ON THE DEHN COMPLEX OF VIRTUAL LINKS

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Abstract. A virtual link comes with a variety of link complements. This article is concerned with the Dehn space, a pseudo manifold with boundary, and the Dehn complex, a 2-dimensional spine of the Dehn space. In the classical case where the link is planar, the Dehn space is the link complement in the 3-sphere. We study topological and geometric properties of the Dehn complex of a virtual link. Among other things, we show that every finitely presented group is the fundamental group of a Dehn complex, and that any alternating triple of an alternating virtual link is a non-positively curved squared complex.

Keywords: Virtual link, Wirtinger complex, Dehn complex, non-positively squared complex, finitely presented group.

Mathematical Subject Classification 2010: 57M20, 57M05, 20F05

1. Introduction

This paper presents results concerning topological and geometric aspects of the Dehn complex of virtual links. Virtual knots and links are an extension of classical knot and link theory and were first introduced by Kauffman [6]. A virtual link diagram ℓ is a 4-regular graph in the plane, with over-crossing and under-crossing information at some nodes. The nodes with this additional information are crossings, and the remaining nodes are called virtual crossings. The latter are indicated by a circle around the node.

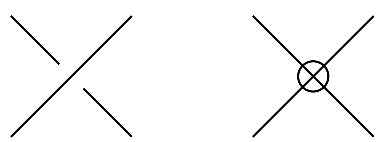

Figure 1. A usual crossing and a virtual crossing.

From a virtual link diagram in the plane, we construct a closed, orientable surface F of minimal genus in which ℓ embeds, so that only crossings appear as nodes. The virtual crossings disappear because we can run the edges of the graph over handles. The details of this construction are as follows. Start with the graph and embed it in $\mathbb{R}^3$ so that the virtual crossings disappear. We obtain a 2-manifold with boundary by thickening every edge of the graph into a band. For every boundary component, glue in a disc to obtain the desired surface. We call the surface thus obtained the projection surface F of ℓ (see Figure 2). The virtual link diagram in the plane thus leads to a link diagram on a surface.

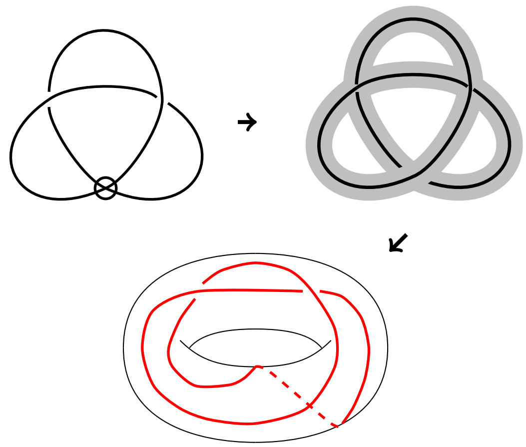

FIGURE 2. A virtual link diagram in the plane gives rise to a link diagram on a surface, a torus, in the case shown; the original image can be found in Kauffman [5].

There are various link complements associated with a virtual link. Given a virtual link diagram ℓ with projection surface F. We can push the diagram into the interior of the thickened surface $F \times I$, and thus obtain an embedding of linked circles in $F \times I$. We denote the image of this embedding by $\hat{\ell}$. If we remove an open neighborhood of $\hat{\ell}$ from $F \times I$, then we obtain a compact manifold with boundary which we denote by $M(\ell)$. If we cone off the bottom surface $F \times \{0\}$ of $M(\ell)$, we obtain a compact pseudo manifold with boundary, which we call the Wirtinger space, $WS(\ell)$. If we further cone off the top surface in the Wirtinger space, we obtain another compact pseudo manifold with boundary, which we call the Dehn space, $DS(\ell)$. In the classical situation where ℓ is a planar link diagram, $M(\ell)$ is a link complement in the thickened 2-sphere, $WS(\ell)$ is a link complement in the 3-ball, and $DS(\ell)$ is a link complement in the 3-sphere. The Wirtinger space and the Dehn space have interesting 2-dimensional spines, the Wirtinger complex $W(\ell)$ and the Dehn complex $D(\ell)$. The Dehn complex is the main object of interest for this article. We will recall the nature of these spines in the next section.

