Notes on Neutral Geometry

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This document contains notes on chapter 3 in the book, Axioms of Plane Geometry.

Initially, we will be talking about neutral geometry, basically Euclidean geometry without any parallel postulate. The theory presented will be consistent with Euclidean geometry or hyperbolic geometry; it will need modifications to support elliptic geometry. The theory as given in the book depends on basic assumptions about sets and about real numbers given in Appendix E. We will try to also give an alternative approach using sets and real numbers less or not at all. There are a couple of reasons for this. One is historical: the ancient Greeks definitely did not explain geometry in terms of number (and definitely not in terms of sets), but rather explained number geometrically. The other has to do with the fact that an adequate understanding of the theory of real numbers is not included in the prerequisites of this course; this is the subject matter of Math 314.

Documents of this type are always filled with typos and errors. I will give points for finding them...

1 Undefined Notions

The undefined notions that the author lists are point, line, distance, half-plane, and angle measure.

We will use more undefined notions in our approach avoiding real numbers, which we will list here as we discover them in our development. For the moment I know that I am going to want incidence (of a point on a line or a line on a point), betweenness and congruence (of line segments) as basic notions (which I will use to avoid talking about real number distances). I will add more undefined notions to this section as I develop them.
2 A Reprise of Incidence Geometry

Axiom 3.1.1 (Existence Postulate): The collection of all points forms a nonempty set. There is more than one point in that set.

Definition 3.1.2 (Plane): The set of all points is called the plane and is denoted by \( \mathbb{P} \).

Set language appears immediately. I will not avoid set language completely, but I will use it less. My alternative axioms will have stars on them,

Axiom 3.1.1\(^\ast\): There are at least two points.

Much more economical. No mention of sets.

Axiom 3.1.3 (Incidence Postulate): Every line is a set of points. For every pair of distinct points \( A, B \) there is exactly one line \( L \) such that \( A \in L \) and \( B \in L \). We denote this line by \( \overleftrightarrow{AB} \): we may more often call it just line \( AB \) because the line notation is really inconvenient in LaTeX.

The notation \( \in \) is of course a primitive notion of set theory, not geometry.

Definition 3.1.4: If a point \( P \) is an element of a line \( L \), we say that \( P \) lies on \( L \), or \( P \) is incident on \( L \), or \( L \) is incident on \( P \). If a point \( P \) is not incident on a line \( L \), we say that \( P \) is an external point for \( L \).

In our parallel treatment, we take the incidence relation as primitive.

Axiom 3.1.3\(^\ast\): For any pair of distinct points \( A, B \), there is a unique line \( L \) such that \( A \) is incident on \( L \) and \( B \) is incident on \( L \). Further, for any line \( M \), there are at least two points \( C, D \) which are both incident on \( M \).

Notice that we put a little more information in our axiom (while taking out the references to sets): their axiom is basically incidence axiom 1 from the last chapter, while ours also includes incidence axiom 2.

It is immediate from properties of sets that if two lines have exactly the same points lying on them, then they are the same line (because sets with the same elements are the same). It is interesting to see that we can prove the same thing, while not relying on the idea that lines are sets of points at all.
**Theorem (alternative approach):** Suppose that $L$ and $M$ are lines and for every point $P$, $P$ is incident on $L$ iff $P$ is incident on $M$. Then $L = M$.

**Proof:** Let $L$ and $M$ be arbitrarily chosen lines and assume that for every point $P$, $P$ is incident on $L$ iff $P$ is incident on $M$. Our goal is to show that $L = M$. By axiom 3.1.3$^*$ we can choose two distinct points $Q$ and $R$ which are both incident on line $L$. Also by axiom 3.1.3 we note that $L \leftrightarrow QR$. Now by assumption $Q$ and $R$ are also both incident on $M$, and by axiom 3.1.3$^*$, $M \leftrightarrow QR$. But we then see that $L = M$.

From the proof of this theorem we can see why we added the assumption that there are at least two points incident on each line: axiom 3.1.3 does not help us to determine uniqueness of a line on the basis of the points incident on it unless there are at least two distinct points on it.

**Definition 3.1.6:** Two lines $L$ and $M$ are said to be parallel iff there is no point $P$ such that $P$ is incident on both $L$ and $M$. [using set ideas, iff $L \cap M = \emptyset$.]

It is useful to observe that in analytic geometry, where we take “is parallel to” to mean “has the same slope as”, we say that a line is parallel to itself; here we do not. Also notice that this notion is not the correct one for three-dimensional geometry: skew lines are “parallel” by this definition, and Euclidean three-space satisfies the Hyperbolic Parallel Postulate if this definition of parallel is used.

Now we can prove a theorem, the final one in the reprise of incidence geometry, and in fact a theorem we have already proved.

**Theorem 3.1.7 (either approach):** If $L$ and $M$ are two distinct nonparallel lines, then there exists exactly one point $P$ such that $P$ is incident on $L$ and $P$ is incident on $M$. [if we treat lines as sets, there is a point $P$ such that $L \cap M = \{P\}$].

**Proof:** Assume that $L$ and $M$ are distinct lines and that $L$ and $M$ are not parallel. Our goal is to show that there is exactly one point $P$ such that $P$ is incident on $L$ and $P$ is incident on $M$. When we show that there is exactly one of something, we often first show that there is at least one and then show that there is at most one. That there is at
least one such point $P$ follows immediately from the fact that $L$ and $M$ are not parallel. We now show that there is at most one point $P$ such that $P$ is incident on both $L$ and $M$. We show this by contradiction: assume that there are two distinct points $Q$ and $R$ incident on both $L$ and $M$, and aim for a contradiction. Notice that this assumption immediately implies that $L = QR = M$ by axiom 3.1.3, so $L = M$, which is a contradiction. The proof is complete.

**Definition:** We say that lines $L$ and $M$ intersect at $P$ iff $L$ and $M$ are distinct and $P$ is incident on both $L$ and $M$.

It is important to note that we cannot yet prove that there are two distinct lines! The two point line is a model of the axioms we have so far.

### 3 The Ruler Postulate: Betweenness, Congruence, and Continuity

**Axiom 3.2.1 (Ruler Postulate–axiom in book approach):** For every pair of points $P, Q$ there is a real number $d(P, Q)$ called the distance from $P$ to $Q$. For each line $L$ there is a bijection $c_L$ from $L$ onto the set $\mathbb{R}$ of real numbers such that $d(P, Q) = |c_L(P) - c_L(Q)|$ for each pair of points $P, Q$ incident on $L$.

