tilde{F}, \tilde{N}]) = m + (|R| + 2)l$ and thus that $\tilde{F}/\tilde{N}$ is efficient. Before we show this, let us make some general remarks. Suppose $F_i/N_i$ is a finite presentation for $G_i, i = 1, 2$, and that $C$ is a finitely generated subgroup of both $G_1$ and $G_2$. Let $F/N$ be a presentation for the amalgamated product $G = G_1 \ast_C G_2$, obtained from the presentations $F_i/N_i$ and a fixed finite generating set for $C$. Then we have an exact sequence (see Hannerbauer [14])

$$0 \rightarrow (ZG \otimes_{G_1} N_1/[N_1, N_1]) \oplus (ZG \otimes_{G_2} N_2/[N_2, N_2]) \rightarrow N/[N, N] \rightarrow ZG \otimes_C IC \rightarrow 0.$$ 

If we apply $Z \otimes_C -$ we obtain the exact sequence

$$H_1(C) \rightarrow N_1/[F_1, N_1] \oplus N_2/[F_2, N_2] \rightarrow N/[F, N] \rightarrow H_1(C) \rightarrow 0.$$ 

In case $C$ is infinite cyclic, $H_1(C) = 0$ and $H_1(C) = Z$ and we obtain

$$N/[F, N] = N_1/[F_1, N_1] \oplus N_2/[F_2, N_2] \oplus Z.$$ 

If we apply this result to our presentation $\tilde{F}/\tilde{N}$ of $S(K, G, l)$ we get

$$\tilde{N}/[\tilde{F}, \tilde{N}] = N/[F, N] \oplus \bigoplus_{i=1}^{l} N_i/[F_i, N_i] \oplus Z.$$ 

This follows from the above discussion and induction on $l$ since

$$S(K, G, l) = S(K, G, l-1) \ast_C G,$$ 

with $C$ infinite cyclic. Since $F_i/N_i$ is an efficient presentation for $G$, we have $d(N_i/[F_i, N_i]) = |R_i| + 1$. Since the first torsion-numbers of $H_2(G)$ and $H_2(K)$ are not relatively prime (in case both $H_2(K)$ and $H_2(G)$ contain torsion), we have $d(H_2(K) \oplus H_2(G)) = d(H_2(K)) + d(H_2(G))$. Since $N/[F, N]$ is the direct sum of $H_2(K)$ and a free abelian group and each $N_i/[F_i, N_i]$ is the direct sum of $H_2(G)$ and a free abelian group (see equation (a) on the first page), we have

$$d(\tilde{N}/[\tilde{F}, \tilde{N}]) = d(N/[F, N]) + \sum_{i=1}^{l} d(N_i/[F_i, N_i]) + l.$$ 

Hence $d(\tilde{N}/[\tilde{F}, \tilde{N}]) = m + (|R| + 2)l$ as claimed. $\square$
4. Groups of dimension 2

A group $G$ has cohomological dimension 2 if the trivial $G$-module $Z$ admits a projective resolution of length 2. The geometric dimension of $G$ is 2 if there exists a 2-dimensional $K(G, 1)$-complex. A group $G$ is of type FL if $Z$ admits a resolution of finite length consisting of finitely generated free $ZG$-modules. A presentation $\mathcal{P}$ is aspherical if the associated 2-complex $K(\mathcal{P})$ modelled on $\mathcal{P}$ is aspherical (that is, it has trivial second homotopy group). Note that in that case $K(\mathcal{P})$ is a $K(G, 1)$-complex and thus $G$ and all its subgroups have geometric dimension 2. These definitions can be found in [9].

Efficient groups of cohomological dimension 2 are of interest in connection with the longstanding open question whether cohomological dimension 2 implies geometric dimension 2. The next proposition shows that subgroups of an efficient group of cohomological dimension 2 that is FL have geometric dimension 2. This result is due to Gutierrez and Ratcliffe [17] (see also Bogley [8, page 329]). In [17] it is stated for subcomplexes of aspherical complexes. Such complexes give rise to presentations which are not only efficient but satisfy the Cockcroft property (see [12, page 149]). For the convenience of the reader we have also included a proof.