2. The Dehn Complex

Both the Wirtinger and the Dehn space, being pseudo 3-manifolds with boundary, have 2-dimensional spines. The procedure for collapsing the Wirtinger and the Dehn space to the Wirtinger complex $W(\ell)$ and the Dehn complex $D(\ell)$ can be found in Harlander/Rosebrock [4]. We recall the cell structures of these 2-complexes. We begin with the Wirtinger complex. First orient the virtual link diagram ℓ. Label the under-crossing arcs of ℓ by $a_1, ..., a_n$. The Wirtinger complex has a single vertex. Its edges are in one-to-one correspondence with the under-crossing arcs $a_1, ...a_n$. The 2-cells, also referred to as faces, are in one-to-one correspondence with the crossings of ℓ. Using the "right hand rule," the orientation of ℓ

yields an orientation of the boundary of the faces. See Figure 3. The Wirtinger complex provides a presentation of its fundamental group, the Wirtinger group. We have

$$\pi_1(W(\ell)) = \langle a_1, a_2, \ldots, a_n \big| R_1, \ldots, R_m \rangle$$

where every relation, R_i, is of the form $a_i a_j = a_j a_k$, corresponding to a crossing in the virtual knot diagram.

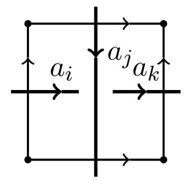

FIGURE 3. The Wirtinger relation $a_i a_j = a_j a_k$ obtained from a crossing in a virtual knot diagram.

We next give a description of the cell structure of the Dehn complex. There are two vertices, v_+ and v_- (which come from the cone points). The edges are in one-to-one correspondence with the connected components, $A_1, \ldots, A_n$, of $F - \ell$. Each edge A_i is oriented from v_+ to v_-. The faces are in one-to-one correspondence with the crossings of ℓ. Considering a particular crossing x of ℓ. Proceed counterclockwise around the crossing x, starting at the end of an over-crossing, and read off the four (not necessarily distinct) components encountered, say $A_{x(1)}$, $A_{x(2)}$, $A_{x(3)}$, and $A_{x(4)}$. A 2-cell is attached to the edge-path, $A_{x(1)}A_{x(2)}^{-1}A_{x(3)}A_{x(4)}^{-1}$ (see Figure 4). Without loss of generality, we may choose the tree of the 1-skeleton of $D(\ell)$ to be the edge A_1. If we collapse the tree to a point, choosing it to be our basepoint d_0, we obtain a presentation of the Dehn group,

$$\pi_1(D(\ell)) = \langle A_1, \ldots, A_n \big| A_1 = 1, A_{x(1)}A_{x(2)}^{-1}A_{x(3)}A_{x(4)}^{-1} = 1, \quad \text{for every crossing } x \in \ell \rangle.$$

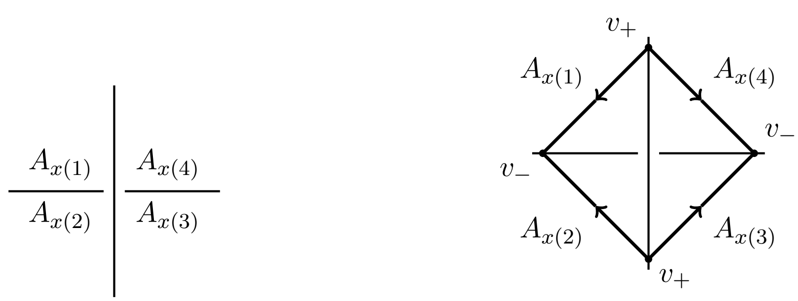

FIGURE 4. A crossing gives rise to a 2-cell in the Dehn complex.

If ℓ is a planar link diagram, then $H_1(D(\ell)) = \mathbb{Z}^m$, where m is the number of circle components of the link. In particular the Dehn group of a classical link is never trivial. The situation is different for virtual links.

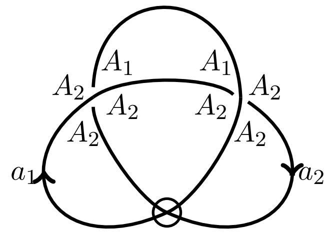

FIGURE 5. The connected components for the virtual knot diagram are labeled A_1 and A_2; under-crossing components are labeled a_1 and a_2.

