Math 175 Test I

Dr. Holmes

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This test will begin at 9:40 am and end at 10:35 pm. Please leave promptly if your paper is collected at 10:35 pm, as a courtesy to the next class. A scientific calculator without graphing or symbolic computation capabilities is allowed; other calculators are not. You may not use a cell phone or PDA as a calculator. Cell phones must be turned off and out of sight.

On your exam paper there will be a number which will be used to post your test grade online. Please record this number. Do not write any answers on your exam paper. Return your exam paper with your test (and make sure your name is on it so I can tell what your number is).

Write your name on your blue book AND on your exam paper (so that I can tell what your magic number is). Cross out any other names on your blue book (if I haven’t already). Take one of the blue books and write 1 on the first pair of facing pages, 2 on the second pair of facing pages, and so forth. Please do problem 1 on the pages numbered 1, problem 2 on the pages numbered 2, and so forth. If you need more space for a problem, do the additional work in your second blue book (and tell me that there is extra work in the second blue book, and clearly label it). If you need another blue book in the course of the exam, please ask for one.
1. Evaluate the given definite integral using the given substitution. You are required to substitute appropriate values in the bounds (you may not return from an expression in terms of $u$ to an expression in terms of $x$). Show all work.

$$\int_0^2 \frac{x}{\sqrt{x^2 + 1}} \, dx; \quad u = x^2 + 1$$

2. A region is pictured. It is bounded by $y = \sqrt{x}$ and $y = \frac{1}{2}x$. You need to set up the integrals in both parts and evaluate one of them.

(a) Determine the area of this region by setting up and evaluating an integral with respect to $x$. Show your integral setup and (maybe) evaluate the integral.

(b) Determine the area of this region by setting up and evaluating an integral with respect to $y$. Show your integral setup and (maybe) evaluate the integral.
3. Volumes by disks and washers.
A region is pictured. It is bounded by \( y = 1 - x^3 \) and the coordinate axes.

(a) Determine the volume of the solid obtained by revolving this region around the \( x \) axis using the method of disks and washers. Show your integral setup and evaluate the integral.

(b) Set up but do not evaluate the integral representing the volume of the solid obtained by revolving this region around the axis \( y = 3 \) using the method of disks and washers.

4. Volumes by cylindrical shells. Set up integrals for all three parts: you only need to evaluate one of part a or part b.
A region is pictured. It is bounded by the line \( y = 2x \), the line \( x = 1 \) and the \( x \)-axis.

(a) Determine the volume of the solid obtained by revolving this region around the \( x \) axis by the method of cylindrical shells. Show setup of the appropriate integral and evaluate it.

(b) Set up but do not evaluate the integral which would be used to determine the volume of the solid obtained by revolving this region about the line \( x = 2 \) by the method of cylindrical shells.

5. Combined arms.
(a) Evaluate the volume of the solid obtained by revolving the region bounded by \( y = 2x - x^2 \) and the \( x \)-axis about the \( y \)-axis by your choice of method. Set up an integral and evaluate it.
(b) Consider the “triangular” region in the first quadrant bounded by \( y = \sin(x) \), \( y = \cos(x) \), and the \( y \)-axis (picture provided). We want to determine the volume of the solid obtained by revolving this region about the \( y \)-axis. Set up but do not evaluate (you won't be able to!) integrals or sums of integrals representing this volume by both the method of disks and washers and the method of cylindrical shells (this is two different setups).

6. Arc length

Set up and evaluate the integral representing the length of the curve parameterized by \( x = t^2 \), \( y = t^3 \), \( 0 \leq t \leq 2 \).

Correct setup is a very large portion of the value of this problem.

Hint: you will need to pull stuff out from under the radical to get the integral into a form where substitution will work.
#1 \int_0^2 \frac{x}{\sqrt{x^2 + 1}} \, dx = \int_0^{2^2 + 1} \frac{1}{\sqrt{u}} \left( \frac{1}{2} \, du \right) = \frac{1}{2} \left[ \frac{u^{1/2}}{(y^2)} \right]_1^{5} = \sqrt{5} - 1.

