Math 301 Test IV

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This exam begins at 9:40 am and ends at 10:35 am. You may use your fancy calculator freely but please make it clear to me what you did with it. Cell phones to be turned off and out of sight.
1. In a certain metropolitan area, three percent of the population emigrates from the city to the suburbs each year, while one percent of the population of the suburbs returns to the city each year.

Compute the stochastic matrix which describes this situation.

If 55 percent of the population lived in the city in 2007, what percentage of the population is expected to live in the city in 2009? Show a matrix calculation.

Find the steady state vector for this stochastic matrix and tell me what percentage of the population is expected to live in the city after a long time.

$$
\begin{bmatrix}
0.99 & 0.01 \\
0.03 & 0.99
\end{bmatrix}
\begin{bmatrix}
0.55 \\
0.45
\end{bmatrix}
= 
\begin{bmatrix}
0.53 \\
0.47
\end{bmatrix}
\quad 53\% \quad \text{in 2009}
$$

$$
\begin{bmatrix}
0.97 & 0.01 \\
0.03 & 0.99
\end{bmatrix}
\begin{bmatrix}
1 \\
-\frac{1}{3}
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-0.3333...
\end{bmatrix}

x_1 - \frac{1}{3}x_2 = 0 \quad x_1 = \frac{1}{3}x_2
$$

$$
\begin{bmatrix}
17 \\
3
\end{bmatrix}
\text{ is a steady state vector}
$$

and

$$
\begin{bmatrix}
25 \\
75
\end{bmatrix}
= 
\begin{bmatrix}
\frac{1}{4} \\
\frac{3}{4}
\end{bmatrix}
\text{ is steady state prob vector.}
$$

25\% \text{ will eventually live in city.}
2. The set of vectors

\[ B = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\} \]

is a basis for \( \mathbb{R}^2 \). Provide a brief justification of this fact.

\( \text{Observe} \) \( \begin{bmatrix} 1 & 1 \end{bmatrix} \) \( \text{and a set of 2 vectors} \ (1, 1) \ \text{in} \ \mathbb{R}^2 \) \( \text{is a basis} \).

The set of vectors

\[ C = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right\} \]

is a basis for \( \mathbb{R}^2 \). Provide a brief justification of this fact.

\( \text{Same justification} \).

Present the matrix which converts \( B \)-coordinates to \( C \)-coordinates (hint: it would be helpful to think in terms of converting from \( B \)-coordinates to standard coordinates, then standard coordinates to \( B \)-coordinates).

\[ B \text{ to standard} \rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

\[ \text{standard to } B \rightarrow \begin{bmatrix} 1 \\ 1 \\ -2 \\ -3 \end{bmatrix} \]

\[ \text{Then} \ P^{-1} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -3 \end{bmatrix} \]

\[ \text{So} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{3} \end{bmatrix} \]

Present the \( C \)-matrix for the transformation “rotate 90 degrees counterclockwise” represented for standard coordinates by the matrix

\[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

\[ \text{over standard to } C \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \text{then } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

\[ \text{rotate} \]

\[ \text{then to standard to } C \rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \]

\[ \text{So} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{3}{3} \end{bmatrix} \]
3. We present the matrix

\[ A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 3 & -4 \\ 1 & 1 & -2 \end{bmatrix} \]

We inform you that this has eigenvalues 1, -1, and 3. Find an eigenvector corresponding to each eigenvalue, and express the matrix \( A \) in the form \( PDP^{-1} \), where \( P \) is an invertible matrix and \( D \) is a diagonal matrix.

\[
\begin{vmatrix} 2-x & 1 & -1 \\ 2 & 3-x & -4 \\ 2 & 1 & 2-x \end{vmatrix} \quad \Rightarrow \quad x = 1 \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
x_1 + x_2 = 0
\]

\[
x_3 = 0
\]

\[
\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{an eigenvector}
\]

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
x_1 = 0
\]

\[
x_2 + x_3 = 0
\]

\[
\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
x_1 + 2x_3 = 0
\]

\[
x_2 - 3x_3 = 0
\]

