Factor Groups of Knot and LOT Groups

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Introduction

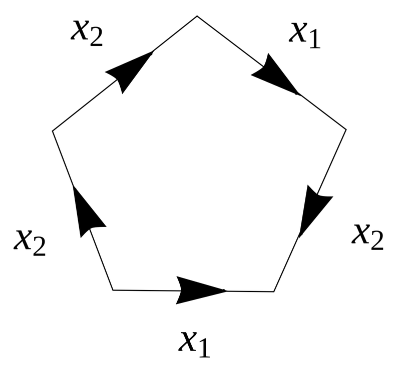
It is difficult to determine whether a group given by a finite presentation is finite or infinite. We study this problem in the context of knot groups and more generally label oriented tree (LOT) groups. More specifically, we are looking at factor groups of knot and LOT groups by powers of meridians. This is in the spirit of Coxeter's work on the factor groups of braid groups. Ultimately we hope our findings will generalize Coxeter's work from the three-strand braid groups to knot groups.

Let $P =< x_1, \ldots, x_n | r_1 = 1, ..., r_m = 1 >$ be a presentation of a group G. Every relation $r_i = 1$ gives rise to a tile. Suppose $r_i = y_1 \ldots y_k, y_j \in \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$. Draw a convex k-gon and fix a vertex on its boundary. Now read around the k-gon in clockwise direction. If $y_l = x_p^{\epsilon}$ then label the lth edge by x_p, and orient the edge clockwise if $\epsilon = 1$ and anticlockwise if $\epsilon = -1$.

Example 1

$$r = x_1 x_2 x_1^{-1} x_2 x_2$$

The above relation gives rise to the following tile:

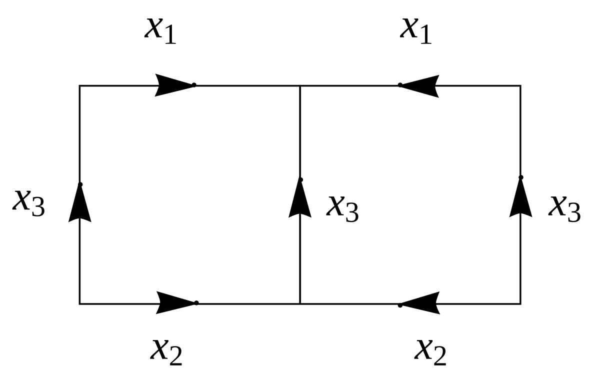


Definition 1

A Reduced Spherical Diagram over P is a tiling of the 2-sphere with the tiles from P and their inverses, so that no tile lies next to its inverse.

Example 2

If a tile and its mirror image lie next to each other in a spherical diagram, then we can perform a mirror image cancellation.



Tilings can be used to show that a group defined by a given presentation is infinite. The following result was obtained by J. Huebschmann in 1979. See also the survey [1] by Bogley and Pride on calculating gernerators of π_2.

Theorem 1 [4]

Let P be a presentation of a group G. If no reduced spherical diagram exists over P and G is not finite cyclic, then G is infinite.

Definition 2

A Label Oriented Tree (LOT) is a cycle-free oriented connected graph with edge labeling. If $x_1, \ldots, x_n$ are the vertices, then each edge is labeled by some x_i. From a LOT T we read off a presentation

$$P(T) =< x_1, \ldots, x_n | \{r_e = 1\}_{e \in E(T)} > .$$

If e is an edge starting at x_i, ending at x_j, labeled by x_h, then $r_e = x_i x_h (x_h x_j)^{-1}$.

Example 3

The LOT shown in Example 4 defines the presentation

$$P =< x_1, x_2, x_3, x_4 \mid x_1 x_3 = x_3 x_2, x_3 x_1 = x_1 x_2, x_4 x_2 = x_2 x_3 > .$$

LOT presentations generalize Wirtinger presentations of long knot groups.

Definition 3

Given a LOT T. Let $Q_k(T)$ be the group defined by

$$P_k(T) =< x_1, \ldots, x_n | x_1^k = 1, \{r_e = 1\}_{e \in E(T)} >$$

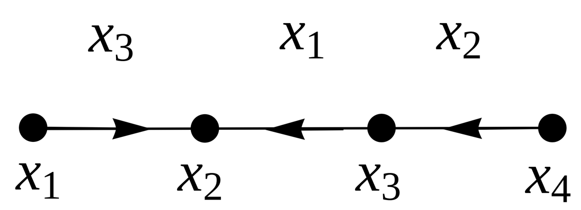
The collection of all $Q_k(T)$ is referred to as the Q series of T.

Objectives

- For which k is $Q_k(T)$ is finite or infinite?
- Which finite groups can occur as $Q_k(T)$ for some LOT T?

Example 4

The long trefoil knot gives rise to the following labeled oriented tree T



The Q series of T is:

- $Q_0 =$ Trefoil Group $= B_3$ (Braid Group on three stands)
- $Q_1 =$ Trivial Group
- $Q_2 = S_3$
- $Q_3 = SL(2, 3)$
- $Q_4 = SL(2, 3) \times C_4$
- $Q_5 = SL(2, 5) \times C_5$
- $Q_n =$ Infinite for $n \geq 6$

This result was obtained by Coxeter [3] in connection with his work on factor groups of braid groups.

