New Foundations is consistent (from the top, new document)

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1 Version Notes

8/18/2015: Further minor revisions.

8/14/2015: Minor revisions, plus a generalization of the “relevant FM construction” allowing unspecified action of the permutation on the parent set (exactly the sort of action which really does occur in the various constructions), which probably should be imported into other versions.

8/3/2015: Labelling things as background or main construction and adding linking material.

8/2/2015: The first version of this series. The sections about the main construction (8,9,10) are actually self-contained, with material from previous versions attached before and after. Conflicts are possible. The material from different sources will be harmonized as I edit.

This version directly constructs a model of tangled type theory. The advantage over a tangled web approach is that there is no elementarity argument.
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2 The Starting Point (background)

The sections labelled “main argument” are in intention self-contained, though probably hard to understand without the “background” sections. There are some “motivational” comments or subsections in the main construction which may refer to the background material because they actually are background material and not part of the main argument.

The beginning of our story is at a point which might be regarded as the end of the story of the *Principia Mathematica* of Russell and Whitehead (PM, [19]). This is the system called TST by Thomas Forster (for example, in [2], the best current monograph on the subject of NF), the simple typed theory of sets. This is a first-order theory with sorts indexed by the natural numbers 0,1,2... and equality and membership as primitive relations. The sorts are traditionally called “types”. Atomic identity sentences $x = y$ are well-formed iff $x$ and $y$ are variables of the same type; atomic membership sentences $x \in y$ are well-formed iff the index of the type of $y$ is the successor of the index of the type of $x$. The axioms are extensionality (objects of each positive type are equal iff they have the same elements) and comprehension (for any formula $\phi(x)$, there is an $A$ such that $x \in A \leftrightarrow \phi(x)$, where $A$, obviously one type higher than $x$ and unique by extensionality, can be denoted by $\{x : \phi(x)\}$).

We regard this as the end of the story of PM (not necessarily an uncontro- versial view) because this is a simple and natural system realizing the aims of PM. The type system of PM is considerably more complicated than that of TST for two reasons: the first reason is that Russell and Whitehead did not know how to implement ordered pairs as sets (Norbert Wiener gave the first implementation in [20], 1914), so PM contains types of $n$-ary relations for every $n$ with arbitrarily complex heterogeneous input types; the second reason was that Russell and Whitehead restricted themselves initially to predicative comprehension, then made their system impredicative by adding an axiom of reducibility; TST follows Ramsey ([11]) in using fully impredicative comprehension and no axiom of reducibility. We do know that Russell tried to abandon reducibility in the second edition of PM, but we also know that much of the mathematics in PM does not survive this. TST and the system of PM with reducibility are mutually interpretable.

It is usual to add axioms of Infinity and Choice to TST, but we do not regard these as part of the basic definition. TST + Infinity (+ Choice) has the same mathematical strength as Zermelo set theory with separation restricted to bounded formulas (a system sometimes called Mac Lane set theory). This is weaker than Zermelo set theory, but easily strong enough for almost all mathematics outside of set theory. TST without Infinity is weaker than arithmetic, but this is no part of our story.

There is a very natural interpretation of TST in terms of the familiar set theory ZFC: let $X_0$ be any set and define $X_{i+1}$ as the power set of $X_i$ for each $i$. Interpret type $i$ variables as ranging over $X_i$. Interpret equality and membership relations between each appropriate pair of types as suitable restrictions of the usual equality and membership relations. Notice that there is no requirement
here that the sets interpreting the types be disjoint: we in fact cannot even ask
the question in the language of TST as to whether objects of distinct types are
equal.

**Definition (initial segments of type theory, natural models):** We define

\[ \text{TST}_n \]

as the theory obtained by restricting the language of TST to men-
tion only types with index less than \( n \) and having as its axioms exactly
the axioms of TST which can be expressed in the restricted language. A

natural model of \( \text{TST}_n \) is one in which type \( i \) is represented by a set \( X_i \),
with \( X_{i+1} \) being of the same size as the power set of \( X_i \) for each appropri-
ate \( i \), with the membership relation being defined using a fixed bijection
\( f_i \) from \( X_{i+1} \) to \( \mathcal{P}(X_i) \): \( a \in b^{i+1} \) in the type theory is implemented as
\( a \in f_i(b) \) in the metatheory. It is an important observation that the first
order theory of a natural model of \( \text{TST}_n \) is completely determined by the

cardinality of the set \( X_0 \) representing type 0 in the model (it is straight-
forward to construct an isomorphism between natural models with base
sets representing type 0 of the same size).
3 The Hall of Mirrors: the formulation of New Foundations (background)

