Math 175 Test IV

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This test begins at 9:40 am and ends at 10:35 am. You may use a plain scientific calculator with no graphing or symbolic computation capabilities. You do not need one. Cell phones must be turned off and out of sight.

Make sure you write your name on this test paper and on any blue books that you use. Do not write any work or answers on this paper. Be sure to record the number on this paper by which your grade will be posted.
1. Numerical estimates of integrals

Estimate

\[ \int_1^4 \frac{1}{x} \, dx \]

using Simpson’s Rule with six partitions. It is sufficient to set up the calculation, but if you have a calculator it would be a good idea to check that the value is actually close to the expected \( \ln(4) \).

2. Trigonometric integrals

Compute

\[ \int \cos^3(x) \sin^2(x) \, dx \]

3. Trigonometric substitution

For each of the following antiderivatives, state the appropriate trigonometric substitution. Evaluate one of the integrals using the trigonometric substitution (some of them may be possible to evaluate in other ways; use of the trig substitution is required here). Use right triangles suitably to simplify applications of trig functions to values of inverse trig functions.

If you evaluate more than one of these integrals, your best work will count and you might get some extra credit. Some of these are harder than others.

(a)

\[ \int \frac{x}{\sqrt{1 - x^2}} \, dx \]

(b)

\[ \int \sqrt{x^2 + 9} \, dx \]

(c)

\[ \int \sqrt{x^2 - 4} \cdot \frac{1}{x} \, dx \]

4. Ratio and Root test

Determine whether

\[ \sum_{n=0}^{\infty} \frac{2^n}{n!} \]

is convergent or divergent, using the Ratio Test. Show all work.

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5. Classify each series as absolutely convergent, conditionally convergent, or divergent. Show all supporting work. If you need to use the Alternating Series Test, state the three conditions required for the test to apply: you do not need to prove them, but you need to state them in terms of the sequence you are working on, not in the abstract. Hint: you only actually need to use the AST in one of the three parts.

(a) \[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \]

(b) \[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} \]

(c) \[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1} \]

6. Power series: do one of the two parts. If you do both parts your best work will count.

(a) Determine the radius of convergence and interval of convergence of the power series \[ \sum_{n=1}^{\infty} \frac{x^n}{n} \]
Remember to check the endpoints of the interval of convergence separately (if the interval has endpoints). Hint: you can use the Ratio or Root Test to determine for what values of \( x \) this converges (make sure you test for absolute convergence).

(b) Write down a power series for \( \frac{1}{1+x^2} \) using the formula for a geometric series. State the interval of convergence of this series (no need to use fancy tests, just facts about geometric series).
Integrate this series term by term (write the first five terms). What function have you just written a series for?
Differentiate this series term by term (write the first five terms). What function have you just written a series for?
7. Compute the first five nonzero terms of the Maclaurin series for \( \sin(x) \) (the Taylor series for \( a = 0 \)). Show all supporting work.

Write the first five nonzero terms of the derivative of this series. What function should this be a series for?
#1 \[ \int_{1}^{4} \frac{1}{x} \, dx \]

\[
\begin{align*}
\frac{1}{1} &+ \frac{1}{1.5} + 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2.5} + 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3.5} + \& 4\frac{1}{4} \\
\frac{1}{2} \left( \frac{1}{1} + \frac{1}{1.5} + 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2.5} + 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3.5} + \& 4\frac{1}{4} \right) \\
\end{align*}
\]

#2 \[ \int \cos^3(x) \sin^2(x) \, dx \]

\[
\begin{align*}
\int (1 - \sin^2(x)) \sin^2(x) \cos(x) \, dx \\
&= \int (1 - u^2) u^2 \, du \\
&= \int u^2 - u^4 \, du \\
&= \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C
\end{align*}
\]
3a. \[ \int \frac{x}{\sqrt{1-x^2}} \, dx \quad x = \sin(u) \]

\[ \text{these variables MUST be different} \]

b. \[ \int \sqrt{x^2 + 9} \, dx \quad x = 3 \tan(u) \]

c. \[ \int \frac{\sqrt{x^2 - 4}}{x} \, dx \quad x = 2 \sec(u) \]

3a. The integral:

\[ x = \sin(u) \]

\[ dx = \cos(u) \, du \]

\[ \int \frac{x}{\sqrt{1-x^2}} \, dx = \int \frac{\sin(u)}{\sqrt{1-\sin^2(u)}} \cos(u) \, du = \int \frac{\sin(u)}{\cos(u)} \, du = \int \sin(u) \, du \]

\[ = -\cos(u) + C \]

\[ = -\sqrt{1-x^2} + C \]
3b. the integral (this one is nasty)

\[ x = 3 \tan(u) \quad \frac{x}{3} = \tan(u) \]

\[ dx = 3 \sec^2(u) \, du \]

\[ \int \sqrt{x^2 + 9} \, dx = \int \sqrt{9 \tan^2(u) + 9} \, 3 \sec^2(u) \, du \]

\[ = \int 3 \sec(u) \cdot 3 \sec^2(u) \, du = 9 \int \sec^3(u) \, du \]

\[ v = \sec(u) \quad w = \tan(u) \]

\[ \int \sec^3(u) \, du = \sec(u) \tan(u) - \int \tan(u) \sec(u) \tan(u) \, du \]

\[ = \sec(u) \tan(u) - \int \tan^2(u) \sec(u) \, du \]

\[ = \sec(u) \tan(u) - \int (\sec^2(u) - 1) \sec(u) \, du \]

\[ \int \sec^3(u) \, du = \sec(u) \tan(u) - \int \sec^3(u) \, du + \int \sec(u) \, du \]

\[ 2 \int \sec^3(u) \, du = \sec(u) \tan(u) + \ln |\sec(u) + \tan(u)| + C \]

