Labeled Oriented Trees That Are Not Diagrammatically Reducible

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1. Introduction

Don't worry about that now. We'll write the introduction last.

2. First examples: the doubling construction

Here we discuss Rosebrock's doubling construction. We also give the first prime example found by computer search. I might write this section.

3. Some general results

**Theorem 3.1.** A reduced labeled oriented interval with less or equal to four vertices is DR. There does exist a reduced labeled oriented interval with five vertices.

**Remark.** One can produce other labeled oriented intervals that are not DR by applying a permutation to the vertex and edge labels. That's too easy.

The labeled oriented interval ant5b (see Figure ??) is also not DR. Note that it is not reduced as a LOI.

**Theorem 3.2.** If the labeled oriented interval displays certain features, then it is not DR.

This is the orange construction that Rachael talked about. Try to make this as general as possible. Use it to come up with new diagrams for Rosebrock's first prime example. Also note that the lois that have the feature all have complexity two and hence are aspherical.

Are thr groups infinite cyclic?

For this paper, the term “peel” denotes the substructure of a LOI with, for some edge $a$ in the LOI, $\cdots a \cdots a^{-1} \cdots$, or $\cdots a^{-1} \cdots a \cdots$, appears in the LOI. For example:

![Diagram](image)

There can, and in Dr. Rosebrock's LOIs there typically are, multiple peels in one LOI. For this construction, peels will be nested. As an example, add one edge $c$ and its inverse $c^{-1}$ to the previous example, as shown below.
Definition 1. In this paper, any LOI is of degree \( p \) provided the LOI contains \( p \) peels.

Notation: \((a, b, c)\) refers to a tile with the labels \( a \), \( b \), and \( c \) in any order and orientation. This shorthand is useful when examining possible diagram reductions, as two tiles must have the same edge labels to be a reducing pair.

Theorem (Keller’s Construction) 1. For any \( m \in \mathbb{N} \), where \( m \) is the degree of the LOI, if the LOI is of one of the following form, then there exists a reduced spherical diagram over the LOI.

Further, fixing the edge labels of either case, if the \([1, 2]\) and \([n - 1, n]\) edges are oriented opposite the \([k, k + 1]\) edge, then there exists a reduced spherical diagram over the LOI.

Proof. We need only prove [i] or [ii], since the two LOIs are isomorphic by the permutation, letting \( n = 2m + 4 \), \((1 \ n \ 2)(n - 1)(3 \ n - 2)\cdots(n/2 \ (n/2) + 1)\).
Thus, without loss of generality, the proof will be on [i]. Fix some \( m \in \mathbb{N} \).
Notice that the only area where we have reductions will be in the bottom three oranges. The topmost orange connects \((7, 1, 8)\) with \((2, 2m + 4, 1)\) of its inverse orange, the third orange, so we certainly cannot have a reduction there. The only reductions will be the second and fourth oranges reducing from the third orange.

The \(2m + 4\) edge that connects the second orange to the third connects a reducing pair, namely the \((2m + 3, 1, 2m + 4)\) tiles. Reducing the pair connects the reducing pair of the \((2m + 2, 1, 2m + 3)\) tiles via the \(2m + 3\) edge, and so on.

We reduce the original \(2m + 4\) edge \(m\) times in the above fashion, obtaining \((2m + 4) - (m) = m + 4\). However, whereas the third orange goes through its \(m\) peels, the second orange reduced along \(m - 1\) of its peels and its \((7, 1, 8)\) tile. As a result, the \(m + 4\) edge that connects the second and third oranges links the second orange's \((m + 3, 1, m + 4)\) tile to the third orange's \((m + 3, m + 4, m + 5)\) tile. Thus, we cannot have any more reductions from there.
We follow the same procedure for the third and fourth oranges, reducing along the $2m + 4$ edge $m$ times.
Checking over the diagram, we see that there are no more reductions. All new connections after reductions do not form reducing pairs, as on both sides of the middle $m + 4$ edge of the second orange, the adjacent tiles connect the $k$th tile with the $(2m + 3) - (k - 1)$ tile, where we enumerate the LOI’s squares by the following method: $[1, 2]$ is number 1, $[2, 3]$ is number 2, ..., $[2m + 3, 2m + 4]$ is number $2m + 3$.

There are no further reductions, and so the spherical diagram is reduced. □
Theorem 3.3. Let $P$ be a labeled oriented tree that contains a labeled oriented interval $Q$ that is a knot. If $P/Q$ is not DR, then $P$ is not DR.