The next chapter in the story is an observation made by Russell about PM (under the heading of “systematic ambiguity”) and made in a much sharper form by Quine about TST. The system is extremely symmetric, in the sense that there is nothing in the system to distinguish the sorts from one another (one has no information at all about the type of individuals, and uniformly more information about types of higher index, but nothing obvious to show that the type 0 of individuals might not have the same structure as the type 1 of sets of individuals, whence each type \( n \) would have the same structure as type \( n + 1 \)). In TST, this can be stated in a very elegant way. For any sentence \( \phi \), let \( \phi^+ \) be a sentence obtained by raising the type of each variable in \( \phi \) by one (without creating any identifications between variables). Since all variables in \( \phi \) are bound, the exact way that the new variables are chosen does not matter. We observe then that if \( \phi \) is provable, so is \( \phi^+ \), and if we define a mathematical construction using a set abstract \( \{ x : \phi \} \) and our scheme of variable type raising sends \( x \) to \( y \), \( \{ y : \phi^+ \} \) will be a precisely analogous mathematical construction one type higher. This is the “hall of mirrors” aspect of TST: for example, it is natural in TST to define the number three as in effect the set of all sets with three elements, following Frege and also PM, but we then get a new version of the number three in each type \( i + 2 \).

Quine made the stronger suggestion that we should simply identify all the types, to avoid the apparently futile recopying of all theorems and defined objects into ever higher types. This gives the theory which is generally called New Foundations (NF) after the title of the paper [10], 1937, where he made the suggestion. NF is the first order single-sorted theory with equality and membership whose axioms are obtained from those of TST by dropping all distinctions of sort between the variables (without introducing any identifications between variables). That is, the axioms are extensionality (objects with the same elements are the same) and stratified comprehension (\( \{ x : \phi(x) \} \) exists for \( \phi \) a stratified formula), where we say that \( \phi \) is stratified iff there is a function \( \sigma \) from variables to natural numbers with the property that for each atomic subformula \( x = y \) of \( \phi \) we have \( \sigma(x) = \sigma(y) \) and for each atomic subformula \( x \in y \) of \( \phi \) we have \( \sigma(x) + 1 = \sigma(y) \). Notice that the function \( \sigma \) (called a stratification of \( \phi \)) is acting on \( x \) and \( y \) as bits of text, not on their values, so we should perhaps put the variables in quotes (but do not here do this). Of course, the condition that a formula of the language of set theory is stratified is equivalent to the condition that it is obtainable from a formula of the language of type theory by dropping type distinctions between variables without introducing identifications between variables.

It is a persistent criticism of NF that it is a syntactical trick. Of course, as phrased here, it does look that way. It is possible to give a finite axiomatization of NF, which eliminates the notion of stratification from the explicit definition of the theory (though the very first thing one would do with such a formulation...
of the theory would be to prove stratified comprehension as a meta-theorem). The standard reference is Hailperin, ([3]).

In terms of the interpretation of TST suggested above, this is a very strange proposal. If type $i + 1$ is represented by exactly the same set as type $i$, it is certainly not represented by (a set the same size as) the power set of the set representing type $i$, which is a larger set by Cantor’s theorem.
4 Good News and Bad News: well-known results about New Foundations (background)

Specker showed in [16], 1962, that NF is equiconsistent with TST + the Ambiguity Scheme which asserts $\phi \iff \phi^+$ for each sentence $\phi$, which is not surprising given the motivation of the theory.

Specker also showed, much more surprisingly, in [15], 1953, that Choice is inconsistent with NF, which implies that Infinity is a consequence of NF (as if the universe were finite, every partition, being finite, would have a choice set). [Quine’s argument for Infinity in the original NF paper is fallacious]. This shows that NF is stronger than expected, but it also shows that it is very strange, and caused considerable doubt about this theory.

On the good news side of things, Jensen showed in [9], 1969, that NFU (New Foundations with urelements) is consistent. We outline his approach.

The idea is to replace TST with TSTU, in which the axiom of extensionality is weakened to apply only to objects with elements, so that each positive type contains at most one element with each nonempty extension, but may contain many elements with empty extension (urelements or atoms). Note that the individuals of type 0 are not atoms, or at least are not considered as atoms: we simply do not ask the question as to what elements they have.

The results of Specker can be extended to show that NFU (New Foundations with the weaker form of extensionality and full stratified comprehension) is equiconsistent with TSTU + Ambiguity.

