ZFC without extensionality interpreted in Zermelo

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version of February 7 2013: I'll keep track of versioning here, adding notes at the top each time I change it. Slight further remarks Feb 7 – credits. Further corrected a silliness about Pairing. Further slight correction re parameters when discussing invariance of formulas ϕ^* under application of permutations π_A.

This is my account of results due to Dana Scott. The proofs are mine and so are the errors. This is an informal set of working notes which I'll be expanding on.

We begin with a model of Zermelo set theory.

Our axioms are

Extensionality: Sets with the same elements are the same.

Pairing: For any x, y there is a set whose elements are exactly x and y.

Union: For any x, there is a set whose elements are exactly the elements of the elements of x.

Power Set: For any x, there is a set y such that z is an element of y exactly if each element of z is an element of x.

Infinity: We say a set x is an empty set if x has no elements. We say that a set x is a successor of y if x is the only element of y. We assert that there is a set which contains all empty sets and contains all successors of each of its elements.

Separation: For each condition $\phi(x)$ [which may have additional parameters] we can express in the language of set theory (not using the variable a) it is an axiom that for each set a there is a set b such that x belongs to b iff x belongs to a and $\phi(x)$.

Choice: For any pairwise disjoint collection P of nonempty sets, there is a set C which contains exactly one element of each element of P.

Notice that using extensionality we can see that there is just one empty set which we can call 0, for each set x a unique successor $\{x\}$, and a minimal set $\mathbb{N}$ which contains 0 and is closed under this successor operation by Infinity and an application of Separation. This is roughly speaking Zermelo's original formulation of set theory. Notice that it does not include Foundation.

Our first move is to swap out this theory for what we claim should be the canonical version of Zermelo set theory.

Extensionality: Sets with the same elements are the same.

Pairing: For any x, y there is a set whose elements are exactly x and y.

Union: For any x, there is a set whose elements are exactly the elements of the elements of x.

Power Set: For any x, there is a set y such that z is an element of y exactly if each element of z is an element of x.

Infinity: We say a set x is an empty set if x has no elements. We say that a set x is an increment of a set y by a set z if the elements of x are exactly z and the elements of y. We claim that there is a set I such that all empty sets are in I and all increments of elements of I by elements of I are in I.

Separation: For each condition $\phi(x)$ [which may have additional parameters] we can express in the language of set theory (not using the variable a) it is an axiom that for each set a there is a set b such that x belongs to b iff x belongs to a and $\phi(x)$.

Choice: For any pairwise disjoint collection P of nonempty sets, there is a set C which contains exactly one element of each element of P.

Rank: We say that a set h is a subhierarchy iff the subset relation on h is well-founded and for any $r \in h$ all elements of h minimal in inclusion among those which do not include r as a subset are power sets of r, and all unions of nonempty subsets of h are elements of h, and all minimal elements of h in inclusion are minimal witnesses to Infinity (i.e. V_ω). We define a rank as a set which belongs to some subhierarchy. We assert that every set is an element of some rank. [notice that we have defined subhierarchies to exclude finite ranks from consideration; we are doing this because this has a technical advantage below, not because we would normally do this in presenting this theory].

It is amusing to note that in the presence of the axiom of Rank, Pairing and Union become redundant. Infinity is redundant in this exact formulation because of the technical modification we made to Rank which we would not normally make. The intersection of all witnesses to Infinity is the collection of hereditarily finite sets.

We indicate how to interpret our formulation of Zermelo in the original formulation (and also in the usual modern formulation). The elements of the domain of the interpretation are well-founded extensional relations with top, where t is a top of a well-founded extensional relation R iff the smallest set which contains t and contains the preimage of each of its elements under R is the field of R. Notice that the empty relation has top (anything) and any other well-founded relation with top has a unique top (an element of its field). The equality of the interpretation is isomorphism. The component of a well-founded extensional relation determined by x in its field is the restriction of R to the smallest set containing x as an element and including all preimages of its elements under R as a subset. The membership relation of the interpretation holds (R is interpreted as an element of S) iff R is isomorphic to a component of S determined by an element of the preimage of the top of S under S. The interpretation thus described with domain the class of all well-founded extensional relations is an interpretation of Zermelo set theory, in which the stronger form of Infinity given above holds; we then restrict its domain to those elements which belong to some rank in the sense of the interpretation; this will still give an interpretation of Zermelo set theory in which the axiom of Rank will hold as well [actually I think Rank holds outright without any restriction].

