Notes on Principia Mathematica

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These are my notes toward an interpretation of the Principia Mathematica of Russell and Whitehead (hereinafter *PM*). Manners forbid saying *the* interpretation, though I think in fact that Russell and Whitehead are very clear (and so usually really admit only one interpretation), mostly correct though often very awkward, and often seriously misinterpreted by those who currently interest themselves directly in the text of *PM*. Mathematical logic has moved past this text: working mathematical logicians do not usually concern themselves directly with this text, and may themselves have inherited misconceptions about what is in it.

p. xii I am very doubtful that it is really possible to eliminate use of real variables in favor of apparent variables in all cases. The Axiom of Reducibility, for example, cannot be stated in this way. I believe that in a computer formalization (parameters for which are a driving factor behind these notes) general propositions which are to hold for all possible type assignments to their real variables (thus excluding the use of quantifiers) and after this for all values to be assigned to the real variables, play an essential role.

p. xiv the extensional view that functions occur only through their values is very useful but does not give all the good things that Russell wants from it.

p. xv A first pass at "atomic propositions". These will consist of a primitive predicate (not to be understood as a pf) with a given arity, applying to a list of terms denoting individuals of that arity. It is desirable to have more generality, and allow introduction of primitive predicates which have more complex types (not just an arity but a specified possibly complex type for each argument).

p. xvi We express our view for these notes of molecular propositions. We disdain the Sheffer stroke as a distracting novelty and take, given atomic or molecular propositions P, Q, $\neg P$ and $P \vee Q$ as further molecular propositions.

p. xviii see note on xii

p. xix but note that what we think is an individual may be relative to context.

p. xx ϕx is literally the result of putting x in place of a. i.e, If ϕ is $a = b$, ϕx is $x = b$.

p. xxii A matrix is a function of variables which may include individuals, propositions or pfs, to which a prefix of quantifiers over each of its variables is applied to get a general statement. The values of a matrix are elementary (unquantified) propositions.

p. xxix a function appears only through its values: this seems to say nothing more than that we do not have primitive operations on functions not ultimately defined in terms of their application to values.

p. xxx "thus the matrix $\phi!x$ has the peculiarity that when a value is assigned to x, this value is a constituent of the result, but when a value is assigned to ϕ, this value absorbed in the resulting proposition and completely disappears". This paragraph supports the convention for substitution of a pf for a variable in applied position: the substitution is immediately carried out, a pf constant does not appear in applied position, ever.

p. xxxi continues the discussion of what happens when one replaces a pf variable in applied position, in a manner entirely consistent with my interpretation.

In our development, we do not use expressions $\phi!\hat{x}$ as variables: we would just use ϕ. This will be seen to simplify matters.

xxxii more support for how substitution works. The following discussion of "variable universals" is mostly notable for the fact that he says this is not what he is doing.

xxxiii I will draw no such distinction between matrices and other propositional functions. This is too convoluted. The distinction that I will draw (which serves all the formal purposes of the weird distinction described here) has to do with the assignment of orders to the types of general pfs. I am firmly convinced that the correct approach to quantification is that in section 10 which allows application of propositional connectives directly to non elementary propositions.

xxxix I do not believe that the assertion that functions occur only through their values has the consequences Russell says it does (causing all functions of pfs to be extensional). I believe that higher order quantification messes this up. Any function that you can actually exhibit will be extensional, this is true. But there is quite a lot of strength in the assertion that all functions are ones that are actually exhibited.

I am skeptical about the assertion that $\phi x \equiv_x \psi x$ implies $\phi\hat{x} = \psi\hat{x}$, because I rather like the interpretation under which a propositional function is actually a bit of text into which substitutions are to be made, and the discussion above (p. xxx) only makes sense from this standpoint. I'll discuss this further when I discuss the introduction of classes. It is not entirely impossible to motivate what he says from a textual standpoint, though. One could view a pf as an equivalence class of bits of text under logical equivalence.

p. xxxix "we now have to distinguish between classes of different orders composed of members of the same order". xli-ii contain implicitly the need for quantification over impredicative variables, I believe.

I need to understand the argument on pp xxxix-xliii.

p. xliv It is known that Appendix B is wrong. I need to examine the argument there and identify the error.

p. 1 We will in a formal development in fact replace the treatments of descriptive expressions and classes and relations with something else. The approaches taken by Russell are unnecessarily cumbersome.

p. 4, a very important statement: "But the limits to which the unrestricted variable is subject do not need to be explicitly indicated, since they are the limits of significance of the statement in whic h the variable occurs, and are therefore intrinsically determined by this statement" – i.e., types are deducible from context. Though as it happens this is a bit deceptive: the rule of modus ponens really requires some type considerations.

p. 5 very important == point (3) indicates that there is no subtyping, in particular applicable to orders within a type.

note the use of propositional letters, variable functions (these are the variable functions of unspecified order mentioned later, I believe).

p. 16 I will always follow the treatment in section 10 which permits propositional connectives to be applied freely to quantified propositions.

p. 17 considerations of type deduction make use of the same letter as a binder in different contexts ill-advised. It will allow additional type deductions and may cause a more specific type than intended to be deduced.