We now argue, following Jensen, for the consistency of TSTU + Ambiguity. We work in some familiar set theory (we use nothing like the full power of ZFC, but we may suppose that to be our working theory). Let $X_0$ be a set and define $X_{i+1}$ as the power set of $X_i$ for each $i$. Let each type $i$ variable in the language of TSTU be interpreted as ranging over elements of $X_i$; interpret equality and membership in TSTU as equality and membership suitably restricted. Thus far we have actually interpreted TST. Now we throw in a refinement. Let $s$ be a strictly increasing sequence of natural numbers. An alternative interpretation takes variables of type $i$ as ranging over elements of $X_{s_i}$, takes equality between type $i$ objects as equality suitably restricted, and interprets membership of type $i$ objects in type $i + 1$ objects as holding where the type $i + 1$ object is an element of $X_{s_{i+1}}$ which has the type $i$ object as an element; note that this interpretation treats all elements of $X_{s_i+1} \setminus X_{s_i+1}$ as urelements (it should be noted that the relation interpreting membership of type $i$ objects in type $i + 1$ objects will not necessarily agree with the relations interpreting membership between other successive types). It is straightforward to establish that this gives an interpretation of TSTU for each increasing sequence $s$. Now choose a finite set $\Sigma$ of formulas of the language of TSTU mentioning no types with index higher than $n − 1$. Define a partition of the $n$ element subsets $A$ of the natural numbers determined by the truth values of the sentences in $\Sigma$ in interpretations of TSTU determined by maps $s$ which have $A$ as an initial segment of their range (the truth value of sentences in $\Sigma$ is entirely determined by the first $n$
elements of the range of the function \( s \) used). This is a partition of the \( n \) element subsets of \( \mathbb{N} \) into finitely many parts (no more than \( 2^{\aleph_0} \)) and so has an infinite homogeneous set \( H \) which is the range of a strictly increasing map \( h \). In the interpretation of TSTU determined by the map \( h \), we have ambiguity for all formulas in \( \Sigma \). This implies by compactness that full Ambiguity is consistent with TSTU, and by the results of Specker that NFU is consistent.

It is useful to note one could use instead of the sets \( X_i \), sets \( X_\alpha \) indexed by elements of any limit ordinal \( \lambda \) (taking unions at limit indices), whereupon the sequences \( s \) would be strictly increasing sequences of ordinals below \( \lambda \) (still indexed by natural numbers); for example, the stages of the cumulative hierarchy up to any limit level could be used. This is relevant to establishing the consistency of strong extensions of NFU.

It is clear that if Choice holds in our working set theory, Choice will hold in all the approximations to NFU obtained by the method above, and so is consistent with NFU, and that if all \( X_i \)'s are finite, the approximations of NFU obtained by the method above will not satisfy Infinity, and so NFU does not prove Infinity. If \( X_0 \) is infinite, Infinity will hold, of course. NFU by itself is weaker than Peano arithmetic, as it happens. NFU + Infinity is a quite usable foundational theory equivalent in strength to TST + Infinity or to Mac Lane set theory.

Other fragments of NF have been shown to be consistent, but the strategy we will follow to show the consistency of full NF follows in its broadest outlines the strategy of Jensen for NFU (though this turns out to be quite hard to do).
5 Tangled Type Theories and Tangled Webs of Cardinals (background, all subsections)

The first section introduces the theory actually implemented in the main construction. The second section is relevant to other approaches to proving the result, as well as having intrinsic interest. The third section is relevant to strong extensions of our result discussed in the conclusion.

5.1 Tangled Type Theories

Tangled type theory TTT is a first order theory with types indexed by the natural numbers, or more generally by ordinals below a fixed limit ordinal \( \lambda \).

Sentences \( x = y \) are well-formed iff the type of \( x \) is equal to the type of \( y \).

Sentences \( x \in y \) are well-formed iff the type of \( x \) is strictly less than the type of \( y \) (it is important to notice that this is not a cumulative type theory). For any formula of the language of the usual TST and any strictly increasing sequence \( s \), if \( \phi \) is a formula of the language of TST we get a formula \( \phi^s \) of the language of TTT by replacing type \( i \) variables in \( \phi \) with type \( s(i) \) variables in \( \phi \) while preserving identifications and distinctions of variables. The axioms of TTT are the formulas \( \phi^s \) for which \( \phi \) is an axiom of TST, for all strictly increasing sequences \( s \).

If NF is consistent, TTT is consistent. This is evident: take each type to be the model of NF (or if you prefer let the types be disjoint labelled copies of the model of NF) and use the membership and equality relations of the model of NF to induce the membership and equality relations between each appropriate pair of types in the obvious way.