It is important to notice that I have changed the notation in the book. My axiom says the same thing, though. This axiom is extremely powerful: it has many important consequences. The properties of the real numbers are very important in deducing those consequences.

Basic algebraic knowledge about the real numbers is presumed in this course; you will see that I will not make formal appeals to axioms in appendix E when I do calculations depending on the Ruler Postulate.

Quite a lot of set theory is presumed here. A bijection from a set $A$ to a set $B$ is a function $f$ from $A$ to $B$ such that $f$ is one-to-one (for any $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$) and onto $B$ (for any $b \in B$ there is $a \in A$ such that $f(a) = b$). This should be a familiar idea. Another word for “bijection” is “one-to-one correspondence” (the phrase that Venema uses in the book).
A function is a set of ordered pairs \( f \) such that if \((x, y)\) and \((x, z)\) belong to \( f \) then \( y = z \) (no two elements of \( f \) have the same first component). We define \( f(x) \) as the unique \( y \) such that \((x, y) \in f\), if there is such a \( y \).

An ordered pair \((x, y)\) can be defined as the set \(\{\{x\}, \{x, y\}\}\). It is possible to prove the basic property of ordered pairs, that \((x, y) = (z, w)\) implies \(x = z\) and \(y = w\), from basic set theory alone.

The point of this brief discussion is to show that the concept of a bijection is supported in principle by the set theory in appendix E. You won’t be tested on set theoretical underpinnings of the notion of bijection.

Notice that the Ruler Postulate implies that there are a lot of points. For any line \( L \), the ruler postulate says that \( L \) can be placed on a one-to-one correspondence with the set of real numbers (an infinite set), and so there are infinitely many points on \( L \). This in itself doesn’t say that there are infinitely many points – we still have to show that there is a line! This is straightforward: by the existence postulate there are at least two points \( P \) and \( Q \), and then by the incidence postulate there is a unique line \( L \) on which \( P \) and \( Q \) are incident, whence by the Ruler Postulate there are at least as many points in the plane as there are real numbers. Notice that the two point line is a model of the existence postulate and the incidence postulate. A real number line is a model of the first three postulates; we still need axioms to tell us that there is a line with an exterior point!

It is important to notice that the map \( c_L \), which is called a coordinate function, is not unique: there are many coordinate functions on any line \( L \).

**Definition 3.2.13 (book):** A coordinate function is a function \( c \) from a subset of the plane to the reals such that for any \( P \) and \( Q \) such that \( c(P) \) and \( c(Q) \) are defined, \( d(P, Q) = |c(P) - c(Q)| \). The Ruler Postulate says that for any line \( L \) there is a coordinate function on \( L \). We will be able to show later that the domain of any coordinate function is a subset of some line.

**Lemma (book approach):** If \( c \) is a coordinate function on a subset of the plane, then so is \(-c\) [defined by \((−c)(P) = −c(P)\)] and so is \(c−r\) for any real number \( r \) [defined by \((c−r)(P) = c(P)−r\)]. \(|(−c)(P)−(−c)(Q)| = |−c(P)−(−c(Q))| = |c(Q)−c(P)| = |c(P)−c(Q)| = d(P, Q)| verifies that \(-c\) is a coordinate function. \(|(c−r)(P)−(c−r)(Q)| = |c(P)−r−(c(Q)−r)| = |c(P)−r−c(Q)+r| = |c(P)−c(Q)| = d(P, Q)| verifies that \(c−r\) is a coordinate function.
**Theorem 3.2.16 (Ruler Placement Postulate in book):** for any distinct points $P$ and $Q$, there is a a coordinate function $c$ on the line $L$ on which $P$ and $Q$ are incident such that $c(P) = 0$ and $c(Q)$ is positive.

**Proof:** Let $c_L$ be a coordinate function on $L$ provided by the Ruler Postulate. By the previous results, the function $c_1$ defined by $c(R) = c_L(R) - c_L(P)$ is a coordinate function, and obviously $c_1(P) = 0$. Now $c_1(Q) \neq 0$ because $c_L(Q) \neq c_L(P)$. If $c_1(Q) > 0$, take $c_1$ to be $c$; otherwise take $-c_1$ to be $c$, which is also a coordinate function and will satisfy $c(P) = (-c_1)(P) = -c_1(P) = -0 = 0$ and $(-c_1)(Q) = -c_1(Q) > 0$ (because $c_1(Q) < 0$).

Below I show that this coordinate function is unique.

I prove some familiar basic statements about distance to show how the Ruler Postulate can be used.

**Theorem 3.2.7 (book):** For any points $P, Q$, $d(P, Q) = 0$ iff $P = Q$. For any $P, Q$, $d(P, Q) \geq 0$. For any points $P, Q$, $d(P, Q) = d(Q, P)$.

**Proof:** First we prove that if $d(P, Q) = 0$ then $P = Q$. Choose points $P$ and $Q$ arbitrarily. Assume that $d(P, Q) = 0$. Our goal is to show $P = Q$: we show this by assuming that $P$ and $Q$ are distinct and reasoning to a contradiction. Since $P$ and $Q$ are distinct, there is a unique line $L$ on which $P$ and $Q$ are incident. There is a coordinate function $c_L$ for this line by the Ruler Postulate. $d(P, Q) = |c_L(P) - c_L(Q)| \neq 0$, because $c_L(P) \neq c_L(Q)$, because $P \neq Q$ and $c_L$ is a bijection. This is a contradiction, as we assumed $d(P, Q) = 0$.

Now we prove that if $P = Q$ then $d(P, Q) = 0$. Assume $P = Q$. By the Existence Postulate there is a point $R \neq P$. By the incidence postulate there is a unique line $L$ on which $P$ and $R$ are incident. By the Ruler Postulate we can choose a coordinate function $c_L$ on $L$. Then we have $d(P, Q) = |c_L(P) - c_L(Q)| = |c_L(P) - c_L(P)| = 0$.

This completes the proof of the first statement.

Let $P$ and $Q$ be arbitrarily chosen points. If $P = Q$ then $d(P, Q) = d(Q, P) = 0 \geq 0$, so we have shown in this case that the second and third statements are true.
If \( P \neq Q \) then there is a unique line \( L \) on which \( P \) and \( Q \) are incident and a coordinate function \( c_L \) on \( L \). \( d(P, Q) = |c_L(P) - c_L(Q)| \geq 0 \) because it is an absolute value.