Proposition. Let $\mathcal{P}$ be a finite presentation of a group $G$ of cohomological dimension 2 that is of type FL. Then $\mathcal{P}$ is efficient if and only if $\mathcal{P}$ is aspherical.

Proof. Let $\mathcal{P} = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ be an efficient presentation for $G$. Let $F$ be the free group on the generators in $\mathcal{P}$ and let $N$ be the normal closure of the relations of $\mathcal{P}$ in $F$. Cohomological dimension 2 together with FL implies that the relation module $N/[N, N]$ of this presentation is finitely generated stably free. This is a consequence of Schanuel's Lemma (see [9, page 192]). So suppose $N/[N, N] \oplus ZG^k = ZG^l$. Replacing $\mathcal{P}$ with $\langle x_1, \ldots, x_n, y_1, \ldots, y_k | r_1, \ldots, r_m, y_1, \ldots, y_k \rangle$, we obtain an efficient presentation with free relation module of rank $l$. In particular $l = m + k$. Since the 2-complex associated with the modified presentation is simply homotopic to the two complex associated with $\mathcal{P}$, asphericity of the new presentation implies asphericity of $\mathcal{P}$. This discussion shows that we may assume that the relation module of $\mathcal{P}$ is free of rank $m$. Let us look at the partial resolution (see Lyndon, Schupp [22, page 100])

$$\pi_3(K(\mathcal{P})) \rightarrow ZG^m \xrightarrow{\partial_3} ZG^k \rightarrow Z \rightarrow 0$$

associated with $\mathcal{P}$ (it arises from the cellular chain complex of the universal covering of $K(\mathcal{P})$). The image of the boundary map $\partial_3$ is the relation module which is free of rank $m$. Thus it follows from Kaplansky's Theorem (see [19], and also [8, page 328]) that $\partial_3$ is an isomorphism and that $\pi_3(K(\mathcal{P}))$ is trivial. Thus $\mathcal{P}$ is an aspherical presentation for $G$. This proves one direction. That asphericity of a presentation implies efficiency is immediate from the partial resolution associated with $\mathcal{P}$. $\square$
The property FL was needed to ensure that every finite presentation of G has stably free relation module. It should be noted that there are no examples known of finitely presented groups of cohomological dimension 2 that are not FL. We know from Corollary 2 of Section 1 that a finitely presented group K of cohomological dimension 2 can be embedded into an efficient group \( S(K, G = Z_n \ast Z_n, l) \), which is of virtual cohomological dimension 2. If we could replace \( Z_n \ast Z_n \) by a group G of cohomological dimension 2 for which our method works, we could eliminate "virtual". If in addition \( S(K, G, l) \) is FL, then K is actually of geometric dimension 2 by the above Proposition. But we believe that such a group G is difficult to find. For our techniques to work we would have to find an efficient presentation \( P = (X|R, w) \) of a group G of cohomological dimension 2 and a word u representing an element of infinite order such that \( [u, w] = 1 \) modulo R. Thus we would have an identity of relations

\[ uu^{-1}w^{-1}\prod_{i=1}^{n}r_i^j = 1, \quad f_i \text{ words in } X^{\pm 1}, \quad e_i \in \{\pm 1\}, \quad r_i \in R, \]

which yields a non-trivial spherical element over P since u is not trivial. So P is an efficient non-aspherical presentation of a group G of cohomological dimension 2. In view of the above Proposition, G could not be FL!

Of course the group \( S(K, Z_n \ast Z_n, l) \) contains a torsion-free subgroup of finite index of cohomological dimension 2. We conclude by remarking that a subgroup of finite index of an efficient group need not be efficient. It was shown in [15] that a finite group can be embedded into a finite efficient group. Since there are non-efficient finite groups (Swan's examples for instance), finite index does not preserve efficiency.

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