We show that the consistency of TTT implies the consistency of NF.

If TTT is consistent it has a model. Enhance the language of TTT with an additional relation \( \leq \). \( x \leq y \) is well-formed iff \( x \) and \( y \) are of the same type and the truth value of \( x \leq y \) is determined by whether \( x \) appears before \( y \) in a fixed well-ordering of the model. Of course \( \leq \) does not appear in instances of the comprehension axiom, but it will satisfy the axioms that \( \leq \) is a total order and that any definable class (including ones defined in terms of \( \leq \)) which is nonempty has a \( \leq \)-minimal element.

Now consider any finite collection of formulas \( \Sigma \) of the language of TST enhanced to contain the predicate \( \leq (x \leq y \text{ well-formed under the same conditions as } x = y) \). Let \( n \) exceed the largest type index appearing in any formula in \( \Sigma \). Partition the \( n \)-element subsets \( A \) of \( \omega \) into \( \leq 2^{\omega} \) compartments by considering the truth values of the formulas \( \phi^s \) for \( \phi \in \Sigma \) and \( s \) such that \( s|n = A \). An infinite homogeneous set \( H \) exists for this partition by the Ramsey theorem and may be chosen to have order type \( \omega \) in the natural order on ordinals, if the types are taken from a \( \lambda > \omega \). The model of TST obtained in the obvious way from the types of the model of TTT with index in the set \( H \) satisfies Ambiguity \((\phi \leftrightarrow \phi^+)\) for each sentence in \( \Sigma \).

By compactness the full Ambiguity Scheme \( \phi \leftrightarrow \phi^+ \) is consistent with TST.
with the additional predicate $\leq$, the new predicate not appearing in instances of comprehension, with the new axioms that $\leq$ is a total order and every definable class has a $\leq$ minimal element, plus Ambiguity for all formulas. Now observe that this extended theory supports the construction of a Hilbert symbol: $(\theta x.\phi)$ can be defined as the $\leq$-minimal element of its type which satisfies $\phi$ if there is one and otherwise as a default object (use the empty set). One can then construct a term model of the theory built entirely from Hilbert symbols, abandon the distinctions of type between terms of different types, and obtain a model of NF.

The use of the external well-ordering removes the necessity to appeal to the rather complex argument for the equiconsistency of NF with TST+Ambiguity given originally by Specker, involving saturated models.

We can get to stronger theories by indexing the types in TTT with larger ordinals (sequences $s$ will then be strictly increasing sequences of ordinals, still indexed by natural numbers). The development above works (as already noted) if the types of TTT are taken to be ordinals less than a fixed limit ordinal $\lambda$. In fact, it works if the types are indexed by any linearly ordered set with no maximum element.

TTT does not embed nicely into ordinary set theory as TST does: there are no natural models of TTT. The reason for this is that each type in TTT has many types “immediately above” it, which cannot represent real power sets of the types, as for any two of these “power sets” of a given type, one is ostensibly a “power set” of the other!

To restore the possibility of thinking of “successor types” as power sets, one might modify the type scheme as follows: types are labelled not by ordinals $\leq \lambda$ but by nonempty finite subsets of $\lambda$. Type $A$ is a copy of type $\min(A)$ of TTT, and its unique “power type” is type $A \setminus \{\min(A)\}$. Sequences of types copied from a given sequence of types in TTT will be elementarily equivalent to one another.
Here is a picture of a part of the type hierarchy, on the left as in TTT and on the right as unfolded.

We show in the following section that we can associate these modified types with cardinals, and embed the entire definition into untyped set theory of the familiar sort.

5.2 Tangled Webs of Cardinals

We articulate a hypothesis about cardinals in ZFA (ZFC sans choice and with extensionality weakened to allow atoms) whose consistency with ZFA implies the consistency of NF. Note that we can use Scott’s trick ([13]) to define cardinals in this theory; the usual definition using initial ordinals will not work when choice is not assumed. The treatment here is derived from our [5], but the notation is greatly improved.

Let $\lambda$ be an infinite limit ordinal (it could be taken to be $\omega$ for the purposes of merely proving $\text{Con}(\text{NF})$ but we aim for more generality). Fix a natural number constant $k$.

