Solutions to propositional logic proof exercises

October 6, 2016

1 Exercises

1. Prove

\[((P \to Q) \land (Q \to R)) \to (P \to R)\]

using the style given here. This should be straightforward.

Assume (1): \((P \to Q) \land (Q \to R)\)

Goal: \(P \to R\)

Assume (2): \(P\)

Goal: \(R\)

(3): \(P \to Q\) simplification, line 1

(4): \(Q \to R\) simplification line 1

(5): \(Q\) modus ponens lines 2,3

(6): \(R\) modus ponens lines 5,4

(7): \(P \to R\) deduction lines 2–6

(8): the exercise by deduction lines 1–7
2. Prove

\[ ((P \to R) \land (Q \to \neg R)) \to (Q \to \neg P). \]

You will want to use the additional rules involving negation.

Assume (1): \[ ((P \to R) \land (Q \to \neg R)) \]

Goal: \[ Q \to \neg P \]

Assume(2): \[ Q \]

Goal: \[ \neg P \] Assume(3): \[ P \]

\begin{align*}
\text{Goal: } & \bot \text{ (a contradiction)} \\
(4): & \ P \to R \text{ simplification 1} \\
(5): & \ Q \to \neg R \text{ simplification 1} \\
(6): & \ R \text{ modus ponens 3,4} \\
(7): & \neg Q \text{ disjunctive syllogism 6,5} \\
(8): & \bot \text{ contradiction, 7,2} \\
(9): & \neg P \text{ negation introduction 3–8} \\
(10): & \ Q \to \neg P \text{ deduction 2–9} \\
\end{align*}

(11): the exercise deduction lines 1–10
3. Prove

\[ ((P \rightarrow R) \lor (Q \rightarrow R)) \rightarrow ((P \land Q) \rightarrow R). \]

This should be relatively straightforward – proof by cases is needed.

**Assume(1):** \((P \rightarrow R) \lor (Q \rightarrow R))

**Goal:** \((P \land Q) \rightarrow R)

**Assume(2):** \(P \land Q\)

**Goal:** \(R\)

(3): \(P\) simplification line 2

(4): \(Q\) simplification line 2

**We start a proof by cases using line 1:**

**Case 1 (5):** \(P \rightarrow R\)

**Goal:** \(R\)

(6): \(R\) mp 3,5

**Case 2 (7):** \(Q \rightarrow R\)

**Goal:** \(R\)

(8): \(R\) mp 4,7

(9): \(R\) proof by cases 1,5–6,7–8

(10): \((P \land Q) \rightarrow R\) deduction 2–9

(11): the exercise deduction 1–10
4. Prove

\[(P \lor Q) \land (\neg Q \lor R) \rightarrow (P \lor R)\]

I see one approach using excluded middle and proof by cases, but I think there is a simpler way using the rules of alternative exclusion and disjunctive syllogism.

**Assume(1):** \[(P \lor Q) \land (\neg Q \lor R)\]

**Goal:** \[P \lor R\]

**Assume(2):** \[\neg R\]

**Goal:** \[P\]

(3): \[P \lor Q\] simplification 1

(4): \[\neg Q \lor R\] simplification 1

(5): \[\neg Q\] disjunctive syllogism lines 5,3

(6): \[P\] disjunctive syllogism lines 4,5

(7): \[P \lor R\] alternative exclusion lines 2–6

(8): the exercise, deduction 1–7

Assuming \[\neg P\] as line 2 and reasoning to \[R\] goes about the same way.
5. Prove

\[ ((P \land Q) \rightarrow R) \rightarrow ((P \rightarrow R) \lor (Q \rightarrow R)) \].

This is quite tricky. You should recognize this problem and a previous one as the two directions of a biconditional you proved using truth tables.

Not assigned
6. Prove

\[ \neg(P \land Q) \leftrightarrow (\neg P \lor \neg Q) \]

This is one of deMorgan’s laws. Notice that this is a biconditional so you have two directions of argument to complete. You can’t use a deMorgan law to prove it; just the rules in this handout.

Part I: Assume(1): \(\neg(P \land Q)\)
Goal: \((\neg P \lor \neg Q)\)

Assume(2): \(\neg\neg P\)
Goal: \(\neg Q\)

Assume(3): \(Q\)
Goal: \(\bot\) (a contradiction)

(4): \(P\) double negation 2
(5): \(P \land Q\) conjunction 4,3
(6): \(\bot\) contradiction, 1,5

(7): \(\neg Q\) negation introduction 3–6

(8): \((\neg P \lor \neg Q)\) alternative exclusion 2–7

Part II: Assume(9): \((\neg P \lor \neg Q)\)
Goal: \(\neg(P \land Q)\)

Assume(10): \(P \land Q\)
Goal: \(\bot\) (a contradiction)

(11): \(P\) simplification 10
(12): \(Q\) simplification 10
(13): \(\neg Q\) disjunctive syllogism 9, 12
(14): \(\bot\) contradiction 12,13

(15): \(\neg(P \land Q)\) negation introduction 10–14

(16): the exercise, biconditional introduction 1–8, 9–15.