Example 1. Consider the virtual knot diagram k in Figure 5. We have

$$\pi_1(D(k)) = \langle A_1, A_2 \big| A_1 = 1, A_2A_2^{-1}A_2A_1^{-1} = 1, A_2A_2^{-1}A_2A_1^{-1} = 1\rangle = 1.$$

Note also that

$$\pi_1(W(k)) = \langle a_1, a_2 \big| a_1a_1 = a_1a_2,\ a_2a_1 = a_1a_1\rangle = \mathbb{Z}.$$

A presentation for the Dehn group can also be obtained from the Wirtinger complex and the projection surface of the virtual link diagram ℓ. Consider the maps

$$F \times \{1\} \overset{\iota}{\hookrightarrow} WS(\ell) \overset{\phi}{\to} DS(\ell).$$

The first map is inclusion, the second map is a topological quotient map induced by coning off the top surface $F \times \{1\}$. It follows that $\phi_* : \pi_1(W(\ell)) \to \pi_1(D(\ell))$ is surjective and the kernel is normally generated by the image of ι_*. This gives another convenient way to construct the presentation of $\pi_1(D(\ell))$ which we describe next. Orient the virtual link ℓ. Consider an oriented curve c on the projection surface F. If c intersects an arc of ℓ labeled a_i in a positive way according to the right hand rule, we record an a_i. If c intersects in a negative way, we record an a_i^{-1}. In this way the curve c gives rise to a word $w(c)$ in the generators a_1,...,a_n of the Wirtinger presentation.

Theorem 2.1 (Byrd [2])**.** *The fundamental group of the Dehn complex $\pi_1(D(\ell))$ is isomorphic to the quotient $\pi_1(W(\ell))\big/K$, where K is normally generated by words $w(c)$, and the curves c range over a set of generating curves of the fundamental group of the projection surface F.*

Example 2. We next give an example of a virtual knot whose Dehn group is non-trivial finite. The Wirtinger group of the virtual knot diagram k in Figure 6 is presented by

$$\langle a_1, a_2, a_3 \big| a_2a_1 = a_1a_3,\ a_3a_2 = a_2a_1,\ a_2a_3 = a_3a_1\rangle,$$

and the Dehn group is presented by

$$\langle A_1, A_2, A_3 \big| A_1 = 1, A_1A_2^{-1}A_1A_3^{-1} = 1, A_1A_2^{-1}A_1A_3^{-1} = 1, A_1A_2^{-1}A_1A_2^{-1} = 1\rangle = \mathbb{Z}_2.$$

Harlander/Rosebrock [4] show that the Wirtinger group of this virtual knot has torsion. Note that the projection surface of k is a torus. The curve indicated in Figure 6 gives the relation $a_1 = a_3^{-1}$ in $\pi_1(D(k))$ under the surjective homomorphism φ_*. Adding the relation from the other generating curve, $a_1 = a_2^{-1}$, we can compute $\pi_1(D(k))$ using Theorem 2.1.

$$\pi_1(D(k)) = \pi_1(W(k)) \Big/ \{a_1 = a_3^{-1},\ a_1 = a_2^{-1}\} = \mathbb{Z}_2$$

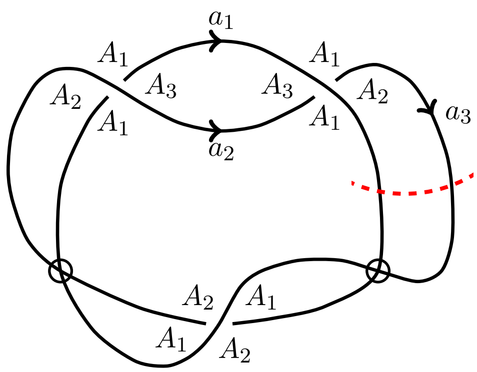

Figure 6. A virtual knot diagram k with Dehn group $\mathbb{Z}_2$. The dashed red line represents a generating curve on the projection surface.

3. Curvature

We begin with the definition of *reducing circle.*

Definition 3.1. Let ℓ be a virtual link diagram with a projection surface F. A simple, closed curve on F that intersects ℓ exactly twice is a reducing circle. When the reducing circle is separating, we require that crossings are contained on both sides of the separating reducing circle.

Separating reducing circles lead to decompositions of the link diagram. If we cut a virtual link diagram ℓ along a reducing circle and connect the four new endpoints with two new small arcs, we obtain two virtual link diagrams, ℓ_1 and ℓ_2, and ℓ is called the composition of ℓ_1 and ℓ_2, $\ell = \ell_1 \# \ell_2$. Cutting along a non-separating reducing circle also simplifies the virtual link diagram. It results in a virtual link diagram whose projection surface is of smaller genus.