We define a tangled web of order $\lambda$ as a function $\tau$ sending nonempty finite
subsets of $\lambda$ to cardinals with two properties:

**naturality:** If $A$ has at least two elements, $2^{\tau(A)} = \tau(A \setminus \{\min(A)\})$

**elementarity:** For each $n$, if $A$ has at least $n+k$ elements, the first-order theory of a natural model of TST$_n$ with type 0 having cardinality $\tau(A)$ is completely determined by the smallest $n+k$ elements of $A$. [the use of $n+k$ here rather than $n$ is a technicality which arose in the course of constructions of tangled webs: the actual value was $k = 1$.]

It may not be immediately evident, but the definition of a tangled web of cardinals is precisely motivated by the desire to replicate the consistency proof of Jensen for NF (the material about tangled type theory at the beginning of the section may make the intellectual genealogy of the proof clearer).

We argue that the existence of a tangled web of cardinals implies the consistency of NF. Let $\Sigma$ be a finite set of sentences of the language of TST mentioning no variable of type $\geq n$. Partition the finite subsets $A$ of $\lambda$ of size $n+k$ by considering the truth values of the sentences in $\Sigma$ in natural models of TST with base type of cardinality $\tau(B)$ where $B$ has at least $n+k$ elements and the smallest $n+k$ elements of $B$ are exactly the elements of $A$. By Ramsey’s theorem there is an infinite homogeneous subset $H$ of $\lambda$ for this partition. Note that for any subset $B$ of $H$ with at least $n+k$ elements, the theory of a model of TST$_n$ with base set representing type 0 of cardinality $\tau(B)$ will assign the same truth values to sentences in $\Sigma$. It follows that for any finite subset $B$ of $H$ with at least $n+k$ elements, we find that ambiguity holds for all sentences in $\Sigma$ in a model with base set representing type 0 of size $\tau(B)$: notice that the power set of this set will be of size $\tau(B \setminus \{\min(B)\})$, and the argument of this expression is also a subset of $H$ with at least $n+k$ elements. Thus the theory of the natural model of TST$_n$ with base type $T$ of size $\tau(B)$ agrees with the model of TST$_n$ with base type $P(T)$ about the truth value of each sentence in $\Sigma$, from which it follows that $\phi \leftrightarrow \phi^+$ holds in the model of TST$_{n+1}$ with base type $T$ for each $\phi$ in $\Sigma$. Thus we find that ambiguity for $\Sigma$ is consistent with TST$_m$ for all $m > n$, so with TST. Thus full Ambiguity is consistent with TST by compactness, and NF is consistent by the results of Specker.

All of the above is an adaptation of Jensen’s proof of the consistency of NFU to the proof of the consistency of NF. Of course, one then actually has to produce a tangled web of cardinals (or a model of tangled type theory) to complete the argument.

### 5.3 $\omega$- and $\alpha$-models from tangled structures

Jensen continued in his original paper by showing that for any ordinal $\alpha$ there is an $\alpha$-model of NFU. We show that if $\lambda$ is taken to be large enough, this argument can be reproduced for NF (given a model of TTT or a tangled web with large enough index $\lambda$).

We quote the form of the Erdös-Rado theorem that Jensen uses: Let $\delta$ be an uncountable cardinal number such that $2^\beta < \delta$ for $\beta < \delta$ (i.e., a strong limit
cardinal). Then for each pair of cardinals $\beta, \lambda < \delta$ and for each $n > 1$ there exists a $\gamma < \delta$ such that for any partition $f : [\gamma]^n \to \lambda$ there is a set $X$ of size $\beta$ such that $f$ is constant on $[X]^n$ ($X$ is a homogeneous set for the partition of size $\beta$).

We assume the existence of a model of TTT in which each type contains a well-ordering of type $\alpha$, witnessed by objects named with the ordinals $< \alpha$ in each type. Note that this does not imply directly that the model of NF we eventually obtain, which comes from an application of compactness, will actually contain such an order: many nonstandard elements might be added to it, and it might cease to be externally a well-ordering at all. We need to do extra work.

