### Introduction

It is difficult to determine whether a group given by a finite presentation is finite or infinite. We study this problem in the context of knot groups and more generally labeled orientable tree (LOT) groups. More specifically, we are looking at factor groups of knot and LOT groups by powers of meridians. This is in the spirit of Coxeter’s work on the factor groups of braid groups. Ultimately we hope our findings will generalize Coxeter’s work from the three-strand braid groups to knot groups.

Let \( P = \langle x_1, \ldots, x_n \mid r_1 = 1, \ldots, r_m = 1 \rangle \) be a presentation of a group \( G \). Every relation \( r_i = 1 \) gives rise to a tile. Suppose \( r_i = y_1 \ldots y_l \in \{x_1^{\pm 1}, \ldots, x_n^{\pm 1} \} \). Draw a convex \( k \)-gon and fix a vertex on its boundary. Now read around the \( k \)-gon in clockwise direction. If \( y_l = x_k \), then label the \( l \)-th edge by \( x_k \), and orient the edge clockwise if \( \epsilon = 1 \) and anticlockwise if \( \epsilon = -1 \).

### Example 1

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### Example 2

If a tile and its mirror image lie next to each other in a spherical diagram, then we can perform a mirror image cancellation.

### Theorem 1 [4]

Let \( P \) be a presentation of a group \( G \). If no reduced spherical diagram exists over \( P \) and \( G \) is not finite cyclic, then \( G \) is infinite.

### Definition 2

A Label Oriented Tree (LOT) is a cycle-free oriented connected graph with edge labeling. If \( x_1, \ldots, x_n \) are the vertices, then each edge is labeled by some \( x_i \). From a LOT \( T \) we read off a presentation \( P(T) = \langle x_1, \ldots, x_n \mid r_1 = 1 \rangle \).

### Example 3

The LOT shown in Example 4 defines the presentation \( P = \langle x_1, x_2, x_3 \mid x_1x_2x_1^{-1}x_2x_1x_2, x_2x_1x_2x_1^{-1}x_2x_1 \rangle \).

### Definition 3

Given a LOT \( T \). Let \( Q(T) \) be the group defined by \( P(T) \). The collection of all \( Q(T) \) is referred to as the \( Q \) series of \( T \).

### Example 4

The long trefoil knot gives rise to the following labeled oriented tree \( T \):

\[
\begin{align*}
& x_1, x_2, x_3, x_4 \\
& \-x_1 \quad x_2 \quad x_1 \quad x_3 \quad x_2 \quad x_4
\end{align*}
\]

### Theorem 2

The \( Q \) series of a Knot Presentation is a Knot Invariant.

### Definition 4

A presentation \( P \) is called a non-positively curved square presentation if all relators \( r \) have length four and in every reduced tiling of a surface with the tiles coming from \( P \) we need at least four tiles around every vertex.

### Results

**Theorem 3**

Suppose \( T \) is a LOT where \( P(T) \) is a non-positively curved square presentation. It follows that, for \( n \geq 4 \), \( Q_n(T) \) is infinite.

### Sketch of proof of Theorem 3

First consider the presentation \( P(T) = \langle x_1, \ldots, x_n \mid r_1 = 1 \rangle \). We have the following spherical diagrams over \( P(T) \):

Using combinatorial curvature arguments, we show the following: A spherical diagram over \( P(T) \) either admits a mirror image cancellation or an \( m_n \)-rewriting that reduces the number of tiles in the diagram (see the figures below Theorem 2). This shows that \( \pi_1(T) \) is generated by the dipoles \( d_i \) and the \( m_n \). Using topological arguments we conclude that \( \pi_1(T) \) is generated by the dipoles \( d_i \).

Our result follows from a more general version of Huebschmann’s Theorem 1.

### Future Work

- It is known (see Bridson and Haefliger [2], page 220, and the references therein) that the Dehn complex of an alternating prime knot is a non-positively curved square complex. We think that the proof of our main theorem can be adapted to show that for every alternating prime knot there exists a number \( N \geq 0 \) so that \( Q_n(T) \) is infinite for \( n \geq N \).
- The \( Q \)-series of the trefoil knot is particularly interesting. We intend to investigate the \( Q \)-series for all torus knots.
- Which finite group can arise as \( Q_n \) of some LOT? Note that every \( Q_n \) has a balanced presentation and this should put some constraints on the types of finite groups that can occur. It is unknown if there are finite groups that require at least four generators that can have balanced presentations. See Johnson [5], Chapter 7 on finite groups with few relations.

### References and Acknowledgements