Definition 3.2. A virtual link diagram is *prime* if there are no reducing circles, separating or non-separating.

A *squared 2-complex* is a 2-complex where every 2-cell has four boundary edges. The Dehn complex and the Wirtinger complex are examples of a squared 2-complex.

Definition 3.3. A squared 2-complex is *non-positively curved* if there are at least four squares grouped around each vertex. That is, if every reduced edge cycle in the link graph of each vertex in the 2-complex has length at least four.

The following result was obtained by Harlander and Rosebrock [3]. It extends well known results from classical knot theory (see Bridson/Haefliger [1], page 220).

Theorem 3.1. *Let ℓ be a prime, alternating virtual link diagram. Then the Dehn complex of ℓ is a non-postively curved squared complex.*

We outline the main ideas of the proof. One first notes that since ℓ is alternating, both vertex links in $D(\ell)$ are isomorphic to the 1-skeleton of the square cell decomposition of the projection surface that is dual to the one induced by ℓ. In particular, each reduces edge path in these links has even length. An edge path of length two would produce a reducing circle. Hence, since ℓ is prime, reduced edge paths in the links have to have length at least four. Thus $D(\ell)$ is a non-positively curved squared complex.

Example 3. The Dehn complex of the knot diagram on the torus in Figure 7 is a non-positively curved squared complex as there are no reducing circles.

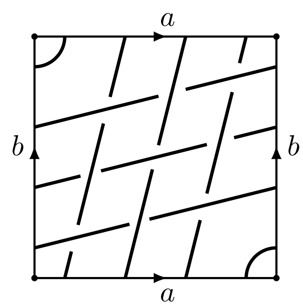

FIGURE 7. A prime alternating knot diagram on a torus.

Byrd [2] investigated the geometry of the Dehn complex for arbitrary alternating virtual links.

Theorem 3.2. [2] *If ℓ is an alternating virtual link diagram drawn on a projection surface F, then we can cut along a finite number of reducing circles to obtain a collection of prime, alternating links, $\ell_1, \cdots, \ell_n$. This gives a decomposition of the Dehn complex $D(\ell)$ into non-positively curved squared complexes, and a decomposition of the fundamental group of $D(\ell)$ into CAT(0) groups.*

The decomposition of the fundamental group is not a decomposition in the graph of groups sense. It means that the fundamental group of $D(\ell)$ is a quotient of a free product of CAT(0) groups and each reducing circle introduces a relation of a particular kind.

Example 4. The knot diagram in Figure 8, drawn on a torus, has one non-separating reducing circle shown in red. Hence its Dehn complex is not a non-positively curved squared complex. We may apply Theorem 3.2 to eliminate the reducing circle and obtain a prime knot diagram on a 2-sphere whose Dehn complex is a non-positively curved squared complex. The resulting prime knot is shown in Figure 9.

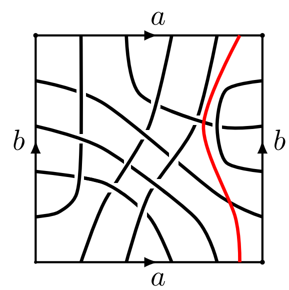

Figure 8. An alternating knot diagram on a torus with one non-separating reducing circle, indicated by the red curve.

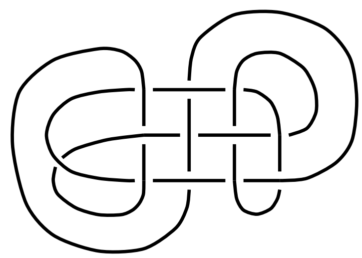

Figure 9. The prime knot obtained from the knot shown in Figure 8 by cutting along the red reducing circle.

Reducing circles can also be eliminated by adding additional link components to a virtual link diagram. One method is tripling a link. We triple a virtual link diagram ℓ by adding components which run parallel on either side of all the original strands of the link. Denote the tripled link by ℓ_T. An alternating example of a tripled virtual knot diagram is shown in Figure 10.