Let $\delta$ be a strong limit cardinal with cofinality greater than $|\alpha|$. Let $\Sigma_n$ be the collection of all sentences of the language of TST$_n$, beginning with an existential quantifier restricted to $\alpha$, in a language which includes constants for all ordinals $< \alpha$ (represented internally as usual in TST) for all sufficiently large types. Let the partition determined by $\Sigma_n$ make use not of the truth value of the formulas, but of the minimal witness in $\alpha$ to their truth, or $\alpha$ if they are false. The Erdős-Rado Theorem in the form cited tells us that we can find homogeneous sets of any desired size less than $\delta$ for this partition, and moreover (because of the cofinality of $\delta$) we can find homogeneous sets of any desired size with the same witnesses taken from $\alpha$ for each sentence in $\Sigma$. This allows us to see that ambiguity of $\Sigma_n$ is consistent, and moreover consistent with standard values for witnesses to each of the formulas in $\Sigma$. We can then extend the determination of truth values and witnesses as many times as desired, because if we add new formulas to bring it up to $\Sigma_{n+1}$ we can take a large enough set homogeneous for the previously given conditions to ensure homogeneity for the partition determined by the new conditions. After we carry out this process for each $n$, we obtain a full description of a model of TST + Ambiguity with standard witnesses for each existential quantifier over $\alpha$. We can reproduce our Hilbert symbol trick (add a predicate representing a well-ordering of our model of TTT to the language as above) to pass to a model of NF with the same characteristics.
6 A motivational remark (background)

In a model of TST, type $\tau + 1$ is the “power set” of type $\tau$. In a natural model, it really is the same size as the power set. In a model of TTT, every type $\tau$ above a type $\sigma$ is a “power set” of type $\sigma$.

Consider type $\sigma - 1$, type $\sigma$ and a top type $\tau$, where we propose to vary $\tau$. A striking point is that however we vary $\tau$ (say, increase it, to obtain “power sets” which must in some sense be “larger”, though we do not know that they increase in a monotone manner), the available pool of set unions of type $\tau$ collections of type $\sigma$ objects qua collections of type $\sigma - 1$ objects will remain the same.

This suggests that to construct a model of TTT, we want to look for ways to have large collections of objects in a type such that we can add new collections of these objects freely in a way which does not create any new set unions in any lower type.

We say that an object of type $n$ in a model of TST is $k$-symmetric iff it is invariant under all permutations of type $n - k$. If $A$ is $k$-symmetric, $\bigcup^k A$ is either the universe or the empty set. So a collection of $k$-symmetric objects, whose $(k + 1)$-st iterated union will be empty or universal, is a candidate for the kind of collection we are looking for.

There are only two 1-symmetric objects. 2-symmetric objects, on the other hand, include cardinals, and we can use Fraenkel-Mostowski methods to construct arbitrarily large collections of indistinguishable cardinals. Cardinals are disjoint objects, so the union of a set of cardinals is uniquely determined by the set. The union of the union of the collection of cardinals is a collection of assorted sets of these cardinalities. The union of the union of the union of a set of cardinals is of course either empty or the universe.

We make a further application of FM methods to ensure that unions of unions of sets of cardinals taken from our large set of indistinguishable cardinals are sets of a highly stereotyped kind, and that adding additional sets of these cardinals and so additional unions of these sets does not add any more unions of unions. The exact combinatorics are described in the next section.
7 A relevant FM construction (background)

The phenomena illustrated in this section happen all through the construction that follows, although the combinatorics revealed here are not explicitly discussed. [The junk objects in each type have a similar structure, although they are not always atoms].

We review the requirements for the Frankel-Mostowski construction (refer to [8]). Any permutation $\pi$ of the atoms in ZFA can be extended to a class permutation of the universe by the convention $\pi(A) = \pi^"A$. A Frankel-Mostowski interpretation is determined by a group $G$ of permutations of the atoms (always considered to be extended to the universe as indicated) and a collection $F$ of subgroups of $G$ (a “normal filter”) satisfying the following conditions:

1. $G \in F$,
2. $H \in F \land H \subseteq K \rightarrow K \in F$,
3. $H \in F \land K \in F \rightarrow H \cap K \in F$,
4. $\pi \in G \land H \in F \rightarrow \pi H \pi^{-1} \in F$ (normality condition).
5. Further, the group of permutations in $G$ such that $\pi(p) = p$ should belong to $F$ for each atom $p$.

We then say that a set $A$ is $F$-symmetric iff the group of permutations in $G$ which fix $A$ belongs to $F$. The objects in the domain of the FM interpretation are the atoms and the sets which are hereditarily $F$-symmetric. The membership relation of the FM interpretation is the restriction of the membership relation of our ambient ZFA to this domain. The grand theorem which we are using but not proving asserts that this class structure satisfies ZFA as well (but generally not choice).

We fix a regular uncountable cardinal $\kappa$. We will refer to sets of cardinality less than $\kappa$ as “small” and all other sets as “large”. We suppose that $P$ is a large set (that is, $|P| \geq \kappa$) and there is a collection of atoms $\mathbb{A}$ equinumerous with $P \times \kappa$ (which are not necessarily all of the atoms present): let $f$ be a bijection from $P \times \kappa$ onto $\mathbb{A}$ and denote $f(p, \alpha)$ as $p_\alpha$.