Proof. Let be a branch of $P$. Insert $Q$ at index $k$ so that the branch now looks like 

And renumber the branch as follows

Because $P$ is not DR, it admits a reduced spherical diagram, call it $S$. To obtain a diagram for $P+Q$, anytime $S$ crosses from $k$ to $k+1$, insert the knot as shown:

This new diagram is reduced, for suppose the new diagram were reducible. $S$ is reduced and so the reduction must occur at some point on the boundary of the inserted knot or within the knot itself. But there can be no cancellations within the knot for each crossing is only used once. Then the cancellation must occur on the boundary of the knot. Then there are two possibilities the first are cancelling crossings of the form:

but then $S$ has crossings of the same labels, contradicting that $S$ was reduced.
The second case is a cancelling pair of the form
But then according to our construction the original diagram had the crossings

which are cancelling crossings contradicting that S was reduced.

Example 1. Examples of this construction can be seen in the first two examples of [?]. The knots are circled in the new diagram and the knot subLOI occurs between edges 5 and 8.
Remark. We can weaken the condition that $Q$ is a knot. All we need is that the boundary vertices of $Q$, $a$ and $b$, say, define the same element of $G(Q)$. Then we have a disc diagram over $Q$ with boundary $ab^{-1}$ and we can insert it when needed instead of inserting the knot.

**Theorem 3.4.** There exists no reduced non-DR LOT on 4 vertices.

**Proof.** Let $P$ be a LOT. If $P$ is not a LOI, then there are three boundary vertices and three edges. Hence since $P$ is boundary reduced each boundary label appears once and only once. Then $P$ is injective and prime, hence DR. Finally suppose $P$ is a LOI. Now suppose $P$ is a reduced non-DR LOI. $P$ must be non-injective and boundary reduced hence the two endpoints of the LOI must be the only edge labels with one endpoint occurring twice as an edge label. In addition, for $P$ to be compressed the edge labels occurring twice must be adjacent and since they are adjacent, the edge labels must be pointing in the same direction. Hence the only options up to permutations are:
the other four possibilities can be obtained from the permutation $(1\ 4)(2\ 3)$. It is easy to see that in all of these cases either the sink or the source part of the link is cycle free, hence all of these LOIs are DR.

4. Labeled oriented intervals with cyclic group

Suppose $P$ is a labeled oriented interval whose underlying group is cyclic. Then, for every pair of vertices $a$ and $b$, there is a reduced disc diagram with boundary $ab^{-1}$. We call such disc diagrams oranges. The present of oranges makes it easier to build reduced spherical diagrams. We illustrate the use of oranges in the following constructions.

Consider the labeled oriented intervals `seagul6a` and `seagul6b`. See Figure ??.

The second drawing (from the top) of Figure ?? is an orange that expresses the fact that the vertices $1$ and $5$ define the same element in the group of `seagul6a`. That orange is then used to construct a reduced spherical diagram for `seagul6a`, shown in the first drawing of Figure ??.

The third drawing (from the top) of Figure ?? is an orange that expresses the fact that the vertices $1 = 6$ in the group of `seagul6b`. That orange is then used to construct a reduced spherical diagram for `seagul6b`, shown in the fourth drawing of Figure ??.
Note that there are DR labeled oriented intervals with cyclic group. Consider the LOI bat3 shown below.
It is easily checked that the group of bat3 is infinite cyclic. Note that the positive part of the link $L^+$ is a tree, hence bat3 is DR.

5. Examples of labeled oriented intervals that are not DR

5.1. 5a. The LOI for 5a
Let us denote the labeled interval ant5 by $P$.

- $G(P) = \mathbb{Z}$
- $P$ is prime
- $P$ is of the smallest possible LOIs
5.2. 5b.

Let us denote the labeled interval ant5b by $P$. Note that $P$ is prime.
Figure ?? shows a reduced spherical diagram for $P$. It is the smallest reduced non-DR LOI.

- $G(P) = \mathbb{Z}$
- $P$ is prime
- $P$ is of the smallest possible LOIs
- $P$ is not reduced as the edge from 3 to 4 has edge label 4
Let us denote the labeled interval $6c$ by $P$.

- $G(P) = \mathbb{Z}$
- $P$ is prime
- We can construct another spherical diagram using construction 1

5.4. 6c.

Let us denote the labeled interval $6c$ by $P$. 

• $G(P) = \mathbb{Z}$
• $P$ is prime

5.5. 7b. Let us denote the labeled interval 7b by $P$. 
- $G(P) = \mathbb{Z}$
- $P$ is prime
- The spherical diagram has only one component

5.6. $7b_2$. 