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d(P, Q) = |c_L(P) - c_L(Q)| = |c_L(Q) - c_L(P)| = d(Q, P)
\]

by properties of absolute value. So we have shown the second and third statements to be true in this case.

The proof of the theorem is complete.

**Definition (book, modified):** A function \( d : A \times A \to \mathbb{R} \) is a metric on \( A \) iff for all \( a, b, c \in A \) we have \( d(a, b) \geq 0 \) with \( d(a, b) = 0 \) iff \( a = b \), \( d(a, b) = d(b, a) \) and \( d(a, b) + d(b, c) \geq d(a, c) \). This last statement is called the Triangle Inequality. We cannot yet prove that our distance function \( d \) is a metric on the plane, because we cannot yet prove the Triangle Inequality for our \( d \). The author leaves it out of his definition of metric because we can’t prove it yet, but “metric” is an extremely important mathematical term and I disagree entirely with the author on the wisdom of defining it in an unconventional way.

**Definition 3.2.2 (either approach):** Three points \( A, B, C \) are collinear iff there exists a line \( L \) such that \( A \) is incident on \( L \), \( B \) is incident on \( L \), and \( C \) is incident on \( L \). We say that \( A, B, C \) are noncollinear iff they are not collinear.

**Definition 3.2.3 (book):** Let \( A, B, C \) be three distinct points. The point \( C \) is said to be between \( A \) and \( B \), written \( A \ast C \ast B \), iff \( d(A, C) + d(C, B) = d(A, B) \).

**Observation:** For real numbers, we would like to say that a real number \( b \) is between real numbers \( a \) and \( c \) iff either \( a < b < c \) or \( c < b < a \). The correct notion of distance on the real line is that the distance from \( a \) to \( b \) is the absolute value of the difference between \( a \) and \( b \), that is, \( |a - b| \). We also believe that if \( b \) is between \( a \) and \( c \) that the distance from \( a \) to \( b \) plus the distance from \( b \) to \( c \) will be the distance from \( a \) to \( c \). We verify this impression.

**Betweenness Lemma for Real Numbers:** For three distinct real numbers \( d, e, f \), it will be true that \( |d - e| + |e - f| = |d - f| \) if and only if either \( d < e < f \) or \( f < e < d \).
Proof: Note that \( |d - e| = d - e \) if \( e < d \) and \( e - d \) if \( d < e \) (from the definition of absolute value). So if \( d < e < f \) we have \( |d - e| + |e - f| = (e - d) + (f - e) = f - d = |d - f| \) (this last because \( d < f \)). If \( f < e < d \) we have \( |d - e| + |e - f| = (d - e) + (e - f) = d - f = |d - f| \) (this last because \( f < d \)).

Now suppose that \( d, e, f \) are three distinct real numbers and \( |d - e| + |e - f| = |d - f| \): our aim is to show that either \( d < e < f \) or \( f < e < d \). Our strategy is to assume in addition that \( d < e < f \) is not true, and show that \( f < e < d \) must follow. Since \( d < e < f \) is assumed not true, and we assume that \( d, e, f \) are distinct, we have either \( e < d \) or \( f < e \) (this is because \( d < e < f \) abbreviates \( d < e \) and \( e < f \)”, so its negation (when the numbers are distinct) is equivalent to “\( e < d \) or \( f < e \)”). We argue by cases on the disjunction \( e < d \) or \( f < e \).

In the first case we have assumed \( |d - e| + |e - f| = |d - f| \) and \( e < d \): our aim is to show that also \( f < e \). Suppose \( e < f \) for the sake of a contradiction: then \( |d - e| + |e - f| = (d - e) + (f - e) \). \((d - e) + (f - e)\) (the sum of two positive quantities) is strictly greater than either \((e - d) + (f - e) = f - d\) or \((d - e) + (e - f) = d - f\) (in which one of the positives is replaced with a negative): one of these is \( |d - f| \), so in either case we have \( |d - e| + |e - f| > |d - f| \), contradicting our assumption.

In the second case we have assumed \( |d - e| + |e - f| = |d - f| \) and \( f < e \): our aim is to show also \( e < d \). Suppose \( d < e \) for the sake of a contradiction: then \( |d - e| + |e - f| = (e - d) + (e - f) \). \((e - d) + (e - f)\) (the sum of two positive quantities) is strictly greater than either \((d - e) + (e - f) = d - f\) or \((e - d) + (f - e) = f - d\) (in which one of the positives is replaced with a negative): one of these is \( |d - f| \), so in either case we have \( |d - e| + |e - f| > |d - f| \), contradicting our assumption.

Theorem (3.2.17, Betweenness Theorem for Points, from the book):

If \( A, B, C \) are incident on a line \( L \) and \( c \) is a coordinate function for \( L \), then \( A \ast B \ast C \) iff either \( c(A) < c(B) < c(C) \) or \( c(C) < c(B) < c(A) \).

Proof: By the Betweenness Lemma for Real Numbers, the statement “\( c(A) < c(B) < c(C) \) or \( c(C) < c(B) < c(A) \)” is equivalent to “\( c(A), c(B), c(C) \) are distinct and \( |c(A) - c(B)| + |c(B) - c(C)| = |c(A) - c(C)| \)” , which is

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equivalent to “$A, B, C$ are distinct points on $L$ and $d(A, B) + d(B, C) = d(A, C)$” (because $c$ is a coordinate function), which is equivalent to “$A, B, C$ are incident on $L$ and $A \ast B \ast C$” by definition.

**Distance Lemma for Real Numbers:** If $a, b, c$ are distinct real numbers (we assume without loss of generality $a < b < c$) and we have for real numbers $d, e$ that $|a - d| = |a - e|$, $|b - d| = |b - e|$, $|c - d| = |c - e|$, then $d = e$: the identity of a real number is exactly determined by its distances from $a, b, c$.

$|a - d| = |a - e| = 0$ iff $a = d = e$. Similarly $|b - d| = |b - e| = 0$ iff $b = d = e$ and $|c - d| = |c - e| = 0$ iff $c = d = e$. This covers the cases where $d$ (and $e$) are equal to one of the three given points.