We refer to the set $\{p_\alpha : \alpha < \kappa\}$ as litter($p$) and refer to such sets as litters. A near-litter is defined as a subset of $\mathbb{A}$ with small symmetric difference from a litter. $p$ is called the “parent” of litter($p$), and of any near-litter with small symmetric difference from litter($p$), and of any atom $p_\alpha$.

We let $G$ be the group of permutations $\pi$ of the atoms (which may include atoms not in $\mathbb{A}$) which have the property that $\pi($litter$(p)) \Delta \text{litter}(\pi(p))$ is small for each $p \in P$, and which further have the property that the action of $G$ on $P$ is restricted to an unspecified $G_0$. Another way of putting this is that a permutation is in $G$ iff it moves only a small number of elements of $\mathbb{A}$ in each litter in a way which is not expected from the action of the permutation on $P$ (and actions on $P$ may be further restricted in unspecified ways). The identity is clearly such a permutation, and inverses of such permutations and
compositions of such permutations are clearly such permutations. It is worth noting that near-litters are mapped to near-litters, and only a small number of elements of any near-litter are moved in a way not deducible from the way their parent is moved.

Let $S$ be a small set of atoms in $A$ and near-litters, such that the intersections of distinct near-litter elements of $S$ are large. Let $G_S$ be the collection of permutations in $G$ which fix each element of $S$ (where we apply the rule $\pi(A) = \pi^+ A$ in the case of the set elements of $S$). The filter $F$ consists of all subgroups of $G$ which include a $G_S$ as a subgroup.

The only condition on $F$ which requires much effort is the normality condition, and it does not really require much. Suppose that $H$ is an element of $F$, and so includes some $G_S$. We claim that $\pi H \pi^{-1}$ includes $G_{\pi(S)}$. Certainly $\pi H \pi^{-1}$ includes $\pi G_S \pi^{-1}$. We claim that $\pi G_S \pi^{-1}$ includes $G_{\pi(S)}$. Suppose that $\sigma \in G_{\pi(S)}$: we would like to show that $\pi^{-1} \sigma \pi \in G_S$: but this is immediate. For $x \in S$ (whether atom or near-litter), $\pi(x) \in \pi(S)$ is fixed by $\sigma$, so $\pi^{-1} \sigma \pi(x) = x$ as required.

It is worth noting that we could have required that our supports consist just of atoms and litters, but the use of near-litters simplifies the proof of the normality condition. From a given support, we obtain a support consisting entirely of atoms and litters by replacing each near-litter element $N$ with the litter $L$ with small symmetric difference from $N$ and in addition the atoms in $L \Delta N$ (which are called the anomalous elements for $N$).

We assume additional conditions on the action of permutations in $G$:

Existence of a special relation on $A$: There is a strictly well-founded transitive relation $\leq$ on $P \cup A$ such that the preimage under $\leq$ of each element $p_\alpha$ of $A$ is $\{p\}$ and for each $p \in P$, the collection of atoms $\leq p$ and litters $\text{litter}(q)$ with $q \leq p$ is a support for $p$, which we will call its strong support. A strong support of a general object is obtained by taking a support of the object made up of atoms and litters and adding the strong support of each parent of an atom or litter in the original support.

A condition on free action of permutations in $G$: For any small collection $\Sigma_1$ of atoms in $A$ and litters and small collection $\Sigma_2$ of elements of $A$ such that each element of $\Sigma_1$ has strong support not meeting $\Sigma_2$, any permutation of $\Sigma_2$ can be extended to a permutation in $G$ fixing each element of $\Sigma_1$.

observation that these conditions are trivially realizable: Notice that these conditions can be made to hold by making all elements of $P$ pure sets or at any rate requiring that no element of $P$ has an element of $A$ in its transitive closure: in this case every element of $P$ has empty support and the permutations in $G$ can be taken to be those which move each litter to a near-litter with the same parent, which clearly has the second property.

We demonstrate some properties of the resulting FM interpretation.
All small subsets of the domain of the FM interpretation are sets of the FM interpretation, which support equal to the union of the supports of their elements.

Any subcollection of the domain of the FM interpretation with small symmetric difference from a set of the FM interpretation will be a set of the FM interpretation, with support equal to the union of the supports of the elements of the small symmetric difference and the support of the set from which the difference is taken.

We say that a set \( C \) is \( \kappa \)-amorphous iff for any \( B \subseteq C \), either \( B \) or \( C \setminus B \) is small. We say that the cardinal of a \( \kappa \)-amorphous set is a \( \kappa \)-amorphous cardinal.