Theorem 2

The Q Series of a Knot Presentation is a Knot Invariant.

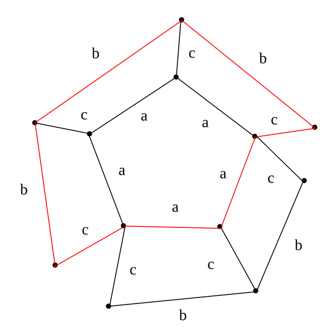


Figure : A Possible Local Tiling

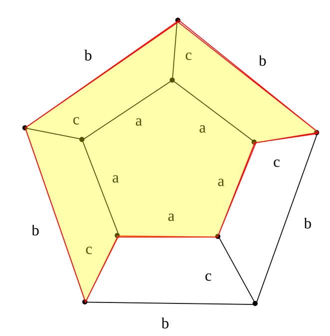


Figure : The m_e-Sphere Shares a Boundary

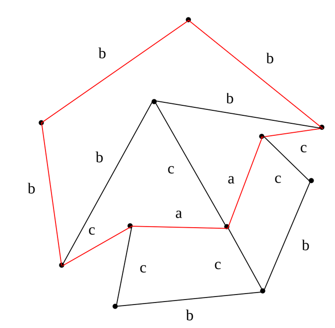


Figure : Insert the Other Side of the m_e-Sphere

Definition 4

A presentation P is called a non-positively curved square presentation if all relators r have length four and in every reduced tiling of a surface with the tiles coming from P we need at least four tiles around every vertex.

Results

Theorem 3

Suppose T is a LOT where $P(T)$ is a non-positively curved square presentation. It follows that, for $n \geq 4$, $Q_n(T)$ is infinite.

Sketch of proof of Theorem 3

First consider the presentation

$$\bar{P}_k(T) =< x_1, \ldots, x_n | x_1^k = 1, \ldots, x_n^k = 1, \{r_e = 1\}_{e \in E(T)} > .$$

We have the following spherical diagrams over $\bar{P}_k(T)$:

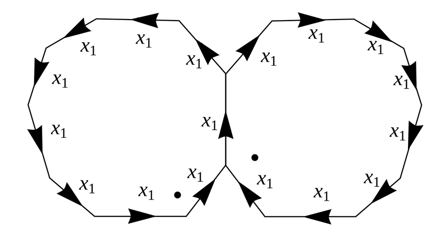


Figure : The dipole d_i

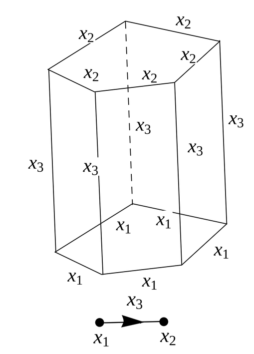


Figure : The sphere m_e, where e is an edge of T

Using combinatorial curvature arguments, we show the following: A spherical diagram over $\bar{P}_k$ either admits a mirror image cancellation or an $m_e -$ rewriting that reduces the number of tiles in the diagram (see the figures below Theorem 2). This shows that $\pi_2(\bar{P}_k)$ is generated by the dipoles d_i and the m_e. Using topological arguments we conclude that $\pi_2(P_k)$ is generated by the dipole d_1. Our result follows from a more general version of Huebschmann's Theorem 1.

Future Work

- It is known (see Bridson and Haefliger [2], page 220, and the references therein) that the Dehn complex of an alternating prime knot is a non-positively curved squared complex. We think that the proof of our main theorem can be adapted to show that for every alternating prime knot there exists a number $N \geq 0$ so that Q_n is infinite for $n \geq N$.
- The Q-series of the trefoil knot is particularly interesting. We intend to investigate the Q-series for all torus knots.
- Which finite group can arise as Q_n of some LOT? Note that every Q_n has a balanced presentation and this should put some constrains on the type of finite groups that can occur. It is unknown if there are finite groups that require at least four generators that can have balanced presentations. See Johnson [5], Chapter 7 on finite groups with few relations.

References and Acknowledgements

[1] W. A. Bogley, S. J. Pride, *Calculating generators of π_2*, in Two -dimensional Homotopy and Combinatorial Group Theory, ed. C. Hog-Angeloni, MetzlerW. Metzler, A. J. Sieradski, LMS Lecture Note Series 197, Cambridge University Press 1993.

[2] M. R. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Grundlehren der mathematischen Wissenschaften, Volume 319, Springer 1999.

[3] H.S.M. Coxeter, *Factor groups of the braid groups*, Proc. 4th Canad. Math. Congress, Banff, 1957 Toronto Univ. Press (1959), 95-122. See also "Kaleidoscopes, Selected Writings of H.S.M. Coxeter", ed. F.A. Sherk, P. McMullen, A.C. Thompson, A.I. Weiss, Canadian Math. Soc. Series of Monographs and Advanced Texts, Wiley 1995.

[4] J. Huebschmann, *Cohomology theory of aspherical groups and of small cancellation groups*, J. Pure Appl. Algebra 14 (1979), 137-143.

[5] D.L. Johnson, Presentations of Groups, LMS Student Texts 15, Cambridge University Press 1990. See chapter 7 Finite groups with few relations.