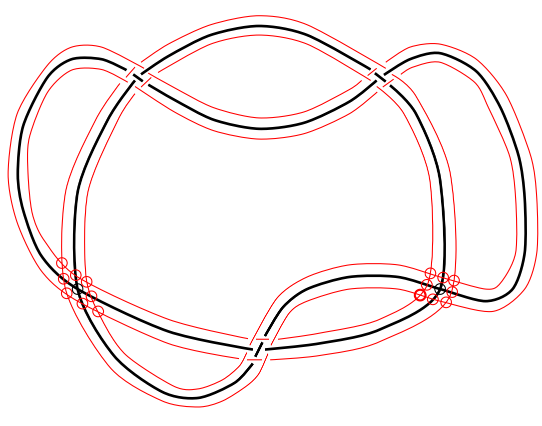

Figure 10. An alternating triple of an alternating virtual knot diagram.

Theorem 3.3. [2] *If ℓ_T is an alternating triple of any alternating virtual link diagram ℓ, then $D(\ell_T)$ is a non-positively curved squared complex.*

4. Dehn Groups

It is well known that the Dehn complex of a classical link is aspherical if the components are linked. In particular Dehn groups of classical links are non-trivial, do not contain torsion, and do not contain free abelian subgroups of rank higher than two. We have already given examples of Dehn groups of virtual links that can not occur as Dehn groups of classical links. In fact, restrictions of any kind do not exist for Dehn groups of virtual links.

Theorem 4.1. *Every finitely presented group is a Dehn group.*

Proof. Let G be a group and $\langle X|R\rangle$ be a finite presentation of G. If R contains m relations then consider a surface of genus m. Choose a relation $r = x_1^{\epsilon_1}...x_k^{\epsilon_k}$ from R. Choose a hole in the surface and place k circles y_1,...,y_k around that hole, y_1 being the outermost and y_k being the innermost circle. Orient the circle y_i clockwise if $\epsilon_i = 1$ and counterclockwise if $\epsilon_i = -1$. See Figure 11.

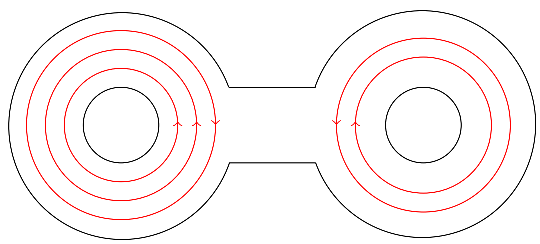

Figure 11. Oriented circles on a surface of genus 2.

If $x_i = x_j$, $i < j$, then we turn the circles y_i and y_j into one circle component using a handle. Figure 12 shows the case $\epsilon_i = 1$ and $\epsilon_j = -1$. The other cases are treated similarly. Relable this new component by x_i. Notice that the handle that was attached introduces new generators and relations in the Dehn group for the circle configuration (see Theorem 2.1). The two crossings on the handle coming from the blue under-crossing component (see Figure 12) yield the relation $ax_i = x_ib$ and its inverse. The meridian of the handle gives the trivial relation $x_i = x_i$, the longitude gives the relation $a = w$, where w is the word obtained from the circles that run through the handle. We can apply Tietze transformations to remove the generator a and the relation $a = w$. The other relation turns into $wx_i = x_ib$. We can use a Tietze transformation to remove the generator b and the relation $wx_i = x_ib$.

Thus the effect of attaching the handle on the Dehn group is changing the relation $y_1^{\epsilon_1}...y_i^{\epsilon_i}...y_j^{\epsilon_j}...y_k^{\epsilon_k}$ into the relation $y_1^{\epsilon_1}...x_i^{\epsilon_i}...x_i^{\epsilon_j}...y_k^{\epsilon_k}$. In the end the relation $y_1^{\epsilon_1}...y_k^{\epsilon_k}$ that held in the Dehn group of the original circle configuration is turned into the relation

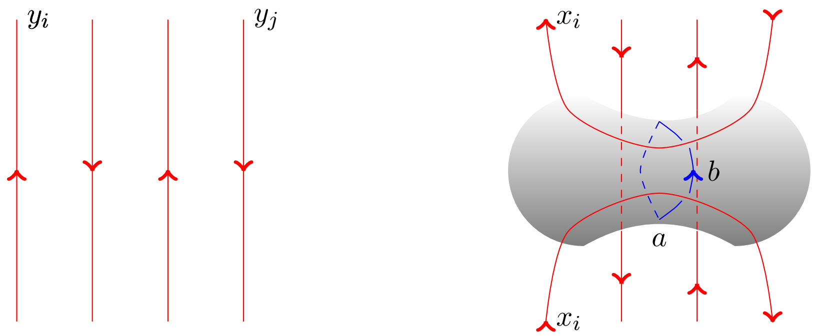

FIGURE 12. Connecting oppositely oriented components using a handle (seen from above).