We show how to interpret ZFC - Extensionality in this improved version of Zermelo set theory (and thus ultimately in the original set theory of Zermelo).

ZFC consists of the original theory of Zermelo (though Infinity is usually formulated differently) plus the axiom scheme of Replacement and the axiom of Foundation, which is a consequence of Rank, which we will show to hold by showing that Rank holds. NOTE: references to Scott's paper frequently say that his result is about ZFC - foundation; this is an accident of his formulation. His result holds with the theory with Foundation and in fact the models he discusses satisfy Foundation. He just doesnt include it in his axiom set.

The axiom scheme of Replacement asserts that for each condition $\phi(x, y)$ in the language of set theory [which may have additional parameters] it is an axiom that if ϕ is functional (for any x, y, z, $\phi(x, y) \wedge \phi(x, z) \rightarrow y = z$) then for any set A there is a set B having exactly those elements b such that $\phi(a, b)$ for some element a of A. ZFC minus extensionality is the theory with the axioms of Zermelo set theory without extensionality plus the scheme of replacement.

In Zermelo set theory with extensionality and rank, we define 0 as the unique empty set and for each x define $x+1$ as the unique singleton $\{x\}$ of x. Define $\mathbb{N}$ as the smallest set containing 0 and closed under successor. Define $\sigma(x)$ as $x+1$ if x is a natural number and x otherwise. For any set x, define x^- as $\{y \mid (\exists z \in x.y = \sigma(z)\}$, which is a set because it is a subset of $y \cup \mathbb{N}$, and define x^+ as $x^- \cup \{0\}$. Define $x \in^* y$ as $\sigma(x) \in y$. For any formula ϕ in the language of equality and membership, define ϕ^* as the formula resulting if $\in$ is replaced everywhere by $\in^*$ in ϕ.

Notice that for any x, y, $(\forall z.z \in^* y \leftrightarrow z \in x)$ is true if and only if $y = x^+$ or $y = x^-$.

A specific point to be made is that $z \in^* x^+ \leftrightarrow z \in^* x^-$, and of course $x^+ \neq x^-$, from which it follows that (the axiom of extensionality)* is false.

For any x and y, both $\{x, y\}^+ = \{0, \sigma(x), \sigma(y)\}$ and $\{x, y\}^- = \{\sigma(x), \sigma(y)\}$ witness the truth of (the axiom of pairing)* and clearly both of these sets exist in Zermelo set theory.

For any x, the set $\bigcup^* x$ defined as $\{y \mid (\exists z.y \in^* z \wedge z \in^* x\}$ exists by separation, as $y \in^* z \wedge z \in^* x$ implies that z belongs to $x \cup \mathbb{N}$ and y belongs either to $\mathbb{N}$ or to some such z, so $y \in \mathbb{N} \cup \bigcup x$ and $\bigcup^* x$ is a definable subcollection of this which exists by Separation. Now both of $(\bigcup^* x)^+$, $(\bigcup^* x)^-$ witness (the axiom of union)*.

Define $x \subseteq^* y$ as $(\forall z.z \in^* x \rightarrow z \in^* y)$. Notice that $x \subseteq^* y$ implies that $x \subseteq y \cup \mathbb{N}$. It follows that $\mathcal{P}^*(x) = \{y \mid y \subseteq^* x\}$ is a definable subcollection of $\mathcal{P}(x \cup \mathbb{N})$, and so exists by Separation, and both $\mathcal{P}^*(x)^+$ and $\mathcal{P}^*(x)^-$ witness (the axiom of power set)*.

