Solutions to propositional logic proof exercises

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1 Exercises

1. Prove
$$((P \to Q) \land (Q \to R)) \to (P \to R)$$
using the style given here. This should be straightforward.

   Assume (1): $(P \to Q) \land (Q \to R)$

   Goal: $P \to R$

      Assume (2): P

      Goal: R

      (3): $P \to Q$ simplification, line 1

      (4): $Q \to R$ simplification line 1

      (5): Q modus ponens lines 2,3

      (6): R modus ponens lines 5,4

   (7): $P \to R$ deduction lines 2–6

(8): the exercise by deduction lines 1–7

2. Prove

$$((P \to R) \wedge (Q \to \neg R)) \to (Q \to \neg P).$$

You will want to use the additional rules involving negation.

Assume (1): $((P \to R) \wedge (Q \to \neg R))$

Goal: $Q \to \neg P$

Assume(2): Q

Goal: $\neg P$ **Assume(3):** P

Goal: $\perp$ (a contradiction)

(4): $P \to R$ simplification 1

(5): $Q \to \neg R$ simplification 1

(6): R modus ponens 3,4

(7): $\neg Q$ disjunctive syllogism 6,5

(8): $\perp$ contradiction, 7,2

(9): $\neg P$ negation introduction 3–8

(10): $Q \to \neg P$ deduction 2–9

(11): the exercise deduction lines 1–10

3. Prove

$$((P \to R) \vee (Q \to R)) \to ((P \wedge Q) \to R).$$

This should be relatively straightforward – proof by cases is needed.

Assume(1): $((P \to R) \vee (Q \to R))$

Goal: $(P \wedge Q) \to R$

> **Assume(2):** $P \wedge Q$
>
> **Goal:** R
>
> **(3):** P simplification line 2
>
> **(4):** Q simplifaction line 2
>
> **We start a proof by cases using line 1:**
>
> > **Case 1 (5):** $P \to R$
> >
> > **Goal:** R
> >
> > **(6):** R mp 3,5
> >
> > **Case 2 (7):** $Q \to R$
> >
> > **Goal:** R
> >
> > **(8):** R mp 4,7
>
> **(9):** R proof by cases 1,5–6,7–8

(10): $(P \wedge Q) \to R$ deduction 2–9

(11): the exercise deduction 1–10

4. Prove

$$((P \vee Q) \wedge (\neg Q \vee R)) \rightarrow (P \vee R)$$

I see one approach using excluded middle and proof by cases, but I think there is a simpler way using the rules of alternative exclusion and disjunctive syllogism.

Assume(1): $((P \vee Q) \wedge (\neg Q \vee R))$

Goal: $P \vee R$

> **Assume(2):** $\neg R$
>
> **Goal:** P
>
> **(3):** $P \vee Q$ simplification 1
>
> **(4):** $\neg Q \vee R$ simplification 1
>
> **(5):** $\neg Q$ disjunctive syllogism lines 5,3
>
> **(6):** P disjunctive syllogism lines 4,5

(7): $P \vee R$ alternative exclusion lines 2–6

(8): the exercise, deduction 1–7

Assuming $\neg P$ as line 2 and reasoning to R goes about the same way.

5. Prove

$$((P \wedge Q) \to R) \to ((P \to R) \vee (Q \to R)).$$

This is quite tricky. You should recognize this problem and a previous one as the two directions of a biconditional you proved using truth tables.

Not assigned

6. Prove

$$\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$$

This is one of deMorgan's laws. Notice that this is a biconditional so you have two directions of argument to complete. You can't use a deMorgan law to prove it; just the rules in this handout.

Part I: **Assume(1):** $\neg(P \wedge Q)$

Goal: $(\neg P \vee \neg Q)$

> **Assume(2):** $\neg\neg P$
>
> **Goal:** $\neg Q$
>
> > **Assume(3):** Q
> >
> > **Goal:** $\bot$ (a contradiction)
> >
> > **(4):** P double negation 2
> >
> > **(5)** $P \wedge Q$ conjunction 4,3
> >
> > **(6):** $\bot$ contradiction, 1,5
>
> **(7):** $\neg Q$ negation introduction 3–6

(8): $(\neg P \vee \neg Q)$ alternative exclusion 2–7

Part II: **Assume(9):** $(\neg P \vee \neg Q)$

Goal: $\neg(P \wedge Q)$

> **Assume(10):** $P \wedge Q$
>
> **Goal:** $\bot$ (a contradiction)
>
> **(11):** P simplification 10
>
> **(12):** Q simplification 10
>
> **(13):** $\neg Q$ disjunctive syllogism 9, 12
>
> **(14):** $\bot$ contradiction 12,13

(15): $\neg(P \wedge Q)$ negation introduction 10–14

(16): the exercise, biconditional introduction 1–8, 9–15.