We now suppose that $d$ is distinct from all of $a, b, c$ (from which it follows that $e$ is as well). So there are three cases, $d < a$, $a < d < b$, $b < d$.

If $d < a$, we have $d < a < b$ which tells us that $|d - a| + |a - b| = |d - b|$ (betweenness lemma for reals), so $|d - a| = |d - b| - |a - b|$, so $a - d = |d - b| - |a - b|$, so $d = a - |d - b| - |a - b|$. Now we know that $|e - a| + |a - b| = |e - b|$, so by the betweenness lemma either $e < a < b$ or $b < a < e$, and the known order of $a$ and $b$ excludes $b < a < e$, so by the same reasoning $e = a - |e - b| - |a - b| = a - |d - b| - |a - b| = d$.

If $a < d < b$, then by the betweenness lemma for real numbers $|a - d| + |d - b| = |a - b|$ so $|a - d| = d - a = |a - b| - |d - b|$ so $d = a + |a - b| - |d - b|$. Now $|a - e| + |e - b| = |a - b|$ tells us by the betweenness lemma that either $a < e < b$ or $b < e < a$ and the known order of $a$ and $b$ excludes the second alternative, so by the same reasoning $e = a + |a - b| - |e - b| = a + |a - b| - |d - b| = d$.

If $b < d$, we have $a < b < d$ so by the betweenness lemma we have $|a - b| + |b - d| = |a - d|$ so $d - a = |a - b| + |b - d|$ so $d = a + |a - b| + |b - d|$. By the betweenness lemma, $|a - b| + |b - e| = |a - e|$ tells us that either $a < b < e$ or $e < b < a$, and the known order of $a$ and $b$ excludes the second alternative, so the same reasoning tells us that $e = a + |a - b| + |b - e| = a + |a - b| + |b - d| = d$.

**Theorem (book, extending the Ruler Placement Postulate 3.2.16):**

If $P$ and $Q$ are distinct points on a line $L$, then there is exactly one coordinate function $c$ on $L$ such that $c(P) = 0$ and $c(Q) > 0$. 

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Proof: We have already shown above that there is such a coordinate function. Our aim here is to show that there is just one such coordinate function. The method of argument is to show that for any point \( R \) we can determine the value of \( c(R) \) from geometric information about \( R \) and the given information about \( c \), which means that there can be only one such function \( c \).

We know that \( c(P) \) is 0 from the given information. We know that \( |c(Q) - c(P)| = |c(Q)| = d(P, Q) \), and because \( c(Q) > 0 \) we then know that \( c(Q) = d(P, Q) \). We may now assume that \( R \) is not \( P \) or \( Q \), as these cases have been covered. We note that we must have \( R \ast P \ast Q \) or \( P \ast R \ast Q \) or \( P \ast Q \ast R \) because these are equivalent respectively to \( c_L(R) < c_L(P) < c_L(Q) \), \( c_L(P) < c_L(Q) < c_L(R) \) and \( c_L(P) < c_L(Q) < c_L(R) \) for any coordinate function \( c_L \) at all: one of these three inequalities must hold if \( R \) is not \( P \) or \( Q \) so one of the three geometric facts must hold (notice that the geometric facts do not depend on the identity of the coordinate function).

If \( R \ast P \ast Q \) we have \( c(R) < c(P) = 0 < c(Q) \) or \( c_L(Q) < 0 < c_L(R) \), but the second is impossible. So \( c(R) < 0 \) and \( d(P, R) = |c(P) - c(R)| = |c(R)| \) is enough to tell us that \( c(R) = -d(P, R) \).

If \( P \ast R \ast Q \) then either \( c(P) = 0 < c(R) < c(Q) \) or \( c(Q) < c(R) < c(P) = 0 \), but the second case is impossible. If \( P \ast Q \ast R \) then either \( c(P) = 0 < c(Q) < c(R) \) or \( c(R) < c(Q) < c(P) = 0 \), but the second case is impossible. In either case \( c(R) > 0 \). That along with \( d(P, R) = |c(R)| \) is enough to show that \( c(R) = d(P, R) \).

In our alternative treatment, betweenness is an undefined notion, for which we now introduce some axioms.

**Betweenness Axiom 1 (alternative):** If \( A \ast B \ast C \) then \( A, B, C \) are collinear and distinct, and also \( C \ast B \ast A \).

**Betweenness Axiom 2 (alternative):** For any distinct collinear points \( A, B, C \), exactly one of \( A \ast B \ast C, A \ast C \ast B, B \ast A \ast C \) is true. For any distinct points \( A, B \) there is a point \( C_1 \) such that \( C_1 \ast A \ast B \), a point \( C_2 \) such that \( A \ast C_2 \ast B \), and a point \( C_3 \) such that \( A \ast B \ast C_3 \).

**Betweenness Axiom 3 (alternative):** We define \( A \ast B \ast C \ast D \) as meaning “\( A \ast B \ast C \) and \( B \ast C \ast D \)”. For any distinct points \( A, B, C, D \), if \( A \ast B \ast C \ast D \) then \( A \ast D \ast C \) is true iff either \( A \ast D \ast B \ast C \) or \( A \ast B \ast D \ast C \).
It might be a useful exercise to prove that each of these statements is true for the betweenness relation as we have defined it in terms of distance. We insert these proofs. Please notice that these are exactly the same statements as the axioms for betweenness of the alternative approach: we show how to prove these statements in the book approach with the book’s definition of betweenness.

Please note that when I label a theorem “book” I do not mean that it is actually in the book, unless I also give a theorem number in the book. I mean that its proof assumes distance as a primitive notion and the Ruler Postulate.

Betweenness Theorem 1 (book): If \( A \ast B \ast C \) then \( A, B, C \) are collinear and distinct, and also \( C \ast B \ast A \).

**Proof:** The book definition of betweenness \( A \ast B \ast C \) is that \( A, B, C \) are collinear and distinct and \( d(A, B) + d(B, C) = d(A, C) \). First, we note that we have already shown that if \( A \ast B \ast C \) then \( A, B, C \) are collinear and distinct. It remains to show that \( C \ast B \ast A \). Since \( A, B, C \) are collinear and distinct, \( C, B, A \) are collinear and distinct. We want to show that \( d(C, B) + d(B, A) = d(C, A) \). But we showed above that \( d(P, Q) = d(Q, P) \), so \( d(C, B) + d(B, A) = d(B, C) + d(A, B) \) [by the theorem] = \( d(A, B) + d(B, C) \) [by the theorem] which is what we needed to show: so \( C \ast B \ast A \).