For any formula ϕ, the objects such that $x \in^* A \wedge \phi^*$ all belong to $A \cup \mathbb{N}$, so $\{x \in A \cup \mathbb{N} \mid x \in^* A \wedge \phi^*\}^+$ and $\{x \in A \cup \mathbb{N} \mid x \in^* A \wedge \phi^*\}^-$ both witness $(\exists B.x \in B \leftrightarrow (x \in A \wedge \phi))^*$, so the starred versions of all instances of Separation are true.

(P is a disjoint collection of nonempty sets)* holds iff P is a collection of nonempty sets, none equal to $\{0\}$, any two of which have intersection either 0 or $\{0\}$. If C is a choice set from $\{p - \{0\} \mid p \in P\}$ (there will be such a set by Choice) then (C is a choice set from P)* holds as well, so (the axiom of choice)* is true.

If V_ω is the set of all hereditarily finite sets (definable as the intersection of all witnesses to Infinity as improved), it is straightforward to show that V_ω also witnesses (the axom of Infinity (improved)*.

If a set A (contains all natural numbers)*, that is, contains all natural numbers except possibly 0, then (B is a power set of A)* is equivalent to "B is the power set of $A \cup \{0\}$ or B is the collection of all nonempty subsets of $A \cup \{0\}$". Noting that no natural number belongs to a subhierarchy, we see that (h is a subhierarchy)* is equivalent to "h is the union of a subhierarchy h_0 and the collection of all $r - \{0\}$ for $r \in h_0$". It should then be evident that (the axiom of rank)* is true.

The shocking thing is that (Replacement)* is also true.

Informally, the reason this is true is that the only way we can uniquely describe an object is by force (by explicitly mentioning it in a formula) because every extension is witnessed by two objects which cannot be told apart. Formally we show this by considering a class of permutations. We want $\pi(x) = \pi(y)$ to be equivalent to $x = y$, which is easily effected by requiring that π be a permutation. We want $\pi(x) \in \pi(y)$ to hold iff $x \in y$, which is effected by having the action of π on a set y consist of having π act on each element (obtaining $\pi``y$) then possibly flipping to the other object with the same extension.

We show how to define these permutations. Let A be a set. We define $\pi_A(x)$ to be $\pi_A^*``(x\Delta\{0\})$ for $x \in A$ and $\pi_A^*``x$ for all other x, where $\sigma(x) = x + 1$ if x is a natural number and x otherwise, $\pi_A^*(0) = 0$, $\pi_A^*(\sigma(x)) = \sigma(\pi_A(x))$. For any set x, $\pi_A(x)$ is a set (proved by induction on rank (the point being that it doesn't raise rank very much)), and is a permutation [fill in details].

Now we see that $x = y \leftrightarrow \pi_A(x) = \pi_A(y)$, $x \in^* y \leftrightarrow \pi_A(x) \in^* \pi_A(y)$. It follows that uniform application of any π_A preserves the truth of any formula ϕ^* if it fixes all parameters in ϕ^*. So if $\phi(x, y)$ is functional*, its truth is preserved by any application of a π_A which fixes all parameters appearing in $\pi(x, y)$. If we have $\phi(x, y)^*$ we will have $\phi(\pi_A(x), \pi_A(y))^*$ for any permutation π_A fixing all parameters in ϕ. It then follows that any such permutation which fixes all the parameters and also fixes x will fix y. But this is only possible if y must in all cases be of rank not exceeding the maximum of the ranks of x and the parameters (any object is moved by some permutation of this class fixing all objects of lower rank), and this implies that the instance of Replacement* can be reduced to an instance of Separation* bounded in a suitable rank.

This is another instance of the general result that Collection is a better axiom than Replacement. From Collection we get a strengthening of Replacement which applies to any formula ϕ such that $\phi(x, y)$ and $\phi(x, z)$ implies not that $y = z$ but that y and z are coextensional, even in the absence of extensionality. ZFC + Collection - Extensionality can thus be shown to interpret full ZFC.