1 2 3 4 5 6 7
Let us denote the labeled interval $7_{b,2}$ by $P$.

- $G(P) = \mathbb{Z}$
- $P$ is prime

5.7. **ex8.** Let us denote the labeled interval $ex8$ by $P$. 
5.8. erz8bsp. In this section we discuss the non-DR LOI erz8bsp. Let us refer to erz8bsp as $P$

Path description of the spherical picture:

Let us explain how to obtain the spherical picture from the path description:
Figure 1. The LOI $P/Q$

$P$ contains a sub-LOI $Q$ that starts at 4 and ends at 7. Note that $Q$ is a LOI for the trefoil knot. Furthermore, the quotient $P/Q$ is a non-DR prime LOI on five vertices. This follows from Theorem 3.2. The fact that $P$ is not DR now follows from Theorem 3.3.
5.9. **erz8k-2.**

Let us denote the labeled oriented interval 8k-2 by $P$. Note that $P$ is prime. At this point we have no general result that would explain why $P$ is not DR.
5.10. ex9.

Let us denote the labeled interval ex9 by $P$.

5.11. erz10a1.

Let us denote the labeled
oriented interval erz10a1 by $P$. Note that $P$ is prime. We can not say much about this one.

5.12. *erz10a2.*

Let us denote the labeled oriented interval *erz10a2* by $P$. Note that $P$ is prime. We can not say much about this one. The situation is very similar to *erz10a1*. 
Let us denote the labeled oriented interval $erz10z$ by $P$. Note that $P$ is prime. We can
not say much about this one.
Let us denote the labeled interval 10a4 by $P$. 
5.15. 10a5.

Let us denote the labeled interval 10a5 by $P$.

5.16. 10c1.

Let us denote the labeled interval 10c1 by $P$. 
5.17. 10g.
Let us denote the labeled interval 10g by $P$. 

\[5.18. \quad 10s.\]

Let us denote the labeled interval 10g by $P$. 

\[\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}\]
Let us denote the labeled interval 10z by $P$. 
Let us denote the labeled interval 7d by $P$.

5.21. erz11g.

Let us denote the labeled oriented interval erz11g by $P$. Note that $P$ contains a sub-LOI on vertices 4 to 7 which gives the trefoil knot. It follows from Mingjia’s result that the quotient $P/Q$ is not $D$. Thus, by Theorem 3.3, $P$ is not DR.
Let us denote the labeled oriented interval erzi by $P$. Note that $P$ contains a sub-LOI on vertices 4 to 9 which is a twisted version of the trefoil knot. It follows
from Rachel's result that the quotient $P/Q$ is not $DR$. Thus, by Theorem 3.3, $P$ is not DR.

5.23. erz11j.

Let us denote the labeled oriented interval erz11j by $P$. Note that $P$ contains a sub-LOI on vertices 3 to 6 which gives the trefoil knot. The quotient $P/Q$ is prime, but non-DR does not seem to follow from Rachel's or Mingjia's results. If we cut the trefoil knot out of the given reduced spherical diagram, we obtain a reduced spherical diagram for $P/Q$. Thus $P/Q$ is a prime LOI that is not DR.
Let us denote the labeled oriented interval erz11k by $P$. Note that $P$ contains a sub-LOI on vertices 1 to 4 which gives the trefoil knot. It follows from Rachel’s result that the quotient $P/Q$ is not $D$. Thus, by Theorem 3.3, $P$ is not DR.
Let us denote the labeled oriented interval erz111 by $P$. Note that $P$ contains a sub-LOI on vertices 3 to 6 which gives the trefoil knot. It follows from Rachel’s result that the quotient $P/Q$ is not $D$. Thus, by Theorem 3.3, $P$ is not DR.
Let us denote the labeled interval 11m by $P$. 

5.26. 

Let us denote the labeled interval 11n by $P$. 

5.27.
Let us denote the labeled interval \( 11n \) by \( P \).

Let us denote the labeled oriented interval \( \text{erz13b} \) by \( P \). Note that \( P \) contains a various sub-LOI's, for example vertices 6 to 11 which gives the trefoil knot with a twist. It follows from Mingjia's result that the quotient \( P/Q \) is not \( DR \). Thus, by Theorem 3.3, \( P \) is not DR.
5.29. LOTSj1.

Let us denote the labeled oriented tree LOTSj1 by $P$. Note that $P$ is a tree and not
an interval. It is prime. We do not have much to say about this one yet. The group is cyclic.