$r = x_1^{\epsilon_1}...x_k^{\epsilon_k}$. The attached handles do not introduce additional relations. We proceed with this construction one relation at a time. Suppose we have relation $r = x_1^{\epsilon_1}...x_k^{\epsilon_k}$ and $s = z_1^{\kappa_1}...z_l^{\kappa_l}$ and $x_i = z_j$. Then we can connect the ith circle around the hole for r with the jth circle around the hole for s using the same handle construction. In the end we have produces a virtual knot drawn on a surface of genus greater or equal to m with desired Dehn group $\langle X|R\rangle$. □

Figure 13 shows a link on a torus with Dehn group the Klein bottle group $\langle x, y|xyx^{-1}y\rangle$. Note that the surface the virtual link is drawn on is not of minimal genus. One could remedy this by adding an under-crossing component along the meridian of the torus. It is not difficult to check that adding such an under-crossing component does not change the Dehn group. The argument is the same as given in the proof of Theorem 4.1, when we added a handle.

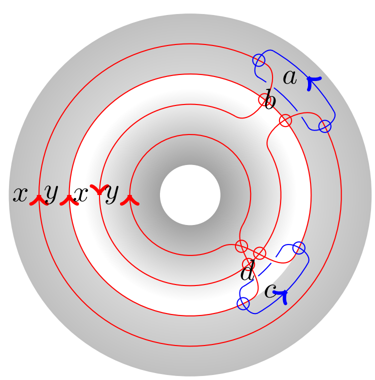

FIGURE 13. A virtual link on a torus; the projection surface has genus 3, the Dehn group is the Klein bottle group.

The general construction for realizing a given presentation as a Dehn presentation typically produces a virtual link on a surface of high genus with many components. Indeed, if we start with the presentation $\langle x_1, ..., x_n | r_1, ..., r_m\rangle$, then the virtual link is drawn on a surface of genus greater or equal to m, and the link has greater or equal to n components. Sometimes ad hoc constructions yield more efficient virtual links that realize a given group as a Dehn group.

Theorem 4.2. *Every cyclic group $\mathbb{Z}_n$ is a Dehn group of a virtual knot with the torus as its projection surface.*

Proof of Theorem 4.2. The trivial group (Example 1), $\mathbb{Z}_2$ (Example 2), and the infinite cyclic group (the unknot) have already been shown to be Dehn groups of virtual links. For any n, consider the knot diagram k drawn on a torus as in Figure 14. By Theorem 2.1, the Dehn group is

$$\pi_1(D(k)) = \pi_1(W(k)) \Big/ \{b_{(n-1)(n-2)}a^{n-1} = ab_{n-2}\cdots b_{(n-1)(n-2)} = 1\}$$

In $\pi_1(W(k))$, we have the relation $a^2 = ab_1$, or $a = b_1$. Continuing along this component (starting at b_1), we see that the Wirtinger relations give $a = b_1 = b_2 = \cdots = b_{n(n-2)}$. Hence,

$$\pi_1(D(k)) = \langle a |\, a^n = 1\rangle = \mathbb{Z}_n.$$

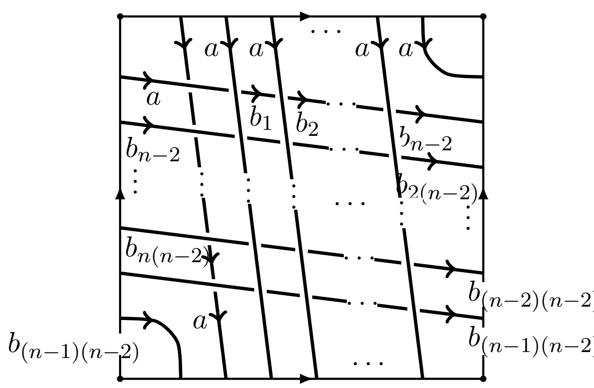

FIGURE 14. A knot diagram k on a torus with the Dehn group $\mathbb{Z}_n$; the knot runs around the meridian and longitude n times.