Betweenness Theorem 2 (book): For any distinct collinear points \( A, B, C \), exactly one of \( A \ast B \ast C, A \ast C \ast B, B \ast A \ast C \) is true. For any distinct points \( A, B \) there is a point \( C_1 \) such that \( C_1 \ast A \ast B \), a point \( C_2 \) such that \( A \ast C_2 \ast B \), and a point \( C_3 \) such that \( A \ast B \ast C_3 \).

**Proof:** Assume that \( A, B, C \) are distinct collinear points. There is then a unique line \( L \) on which \( A, B, C \) are incident. Choose a coordinate function \( c_L \) for this line \( L \). Because \( c_L \) is a bijection, \( c_L(A) = a, c_L(B) = b, c_L(C) = c \) are three distinct real numbers. There are six different alternative orders for the three numbers, \( a < b < c, a < c < b, b < a < c, b < c < a, c < a < b, c < b < a \).

Now we use the Betweenness Lemma for Real Numbers.

Since we know that \( A, B, C \) are distinct and collinear, notice that \( A \ast B \ast C \) will be true iff \( d(A, B) + d(B, C) = d(A, C) \), that is if \( |a - b| + |b - c| = \)
Betweenness Theorem 3 (book): We define

**Proof:** Suppose that

Similarly, we know that $A * C * B$ will be true iff $a < c < b$ or $b < c < a$.

and that $B * A * C$ will be true iff $b < a < c$ or $c < a < b$.

Since one of the six alternative orders of $a, b, c$ must hold, and no two of them can hold at the same time, we see that one of the three alternatives $A * B * C$, $A * C * B$, $B * A * C$ must hold, and no two of them can hold at the same time.

Now we prove the last part of the theorem. Suppose that $A$ and $B$ are two distinct points. There is a line $L$ on which $A$ and $B$ are incident, and a coordinate function $c_{L}$ for $L$. $c_{L}(A) = a$ and $c_{L}(B) = b$ are distinct real numbers. Either $a < b$ or $b < a$, If $a < b$, choose real numbers $c_{1}, c_{2}$ and $c_{3}$ such that $c_{1} < a < c_{2} < b < c_{3}$. If $b < a$ choose numbers $c_{1}, c_{2}, c_{3}$ such that $c_{3} < b < c_{2} < a < c_{1}$. The points $c_{L}^{-1}(c_{1}) = C_{1}$, $c_{L}^{-1}(c_{2}) = C_{2}$, and $c_{L}^{-1}(c_{3}) = C_{3}$ will satisfy $C_{1} * A * B$, $A * C_{2} * B$, and $A * B * C_{3}$. $c_{L}$ has an inverse function because it is a bijection. We show for example that $A * C_{2} * B$. $A, C_{2}, B$ are collinear because they all lie on $L$; they are distinct because they have distinct images $a, c_{2}, b$ under $c_{L}$. $d(A, C_{2}) + d(C_{2}, B) = |c_{L}(A) - c_{L}(C_{2})| + |c_{L}(C_{2}) - c_{L}(B)| = |a - c_{2}| + |c_{2} - b| = |a - b|$ (because we know that either $a < c_{2} < b$ or $b < c_{2} < a$) $= |c_{L}(A) - c_{L}(B)| = d(A, B)$, which is what needed to be shown.

**Betweenness Theorem 3 (book):** We define $A * B * C * D$ as meaning “$A * B * C$ and $B * C * D$”. For any distinct points $A, B, C, D$, if $A * B * C$ then $A * D * C$ is true iff either $A * D * B * C$ or $A * B * D * C$.

**Proof:** Suppose that $A, B, C$ are distinct collinear points, that $D$ is distinct from the previous three points, and that $A * B * C$. Note that there is a uniquely determined line $L$ such that $A, B, C$ are incident on $L$, and let $c$ be a coordinate function for $L$.

Assume that $A * D * C$ is true with the aim of showing that either $A * D * B * C$ or $A * B * D * C$ is true.

Because $A * D * C$, $D$ is incident on a common line with $A$ and $C$, which must be $L$ because $A, C$ are distinct. Thus $c$ is defined at $D$. We have either $c(A) < c(B) < c(C)$ or $c(C) < c(B) < c(A)$. We also
have either \( c(A) < c(D) < c(C) \) or \( c(C) < c(D) < c(A) \). All of these inequalities are true by the Betweenness Theorem for Points.

If \( c(A) < c(B) < c(C) \) we cannot have \( c(C) < c(D) < c(A) \), so we must have \( c(A) < c(D) < c(C) \), and we must have either \( c(A) < c(D) < c(B) < c(C) \) or \( c(A) < c(B) < c(D) < c(C) \) by order properties of the real numbers, so we must have either \( A \ast D \ast B \ast C \) or \( A \ast B \ast D \ast C \) by the Betweenness Theorem for Points.

If \( c(C) < c(B) < c(A) \) we cannot have \( c(A) < c(D) < c(C) \), so we must have \( c(C) < c(D) < c(A) \), and we must have either \( c(A) < c(D) < c(B) < c(C) \) or \( c(A) < c(B) < c(D) < c(A) \) by order properties of the real numbers, so we must have either \( A \ast D \ast B \ast C \) or \( A \ast B \ast D \ast C \) by the Betweenness Theorem for Points.

Assume that either \( A \ast D \ast B \ast C \) or \( A \ast B \ast D \ast C \) is true with the aim of showing that \( A \ast D \ast C \). \( A \ast D \ast B \ast C \) implies \( c(A) < c(D) < c(B) < c(C) \) or \( c(C) < c(D) < c(B) < c(A) \), which implies either \( c(A) < c(B) < c(C) \) or \( c(C) < c(B) < c(A) \), which implies \( A \ast B \ast C \).

The proof is complete.

We now give definitions of segments and rays which are valid in either the book approach or the alternative approach: segments and rays are defined in terms of betweenness, which we have seen is a basic notion in the alternative approach and defined in terms of collinearity and distance in the book approach.

**Definition 3.2.4 (either approach):** We define the segment \( \overline{AB} \) as the set \( \{ P \mid P = A \lor P = B \lor A \ast P \ast B \} \). In non-set language, we say that \( P \) is incident on the segment \( \overline{AB} \) if \( P = A \) or \( P = B \) or \( A \ast P \ast B \).