So

\[ 9 \int \sec^3(u) \, du = \frac{9}{2} \left[ \sec(u) \tan(u) + \ln |\sec(u) + \tan(u)| \right] + C \]

\[ = \frac{9}{2} \left[ \frac{\sqrt{9 + x^2}}{3} \left( \frac{x}{3} \right) + \ln \left| \frac{\sqrt{9 + x^2}}{3} + \frac{x}{3} \right| \right] + C \]
3c. the integral

\[
\int \frac{\sqrt{4\sec^2(u)-4}}{2\sec(u)} \, du = \int \frac{2\tan(u)}{2\sec(u)} \sec(u) \tan(u) \, du
\]

\[
= 2 \int \frac{\tan^2(u)}{2} \, du = 2 \int \frac{\sec^2(u) - 1}{2} \, du
\]

\[
= 2 \left[ \frac{\tan(u) - u}{2} \right] + C
\]

\[
= 2 \left[ \frac{\sqrt{x^2-y} - \arccsc\left(\frac{x}{2}\right)}{2} \right] + C
\]
# 4

\[
\sum_{n=0}^{\infty} \frac{2^n}{n!} = \lim_{n \to \infty} \left( \frac{2^{n+1}}{(n+1)!} \right) = \lim_{n \to \infty} \frac{2}{n+1} = 0 < 1
\]

So this converges by the Ratio Test.

# 5

a. \[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\] Since \(\sum \frac{1}{n}\) diverges (p-series, \(p \leq 1\))

Since \(\frac{1}{n} \geq 0\) \(\Rightarrow\) non-negative \(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}\) converges by AST

Since \(\frac{1}{n+1} \leq \frac{1}{n}\) \(\Rightarrow\) decreasing \(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}\) converges conditionally

b. \[\lim_{n \to \infty} \frac{1}{n} = 0 \Rightarrow\] goes to 0

\[\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}\]

So this converges absolutely

\[\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}\]

Thus, this diverges by n-th term test.
6.

a. \[ \sum_{n=1}^{\infty} \frac{x^n}{n} \]

apply Ratio Test: \[ \lim_{n \to \infty} \left| \frac{x^{n+1}}{n+1} \right| = \lim_{n \to \infty} \frac{|x|}{n+1} < 1 \]

for \(-1 < x < 1\),

So this converges absolutely for \(-1 < x < 1\).

For \(x = 1\), we have \[ \sum_{n=1}^{\infty} \frac{1}{n} \] - this diverges \(p \leq 1\).

For \(x = -1\), we have \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \], which converges conditionally.

So radius of convergence is 1 and interval of convergence is \([-1, 1]\).
\[ \frac{1}{1 + x^2} = \frac{a}{1 - r} \quad \text{with} \quad a = 1, \quad r = -x^2 \]

So a series for this is

\[ 1 - x^2 + x^4 - x^6 + x^8 - x^{10} \ldots \]

\[ |r| < 1 \quad \text{when} \quad |1 - x^2| < 1,\quad \text{when} \quad -1 < x < 1. \]

\[ \text{r.o.c. is 1. Interval is } (-1, 1), \]

\[ \int \frac{1}{1 + x^2} \, dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \quad (+C) \]

\[ = \arctan(x) + C \]

\[ \frac{d}{dx} \left[ \frac{1}{1 + x^2} \right] = -2x + 4x^3 - 6x^5 + 8x^7 - 10x^9 \ldots \]

\[ = -\frac{2x}{(1 + x^2)^2} \]
\[ f(x) = \sin(x) \]
\[ f'(x) = \cos(x) \]
\[ f''(x) = -\sin(x) \]
\[ f'''(x) = -\cos(x) \]
\[ f^{(4)}(x) = \sin(x) \]
\[ f^{(5)}(x) = \cos(x) \]
\[ f^{(6)}(x) = -\sin(x) \]
\[ f^{(7)}(x) = -\cos(x) \]
\[ f^{(8)}(x) = \sin(x) \]
\[ f^{(9)}(x) = \cos(x) \]

\[ f(0) = 0 \quad c_0 = 0 \]
\[ f'(0) = 1 \quad c_1 = 1 \]
\[ f''(0) = 0 \quad c_2 = 0 \]
\[ f'''(0) = -1 \quad c_3 = -\frac{1}{3!} \]
\[ f^{(4)}(0) = 0 \quad c_4 = 0 \]
\[ f^{(5)}(0) = 1 \quad c_5 = \frac{1}{5!} \]
\[ f^{(6)}(0) = 0 \quad c_6 = 0 \]
\[ f^{(7)}(0) = -1 \quad c_7 = -\frac{1}{7!} \]
\[ f^{(8)}(0) = 0 \quad c_8 = 0 \]
\[ f^{(9)}(0) = 1 \quad c_9 = \frac{1}{9!} \]

\[ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots \]

\[ \frac{d}{dx} \left[ \sin(x) \right] = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} + \cdots \]

\[ = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \]

\[ = \cos(x) \]