pp. 17-18 Important discussion of assertion of propositions with real variables in them, i.e., assertions of propositional functions.

p. 22 the functions of functions with which we are concerned are the extensional functions of functions.

p. 24 It is my intention to identify classes with certain pfs in a way which clearly works out as compatible with PM.

p. 30, top safety of using typically ambiguous names for universe and empty set.

p. 31 consider possibility of treating $R`y$ itself as primitive instead of using the admittedly awkward descriptions.

p. 48 on – discussion of the existence of different types of functions applicable to a fixed a and the impossibility of quantifying over all such functions.

p. 51 everything is individual, proposition, or function. Note that here we do not have the restriction of notation $\phi!x$ to matrices, an annoyance in the intro to the second edition. Here it just indicates predicativity.

Note the use of "predicate" for a first order function on p. 51, and note that it is strictly nonce, not to be used outside the current discussion.

p. 52 Here we are using the one step at a time approach to converting matrices to quantified functions, which makes much more sense. This avoids the horrors referred to in the note on p. xxxii.

p. 53 order defined, quite clearly and correctly.

pp. 53-4 The reasons why one does not need nonpredicative variables which are given appear to be spurious. Something like this is correct as soon as one has the axiom of reducibility. It is quite easy to define nonpredicative functions using nonpredicative variables, and such constructions might be needed in the absence of reducibility.

The given reasons why quantification over propositions can be dispensed with appear to be valid.

p. 58 The argument from existence of classes to reducibility is in my opinion a howler ;-) It also however brings to the fore my discomfort with the discussion on p. 25 re types and orders. classes *of the same order* with the same elements are equal (that is, of the same type – two "orders in the same type" are disjoint types).

p. 65 Here is an important discussion of typical ambiguity.

p. 72 discussion of extensional predicates of pfs. Note that "formally equivalent" must also include being of the same type.

p. 74 discusses the motivation of the contextual definition of classes.

p. 75 Russell frankly disagrees with what I say above – he allows formal equivalence between pfs of different types. He provides some support for the idea that functions of a given order in a simple type can be identified with particular functions of any higher order in the same type. This may require reducibility to be sensible, though. And he indeed appeals to reducibility.

p. 76 comprehension for classes seen to be motivated by reducibility – by Russell – but note that this is impredicative comprehension. As he says in the intro to the second edition, if one does not have reducibility one needs classes of many orders with the same elements – but the considerations here might suggest that classes of different orders with the same elements could be identified, so one would have subtyping with classes.

p. 77 Cls $\in$ Cls discussed, though I dont see the resolution.

pp. 77-84 read carefully to understand use of his class symbols.

p. 141 prop 10.14 he does know about perils of type information interacting with inference.

p. 164 the definition of "predicative" given here must be characterized as wrong. The definition given on p. 53 is workable, this one is not. A predicative function can sensibly be understood as one which is of the lowest order possible compatible with its arguments. This supports sensible formal reasoning about pfs or classes with predicativity restrictions in a familiar way. Defining "predicative" as "is a matrix" is completely destructive. Any statement about particular pfs can be reduced to statements about underlying matrices, that is true; but all general statements about pfs would be divided into cases depending on patterns of quantification, and there are *infinitely many* possible patterns of quantification, even for first order propositions. This is not a brilliant refinement; it can only be characterized as an error. On p. 162 he says that in practice we may without danger of reflexive fallacies treat first order functions as a type; in fact, I would say that we must in order to have a sensible system treat first order functions as a type, and the p. 164-5 developments would destroy this possibility as we could not in fact make any general statement about first order functions or even define any second order pf of interest.

p. 165 use of ϕ instead of $\phi!\hat{x}$ as a binder is a mere abbreviation.

p. 165 in practice we never need to know absolute types but only relative types.

The reason he gives on p. 165 for the possibility of using only predicative functions is actually spurious. [discussion required]. Putting a nonpredicative function in a particular context is equivalent to putting some predicative function in a *different* context. This is very doubtful. At the very least, if one does not quantify over functions in general, one needs many many additional cases (infinitely many, in fact) in situations where only one is really needed. Now, the axiom of reducibility has the full effect of making every function equivalent to a predicative function, so that this is harmlessly true.

I hadnt noticed at the time of writing the previous paragraph that the reduction of all variables to predicative variables actually only works well if the p. 53 definition is in use. It is also worth noting that the axiom of reducibility does NOT make every function equivalent to a predicative function in the p. 164 sense. One would need to be able to handle matrices of all arities, and this is not supported.

last paragraph of p. 165, again, present serious difficulties. Symbolizing a function whose order is not assigned, we can only wlog conclude that it is predicative if we already assume reducibility.

p. 167 The axiom of reducibility uses a real variable in a way which cannot be reduced to use of an apparent variable, becayse many assignments of type are possible.

p. xliii in the discussion of similarity: the problem arises because R may belong to a type with arbitrarily high order – note that types of arbitrarily high order (ie nonpredicative) are clearly supposed quantified over in this discussion. As soon as reducibility is abandoned, the restriction to quantification over predicative variables no longer makes sense, and he no longer thus restricts himself.