We define the ray \( \overrightarrow{AB} \) as the set \( \{ P \mid P = A \lor P = B \lor A \ast P \ast B \lor A \ast B \ast P \} \). In non-set language, we say that \( P \) is incident on \( \overrightarrow{AB} \) if either \( P = A \) or \( P = B \) or \( A \ast P \ast B \) or \( A \ast B \ast P \).

Notice that “\( P \) is incident on \( \overrightarrow{AB} \)” is trivially exactly equivalent to “\( B \) is incident on \( \overrightarrow{AP} \)” if \( P \) is assumed distinct from \( A \).

**Definition 3.2.5 (book):** The length of the segment \( \overline{AB} \) is defined as \( d(A, B) \).

We say that two line segments \( \overline{AB} \) and \( \overline{CD} \) are congruent, written \( \overline{AB} \cong \overline{CD} \), if \( d(A, B) = d(C, D) \). Recall that in our alternative treatment, congruence of line segments is a primitive notion., and you will
not be surprised to see some axioms for congruence (also involving betweenness) shortly.

**Definition 3.2.6 (either approach):** We say that \( A \) and \( B \) are endpoints of \( \overrightarrow{AB} \) and \( A \) is an endpoint of \( \overrightarrow{AB} \). The other points incident on \( \overrightarrow{AB} \) and \( \overrightarrow{AB} \) are said to be interior points of the segment or ray.

Before introducing the axioms for congruence, we discuss the book proof of the Midpoint Theorem.

**Definition 3.2.21 (book):** A point \( M \) is said to be a midpoint between points \( A \) and \( B \) iff \( A * M * B \) and \( d(A, M) = d(M, B) \). Notice that \( A \) and \( B \) need to be distinct for there to be a midpoint between them on this definition.

**Theorem 3.2.22 (book):** If \( A, B \) are distinct points, there is a midpoint between them.

**Proof:** Let \( A, B \) be distinct points. There is a unique line \( L \) on which they are both incident. Let \( c \) be a coordinate function for \( L \). Let \( M = c^{-1}(\frac{c(A) + c(B)}{2}) \). The coordinate function \( c \) on \( L \) provided by the Ruler Postulate is a bijection, so it has an inverse. Now we know that either \( c(A) < \frac{c(A) + c(B)}{2} < c(B) \) or \( c(B) < \frac{c(A) + c(B)}{2} < c(A) \), so either \( c(A) < c(M) < c(B) \) or \( c(B) < c(M) < c(A) \) so \( A * M * B \). \( d(A, M) = \frac{|c(A) - \frac{c(A) + c(B)}{2}|}{2} = \frac{|c(A) - c(B)|}{2} = c(B) \) = \( d(M, B) \), completing the proof.

We think that this proof sums up our worry about the Ruler Postulate: is this geometric reasoning?

We now present axioms for congruence.

**Congruence Axiom 1 (alternative):** Congruence is an equivalence relation. \( \overrightarrow{AB} \cong \overrightarrow{BA} \)

**Congruence Axiom 2 (alternative):** For any ray \( \overrightarrow{AB} \) and segment \( \overrightarrow{CD} \), there is exactly one point \( E \) incident on \( \overrightarrow{AB} \) such that \( \overrightarrow{AE} \cong \overrightarrow{CD} \).

**Congruence Axiom 3 (alternative):** If \( A * B * C \) and \( D * E * F \) and \( \overrightarrow{AB} \cong \overrightarrow{DE} \) and \( \overrightarrow{BC} \cong \overrightarrow{EF} \) then \( \overrightarrow{AC} \cong \overrightarrow{DF} \).
We prove the congruence axioms of the alternative approach as theorems of the book approach:

**Congruence Theorem 1 (book):** Congruence is an equivalence relation. \( \overline{AB} \equiv \overline{BA} \)

**Proof:** The book definition of congruence makes it obvious that it is an equivalence relation: the equivalence class of a segment \( \overline{AB} \) under congruence is the set of all segments with the same length as \( \overline{AB} \). The second statement follows from \( d(A, B) = d(B, A) \).

**Theorem 3.2.23 (Point Construction Postulate from the book):** If \( A, B \) are distinct points and \( d \) is a nonnegative real number, there is a unique point \( C \) incident on \( \overrightarrow{AB} \) such that \( d(A, C) = d \).

**Proof:** Choose a coordinate function \( c \) for the unique line \( L \) on which \( A \) and \( B \) are incident such that \( c(A) = 0 \) and \( c(B) > 0 \). Let the point \( C \) be \( c^{-1}(d) \). \( d(A, C) = |c(A) - c(C)| = |0 - d| = d \). Notice that if \( c(B) = d \) then \( B = C \). Suppose \( D \) is incident on \( \overrightarrow{AB} \) and \( d(A, D) = d \). We have three cases: either \( A \ast D \ast B \) or \( A \ast B \ast D \) or \( D = B \). If \( A \ast D \ast B \) we have either \( 0 = c(A) < c(D) < c(B) \) or \( c(B) < c(D) < c(A) = 0 \); the latter is ruled out by assumed properties of \( c \). If \( A \ast B \ast D \) we have either \( 0 = c(A) < c(B) < c(D) \) or \( c(D) < c(B) < c(A) = 0 \); the latter is ruled out by assumed properties of \( c \). If \( B = D \) we have \( c(B) = d \). In all three cases, we have \( c(D) > 0 \). \( c(D) > 0 \) and \( d(A, D) = |c(A) - c(D)| = |c(D)| \) is enough to establish \( c(D) = d \) and so \( C = D \).

**Congruence Theorem 2 (book):** For any ray \( \overrightarrow{AB} \) and segment \( \overline{CD} \), there is exactly one point \( E \) incident on \( \overrightarrow{AB} \) such that \( \overline{AE} \equiv \overline{CD} \).

**Proof:** Let \( d = d(C, D) \). By the previous theorem there is a unique point \( E \) on \( \overrightarrow{AB} \) such that \( d(A, E) = d \), that is \( d(A, E) = d(C, D) \), that is \( \overline{AE} \equiv \overline{CD} \). Further, if \( F \) is incident on \( \overrightarrow{AB} \) and \( \overline{AF} \equiv \overline{CD} \) then \( d(A, F) = d(C, D) = d \) so by the previous theorem \( F = E \), establishing uniqueness.