□

Theorem 4.3. *Every direct sum of two finite cyclic groups $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is the Dehn group of a virtual link with two components with genus 2 projection surface.*

Proof. We can produce two knots k_1 and k_2 on tori with Dehn groups $\mathbb{Z}_m = \langle a_1 |\, a_1^m = 1\rangle$ and $\mathbb{Z}_n = \langle a_2 |\, a_2^n = 1\rangle$, respectively, using Theorem 4.2. From these two knot diagrams, we construct a link diagram on a genus 2 surface with Dehn group $\mathbb{Z}_m \oplus \mathbb{Z}_n$. Connect the

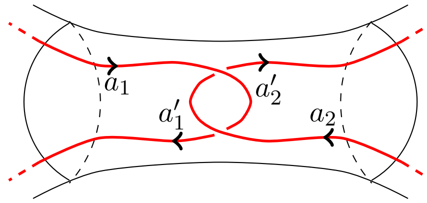

FIGURE 15. The handle with the linked components that connect the two tori.

two tori by a handle. Pull a strand of k_1 and a strand of k_2 onto the handle and link the two strands (see Figure 15). Call the new link diagram ℓ.

If we include the relations from the new crossing and the one coming from curves around the new handle, we obtain as the Dehn group of ℓ

$$\begin{aligned}\pi_1(D(\ell)) &= \left\langle a_1, a_1', a_2, a_2' \middle|\ a_1^m = a_2^n = 1, a_2a_1 = a_1a_2', a_1a_2 = a_2a_1', a_1 = a_1', a_2 = a_2' \right\rangle \\ &= \left\langle a_1, a_2 \middle|\ a_1^m = a_2^n = [a_1, a_2] = 1 \right\rangle.\end{aligned}$$

□

Theorem 4.4. *Every direct sum of a finite cyclic group and an infinite cyclic group, $\mathbb{Z}_n \oplus \mathbb{Z}$, can be expressed as a Dehn group of a virtual link.*

Proof. Consider a knot diagram on a torus as in Theorem 4.2 with a Dehn group $\mathbb{Z}_n$ and the unknot drown on a sphere. We can use the same construction as in Theorem 4.2. Note that the connected sum of a torus with a sphere is a torus, we end up with the desired 2-component link on a torus. □

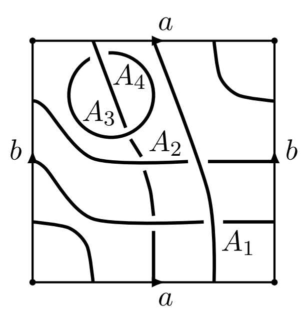

FIGURE 16. A 2-component link on a torus with the Dehn group $\mathbb{Z}_3 \oplus \mathbb{Z}$.

Figure 16 shows a 2-component link ℓ on a torus with Dehn group $\mathbb{Z}_3 \oplus \mathbb{Z}$. We can also compute the Dehn group of ℓ directly from the Dehn complex. We obtain

$$\begin{aligned}\pi_1(D(\ell)) &= \langle A_1, A_2, A_3, A_4 \big| A_1 = 1, A_2^n = 1, A_1A_3^{-1}A_4A_2^{-1} = 1, A_1A_2^{-1}A_4A_3^{-1} = 1\rangle \\ &= \mathbb{Z}_3 \oplus \mathbb{Z}\end{aligned}$$

Two simply linked circles provides an obviously example of a knot diagram with Dehn group $\mathbb{Z} \oplus \mathbb{Z}$. We have now shown that every cyclic group is a Dehn group of a virtual knot and every abelian group that is generated by two elements is the Dehn group of a virtual 2-component link.

References

[1] M. R. Bridson and A. Haefliger. *Metric Spaces of Non-Positive Curvature*. Springer, Berlin, 1999.
[2] R. E. Byrd. On the geometry of virtual knots. Master's thesis, Boise State University, 2012.
[3] J. Harlander and S. Rosebrock. Generalized knot complements and some aspherical ribbon disc complements. *Journal of Knot Theory and Its Ramifications*, 12(7):947–962, 2003.
[4] J. Harlander and S. Rosebrock. On distinguishing virtual knot groups from knot groups. *J. of Knot Theory and Its Ramifications*, 19(5):695–704, 2010.
[5] L. H. Kauffman. Introduction to virtual knot theory. arXiv:1101.0665v2 [math.GT], 2012. Submitted.
[6] L.H. Kauffman. Virtual knot theory. *Europ. J. Combinatorics*, 20:663–691, 1999.

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