u = x^2 + 1 \quad du = 2x \, dx \quad x \, dx = \frac{1}{2} \, du

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#2

\begin{align*}
\text{a.} & \quad \int_0^4 \sqrt{x} - \frac{1}{2} x \, dx \\
& = \left[ \frac{x^{3/2}}{\frac{3}{2}} \right]_0^4 - \left[ \frac{x^2}{4} \right]_0^4 \\
& = \frac{2}{3} \cdot 4^{3/2} - \frac{16}{4} \\
& = \frac{2}{3} \cdot 8 - 4 = \frac{4}{3}
\end{align*}

\begin{align*}
\text{b.} & \quad \int_0^2 2y - y^2 \, dy \\
& = \left[ y^2 - \frac{y^3}{3} \right]_0^2 \\
& = 4 - \frac{8}{3} = \frac{4}{3}
\end{align*}
#3

\[ y = 3 \]

\[ y = 1 - x^3 \]

\[ (0,0) \]

\[ (\pi, \pi) \]

\[ \int_0^1 \pi (1 - x^2)^2 \, dx = \pi \int_0^1 1 - 2x^2 + x^6 \, dx \]

\[ = \pi \left[ x - \frac{x^4}{2} + \frac{x^7}{7} \right]_0^1 \]

\[ = \pi \left[ 1 - \frac{1}{2} + \frac{1}{7} \right] \]

\[ = \pi \left[ \frac{1}{2} - \frac{1}{7} + \frac{2}{7} \right] \]

\[ = \frac{9}{14} \pi \]

b. \[ \int_0^1 \pi (3^2) - \pi (3 - (1 - x^3))^2 \, dx \]

Equivalently:

\[ 9\pi - \pi (2 + x^3)^2 \]
Test I solution, p. 3

1. \[
\int_0^2 2\pi y \left(1 - \frac{y}{2}\right) \, dy
\]

\[
= 2\pi \int_0^2 \frac{y^3}{2} - \frac{y^2}{2} \, dy
\]

\[
= 2\pi \left[ \frac{y^4}{8} - \frac{y^3}{6} \right]_0^2
\]

\[
= 2\pi \left[ \frac{16}{8} - \frac{8}{6} \right]
\]

\[
= 2\pi \left[ \frac{12}{6} - \frac{8}{6} \right] = 2\pi \cdot \frac{2}{3} = \frac{4}{3}\pi
\]

b. Note we are now writing in terms of x.

\[
\int_0^1 2\pi (2-x) \cdot 2x \, dx
\]
5.

a.

\[ y = 2x - x^2 \]

\[
\int_0^2 2\pi x (2x - x^2) \, dx
\]

\[ = 2\pi \int_0^2 2x^2 - x^3 \, dx \]

\[ = 2\pi \left[ \frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 \]

\[ = 2\pi \left( \frac{16}{3} - \frac{16}{4} \right) \]

\[ = 2\pi \cdot \frac{16}{12} = 2\pi \cdot \frac{4}{3} = \frac{8\pi}{3} \]

b.

\[ x = \cos y \]

\[ y = \sin(x) \]

\[ x = \pm \sin(y) \]

\[
\text{cylindrical shells:}
\]

\[ \int_0^{\sqrt{2}} 2\pi x (\cos(x) - \sin(x)) \, dx \]

\[ \int_0^{\sqrt{2}} \pi (\cos(x))^2 - \pi (\sin(x))^2 \, dx \]

\[ \int_0^{\sqrt{2}} \pi \text{ (cylindrical pieces)} + \int_0^{\sqrt{2}} \pi (\cos(x))^2 \, dx + \int_0^1 \pi (\cos(x))^2 \, dx \]
\[
\int_0^2 \sqrt{(2t)^2 + (3t^2)^2} \, dt
\]

\[
= \int_0^2 \sqrt{4t^2 + 9t^4} \, dt
\]

\[
= \int_0^2 t\sqrt{4 + 9t^2} \, dt
\]

Let \( u = 4 + 9t^2 \)

Then \( du = 18t \, dt \)

\[ t \, dt = \frac{1}{18} \, du \]

\[
= \int_4^{40} \sqrt{u} \left( \frac{1}{18} \, du \right)
\]

\[
= \frac{1}{18} \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_4^{40}
\]

\[
= \frac{1}{18} \left[ \frac{2}{3} (40^{\frac{3}{2}} - 4^{\frac{3}{2}}) \right]
\]

\[
= \frac{1}{18} \left[ \frac{8}{27} \left( 100\sqrt{10} - 1 \right) \right]
\]

\[
= \frac{8}{27} \left( 100\sqrt{10} - 1 \right)
\]