**Congruence Theorem 3 (book):** If \( A \ast B \ast C \) and \( D \ast E \ast F \) and \( \overline{AB} \equiv \overline{DE} \) and \( \overline{BC} \equiv \overline{EF} \) then \( \overline{AC} \equiv \overline{DF} \).
Proof: We have \( d(A,B) = d(D,E) \). We have \( d(B,C) = d(E,F) \). Then (using equations on distances justified by the betweenness hypotheses) \( d(A,C) = d(A,B) + d(B,C) = d(D,E) + d(E,F) = d(D,F) \), so \( AC \cong DF \).

We remind you again that when we label a theorem “book” we mean that it relies on the primitive notion of distance and the Ruler Postulate, not that it actually appears in the book.

We give the final axiom of the alternative approach.

Definition: We say that a set of points \( A \) is **convex** iff for every \( P, Q \in A \) and any point \( R, P \star R \star Q \) implies \( R \) is in \( A \).

Interval Axiom (alternative): A convex subset \( C \) of a line \( L \) is either the empty set, or a point, or the line \( L \), or there is a segment \( \overrightarrow{AB} \) such that \( C \) is a subset of \( \overrightarrow{AB} \) containing all its interior points, or there is a ray \( \overrightarrow{AB} \) such that \( C \) is a subset of \( \overrightarrow{AB} \) containing all its interior points.

The Interval Axiom does the work of the least upper bound property of the real numbers: it is our continuity axiom, ensuring that there are no “holes” in our lines. If we use the official Ruler Postulate, the least upper bound axiom for the reals has the same effect. Notice that in the Interval Axiom we have found it necessary to talk about sets of points. We will follow the convention that we identify a line, segment or ray with the set of points incident on it, in spite of our preference for avoiding set language.

Our aim here is to show that our axioms for betweenness and congruence are enough to allow us to talk about geometric quantity (length of line segments) without actually talking about real numbers. This agrees with the historical practice of the Greeks, and it is also useful to be able to think about geometric quantities...geometrically.

Lemma: Suppose that \( X \) and \( Y \) are distinct from \( A \) and each other and incident on ray \( \overrightarrow{AB} \). Then either \( A \star X \star Y \) or \( A \star Y \star X \).

Proof: If one of \( X \) or \( Y \) is equal to \( B \), without loss of generality \( X \), then either \( A \star B \star Y \) or \( A \star Y \star B \) is true by definition of a ray, and so \( A \star X \star Y \) or \( A \star Y \star X \) is true. If \( Y = B \) the proof is exactly the same.

If \( X \) and \( Y \) are both distinct from \( B \) and \( A \star X \star Y \) and \( A \star Y \star X \) are both false, then the only alternative is \( X \star A \star Y \). Now we have either
A∗Y*B or A*B*Y. If we have A*Y*B, we have X*A*Y*B, from which we can deduce X*A*B, which is a contradiction (X would not be on the ray). If we have A*B*Y, we know that either A*X*Y, from which we get A*X*Y and contradiction, or A*B*X. From X*A*Y and A*B*X we get X*B*A and so B*A*Y, which contradicts Y being on the ray.

Identity conditions for rays: We observed above that if P is incident on A*B and distinct from A then it is trivial from symmetry of the definition that B is incident on A*P. We would like to show here that in fact A*P and A*B are the same (as sets) if P is incident on A*B and distinct from A. Suppose that Q is incident on A*B and distinct from A. It follows from the Lemma that A*Q*P or A*P*Q, from which it follows that Q is incident on A*P. Now suppose that Q is incident on A*P and distinct from A. We know that B is also incident on A*P by the symmetry of the definition, and then by the lemma we know that either A*Q*B or A*B*Q, so Q is incident on A*B. We have shown that exactly the same points are incident on the two rays. So a ray is uniquely specified by its endpoint and any other point on it.

Theorem: Suppose that A, B are distinct points and C, D are distinct points. Suppose that the unique point X on ray E*F such that AB ≃ EX and the unique point Y on ray E*F such that CD ≃ EY satisfy E*X*Y. Then the unique point W on ray G*H such that CD ≃ GW satisfy G*Z*W.

Proof: Suppose that A, B are distinct points and C, D are distinct points. Suppose that the unique point X on ray E*F such that AB ≃ EX and the unique point Y on ray E*F such that CD ≃ EY satisfy E*X*Y. Let Z be the unique point on G*H such that GZ ≃ AB. There is a point I such that G*Z*I. Let W be the point on ZI such that ZW ≃ XY. Then by congruence axiom 3 it follows that GW ≃ EY (compare EX to GZ and AB; compare XY to ZW; note that E*X*Y and G*Z*W). And so GW is congruent to CD, completing the proof: the points Z and W are the objects they are supposed to be and stand in the right order on the line.

This theorem justifies a definition:
**Definition:** We say that a segment $AB$ is shorter than a segment $CD$ (written $AB < CD$) if it is true that for any ray $XY$ the unique point $Z$ incident on $XY$ such that $AB \cong XZ$ and the unique point $W$ incident on $XY$ such that $XW \cong CD$ satisfy $X \ast Z \ast W$: the theorem above tells us that if this is true for any ray it will be true for all rays. We say that $AB > CD$ iff $CD < AB$.

**Observation:** It follows readily from what was shown above that for any pair of segments $AB$ and $CD$ exactly one of the statements $AB < CD$, $AB > CD$ and $AB \cong CD$ is true, because for any points $X$ and $Y$ incident on a ray $EF$, exactly one of $E \ast X \ast Y$, $E \ast Y \ast X$, or $X = Y$ must hold.

We now construct a special ray which we will use to represent lengths. What we call the length of a ray in the alternative treatment will not be a real number but a point on this ray.

**Definition (alternative):** Choose two distinct points $P_0$ and $P_1$. These points are fixed – whenever we write $P_0$ or $P_1$ hereinafter we are talking about these exact two points. We will refer to $P_0P_1$ as the number ray. Define $|AB|$ as the unique point $Q$ on $P_0P_1$ such that $P_0Q$ is congruent to $AB$. For any rays $AB$ and $CD$, define $|AB| + |CD|$ as the unique point $R$ constructed as follows: find the point $|AB|$ (call it $Q$); find a point $S$ such that $P_0 \ast Q \ast S$; let $R$ be the unique point on $QS$ such that $QR \cong CD$. Notice that for any points $Q, R$ on $P_0P_1$ neither of which are $P_0$ we can read $Q + R = |P_0Q| + |P_0R|$; we add “lengths of segments”, but any point on the number ray is the length of a segment, so we can add points on the number ray. We define $|AB| < |CD|$ as equivalent to the notation $AB < CD$ defined above.