Thus \( x = 1 \) gave \( x = -1 \)

\[
4x = 3
\]

\[
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]
4. Find the characteristic equation and the (complex) eigenvalues of

\[
\begin{bmatrix}
1 & -5 \\
1 & 5
\end{bmatrix},
\]

find the corresponding eigenvectors (be sure to make the correspondence between eigenvalues and their eigenvectors clear), and express the matrix in the form

\[ PCP^{-1}, \]

where \( C \) is of the form

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}.
\]

\[
\text{det}\begin{bmatrix}
1 & -5 \\
1 & 5
\end{bmatrix} = (1-x)(5-x) + 5
\]

\[= 5 - 6x + x^2 + 5\]

\[= (x^2 - 6x + 10) \text{ char poly} \]

\[x^2 - 6x + 10 = 0\]

\[(x-3)^2 + 1 = 0\]

\[(x-3)^2 = -1\]

\[x = 3 \pm i\]

\[
\text{ref } \begin{bmatrix}
1 & -5 \\
1 & 5
\end{bmatrix} \mid x = 3 - i \\
\begin{bmatrix}
1 & 2+2i \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2 - i \\
1
\end{bmatrix} \text{ is eigenvector for } \lambda = 3 - i
\]

\[
\begin{bmatrix}
-2 + i \\
1
\end{bmatrix} \text{ for } \lambda = 3 + i
\]
5. A transformation of $\mathbb{R}^2$ is determined by reflecting the plane through the line $y = x$. Without doing any calculations, describe two eigenvalues for the matrix representing this transformation, and give specific vectors in numerical form which are eigenvectors for these eigenvalues. Drawing a picture is encouraged.
6. Estimate an eigenvector and eigenvalue of

\[ A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 3 & -4 \\ 1 & 1 & -2 \end{bmatrix} \]

by the power method. You have already seen this matrix above. You should know which eigenvalue you will find at the outset: which one, and why that one?

the eigenvector for \( \lambda = 3 \), because 3 is largest (in abs value) of the eigenvalues.

Please give the first two or three steps of your iterative calculation, then give the final answer to which repeated applications of the procedure tend. I should be able to tell from what you write what you did with your calculator.

Use \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) as starting vector.

INSTRUCTOR: given in class.

\[ A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 2.5 \\ 3.5 \\ 1.5 \end{bmatrix} \]

\[ \begin{bmatrix} 2.8 \\ 3.8 \\ 1.2 \end{bmatrix} \]

\[ \begin{bmatrix} 3 \\ 4.5 \\ 1.5 \end{bmatrix} = 1.5 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \]

which shows \( \lambda = 3 \), a multiple (in this \( A \))

1.5 times the eigenvector found earlier.
7. Verify that

\[
B = \begin{bmatrix}
1 & 0 & 1.2 \\
-2 & 1 & 2 \\
0 & 1 & -5 \\
1 & 2 & 1
\end{bmatrix}
\]

is an orthogonal set. Briefly indicate why this means that we know it is a basis for \( \mathbb{R}^3 \).

Orthogonal set of nonzero vectors are linearly independent. A set of 3 l.i. vectors in \( \mathbb{R}^3 \) is a basis.

Find the \( B \)-coordinate vector for

\[
\begin{bmatrix}
3 \\
3 \\
-1
\end{bmatrix}
\]

using a calculation with inner products (Hint: this involves the same calculations as computing the orthogonal projection of

\[
\begin{bmatrix}
3 \\
3 \\
-1
\end{bmatrix}
\]

onto each of the elements of the basis.)

\[
\begin{align*}
3 - 6 - 1 &= q \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} + r \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ -7 \\ 1 \end{bmatrix} \\
&= -\frac{72}{30} \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}
\end{align*}
\]

Express \( \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} \) as a linear combination of the elements of the basis.

\[
\begin{align*}
-\frac{2}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} - \frac{1}{15} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}
\end{align*}
\]
8. Suppose we know that the nonzero vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbb{R}^2$ are orthogonal to one another. It follows that $\mathbf{u}_1$ and $\mathbf{u}_2$ form a basis of $\mathbb{R}^2$.

This means that for any vector $\mathbf{y}$ in $\mathbb{R}^2$, we can find scalars (real numbers) such that

$$\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2.$$ 

Prove that $c_1 = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}$. Hint: take the inner product of both sides of the equation above with $\mathbf{u}_1$ and do a little calculation (with a couple of comments needed). Write the formula for $c_2$ as well.

\[ \mathbf{y} \cdot \mathbf{u}_1 = (c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2) \cdot \mathbf{u}_1 \]

\[ = c_1 \mathbf{u}_1 \cdot \mathbf{u}_1 + c_2 (\mathbf{u}_2 \cdot \mathbf{u}_1) \]

\[ \mathbf{y} \cdot \mathbf{u}_1 = c_1 \mathbf{u}_1 \cdot \mathbf{u}_1 \quad \text{since } \mathbf{u}_1 \cdot \mathbf{u}_1 \neq 0 \]

\[ \text{So } c_1 = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \]

\[ c_2 = \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \quad \text{same proof, } \text{"multiply by } \mathbf{u}_2 \text{ instead.} \]