We state some theorems about addition and order on points of the number ray which look like familiar arithmetic facts.

**Theorem (alternative):** For any points $P, Q, R$ on the number ray other than $P_0$, $P + Q = Q + P$, $(P + Q) + R = P + (Q + R)$, and $P + R = Q + R \rightarrow P = R$.

**Theorem (alternative):** For any points $P, Q, R, S$ on the number ray other than $P_0$, exactly one of $P < Q$, $P = Q$, $Q < P$ is true, and if $P < Q$
and \( Q < R \), it follows that \( P < R \), and \( P < Q \iff P + R < Q + R \), and finally \( P < R \) and \( Q < S \) implies \( P + Q < R + S \).

These have proofs from the axioms of betweenness and congruence (and the theorems we have proved from these axioms), which I will add here. The proofs are not hard, just boring.

**Definition:** We call a set of points \( I \) on the number ray *inductive* if it contains \( P_1 \) and for any \( P \in I \), \( P + P_1 \) is also in \( I \). We say that a point is a *counting point* iff it belongs to every inductive set. Unofficially, we define \( 1P_1 \) as \( P_1 \) and \( (n + 1)P_1 \) as \( nP_1 + P_1 \) for each positive integer \( i \): the set of counting points is exactly the set of points \( nP_1 \) for \( n \) a positive integer. This is unofficial because we are not assuming familiarity with the integers.

**Theorem:** For every point \( Q \) on the number ray there is a counting point \( P \) such that \( P_0 \ast Q \ast P \).

**Proof:** Mildly tricky using the Interval Postulate. I will put it here.

**Definition:** For any point \( Q \) on the number ray which is not \( P_0 \), we say that a set of points \( I \) on the number ray is \( Q \)-inductive iff it contains \( Q \) and for any \( R \in I \) we have \( R + Q \) in \( I \). A \( Q \)-counting point is a point which belongs to every \( Q \)-inductive set. Informally, the \( Q \)-counting points are defined as the collection of \( nQ \)'s where \( 1Q = Q \), \( (n + 1)Q = nQ + Q \), but we are avoiding numbers. If \( P \) is a counting point, we can define \( Q_P \) as the \( Q \)-counting point \( R \) such that there is a one-to-one correspondence between the set of counting points incident on \( P_0P \) and the set of \( Q \)-counting points incident on \( P_0R \). Informally \( nQ = Q_{nP_1} \) is obvious.

**Theorem (the Archimedean Property):** For every point \( Q \) on the number ray and for every point \( R \) there is a \( Q \)-counting point \( S \) such that \( P_0 \ast R \ast S \).

**Proof:** basically the same.

**Definition of Proportion:** We say \( P : Q :: R : S \), where \( P, Q, R, S \) are points on the number ray other than \( P_0 \), iff for any counting points \( p, q \) we have \( P_p < Q_q \iff R_p < S_q \); informally, for any pair of positive integers \( m, n \), we have \( mP < nQ \iff mR < nS \).
Definition of midpoint (alternative): If $A, B$ are points, we say that $M$ is a midpoint between $A$ and $B$ iff $A \ast M \ast B$ and $\overline{AM} \cong \overline{MB}$. We need a different definition because the one in the book mentions distance.

Midpoint Theorem (alternative): For any distinct points $A, B$, there is a midpoint between $A$ and $B$.

Proof: Let $Q = |\overline{AB}|$. Our strategy will be to prove that there is a midpoint between $P_0$ and $Q$, then use this to construct a midpoint between $A$ and $B$. We consider the set $H$ of all points $X$ on the number ray such that $X + X < Q$. It is straightforward to prove that $X < Y$ implies $X + X < Y + Y$. From this it follows that if $X$ and $Z$ are in $H$ and $X \ast Y \ast Z$ we have either $X < Y < Z$ or $Z < Y < X$ from which we get $X + X < Y + Y < Z + Z < Q$ or $Z + Z < Y + Y < X + X < Q$, and in either case $Y \in H$, so $H$ is a convex subset of a line. For any $X \in H$, we have $Y \in H$ for any $P_0 \ast Y \ast X$, so this set contains more than one point. It does not contain $Q$, so it is not a ray. We show that for any $X$ there is $Y$ such that $Y + Y < X$. Choose any $Z < X$: if $Z = |\overline{XZ}|$ then choose any $W < Z$ and $W + W < Z + Z = X$ follows. Otherwise let $W$ be the minimum of $Z$ and $|\overline{XZ}|$: then $W + W < Z + |\overline{XZ}| = X$. By the Interval Postulate there are points $U$ and $R$ such that $\overline{UR}$ contains all elements of $H$ and the only points in this interval that might not be in $H$ are $U$ and $R$. That $U = P_0$ is immediate: $P_0$ is not in $H$, and any point on the number ray $X$ has $Y \in H$ such that $P_0 \ast Y \ast X$, so nothing “above” $P_0$ can be $U$.

We claim that $R + R = Q$. We prove this by contradiction.

Suppose that $h = R + R < Q$. Choose $y$ such that $y + y < |\overline{hQ}|$. Then $(R + y) + (R + y) = R + R + y + y < R + R + |\overline{hQ}| = Q$, which is impossible, because this would cause $R + y$ to be in $H$, and $R + y$ is not in the interval $\overline{P_0R}$.

Suppose that $h = R + R > Q$. Choose $y < R$ such that $y + y < |\overline{hQ}|$. Now consider the point $R - y = |\overline{Ry}|$. $R - y$ is on the number ray. $(R - y) + (R - y) + (y + y) = R + R$. We have $Q + |\overline{hQ}| = h$ and $(R - y) + (R - y) + (y + y) = h$: if we had $(R - y) + (R - y) \leq Q$ we would have $h < h$, which is absurd. So $R - y$ is not an element of $H$. And this is absurd, because $R - y$ is in the interval $\overline{P_0R}$ which must be included in $H$. 

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Now the midpoint between $A$ and $B$ can be constructed as the unique point $M$ on $\overline{AB}$ such that $\overline{AM} \cong \overline{P_0H}$. 