Every litter \( \text{litter}(p) \) is a set of the FM interpretation with support its own singleton. Further, every litter is a \( \kappa \)-amorphous set in the FM interpretation.

Let \( C \) be a subset of \( \text{litter}(p) \) with strong support \( S \), such that neither \( C \) nor \( \text{litter}(p) \setminus C \) is small. We can then choose two atoms \( p_\alpha, p_\beta \) from the litter, one in \( C \) and one not in \( C \), neither belonging to the strong support \( S \). There will be a permutation in \( G \) exchanging \( p_\alpha \) and \( p_\beta \) and fixing all elements of \( S \). This permutation moves \( C \) without moving any element of its support \( S \), which is a contradiction.

Not only is every subset of a litter either small or co-small, but every subset of the collection \( A \) of atoms of interest has small symmetric difference from either a small or a co-small union of litters. We prove this in stages.

Suppose that a subset \( C \) of \( A \) cuts a large number of litters (that is, there is a large collection of litters \( L \) such that \( L \cap C \) and \( L \setminus C \) are both nonempty). Suppose further that \( C \) has strong support \( S \). Any litter cut by \( C \) is cut into a small part and a large part. We can choose a litter cut by \( C \) which is not in \( S \) and no element of the small part of which belongs to \( S \) (because we are only ruling out a small collection of litters); we can then choose from this litter an element of \( C \) which is not in \( S \) and a non-element of \( C \) which is not in \( S \). A permutation in \( G \) exchanging these two atoms and fixing all elements of \( S \) will exist and will move \( C \) but not any element of \( S \), which is a contradiction. This shows that any subset of \( A \) in the FM interpretation has small symmetric difference from a union of litters which is a set.

Suppose that a union of litters \( C \) is a set of the FM interpretation including a large collection of litters and excluding a large collection of litters, with a strong support \( S \). We can then choose two litters, one included in \( C \) and one not included in \( C \), neither of which is in \( S \), and choose an element from each litter which does not belong to \( S \). A permutation in \( G \) which interchanges these two atoms and fixing all elements of \( S \) will exist and will move \( C \) to a non-union of litters (so certainly move it) and will not move any element of \( S \). This is a contradiction.

If \( B \subseteq A \) is a set of the FM interpretation, we give a precise description of the union of a large collection of litters with small symmetric difference from \( B \). We know that for all but a small collection of litters \( L \), either \( L \) is included in \( B \) or \( L \) is disjoint from \( B \). We also know that for each litter \( L \), exactly one of the sets \( L \setminus B \) and \( L \cap B \) is large. The union of litters \( C \) that we specify is the union of all litters \( L \) such that \( L \cap B \) is large. The symmetric difference of \( L \) and \( B \) is
the union of the small collection of nonempty \( L \cap B \)'s for litters \( L \) not included as subsets in \( C \) and the small collection of nonempty \( L \setminus B \)'s for \( L \) included as a subset in \( C \). This is a small union of small sets, and so is small. Moreover, \( C \) is a set in the FM interpretation, as either the collection of litters included in \( C \) or its complement is a small set of litters which can serve as support for \( C \).

We have completed the description of the subsets of \( \mathcal{A} \) in the FM interpretation, being exactly those sets with small symmetric difference from the union of a small or co-small collection of litters.

We add a remark which is useful below. Note that a large collection of atoms which is a set of the FM interpretation must have large intersection with some litter. Otherwise, it would have to have small intersection with each of a large collection of litters, and we have seen above that a set of the FM interpretation cannot cut each of a large collection of litters.

We argue that if \( B \) and \( C \) are subsets of \( \mathcal{A} \) which are of the same cardinality in the FM interpretation, \( B \Delta C \) is small. We also observe that if \( B \) and \( C \) are large sets of the FM interpretation with small symmetric difference, it is evident that they are of the same cardinality in the FM interpretation. Suppose that \( B \) and \( C \) are of the same cardinality in the FM interpretation and \( B \Delta C \) is large: this is to be witnessed by a bijection \( f \) with strong support \( S \). We may suppose without loss of generality that \( C \setminus B \) is large (the case where \( B \setminus C \) is large is handled symmetrically). The preimage of \( C \setminus B \) is large, and so has large intersection with some litter. The image of the intersection of the preimage of \( C \setminus B \) and this litter is large, and so has large intersection with some litter. So we have a large subset of a litter in the preimage of \( C \setminus B \) and this litter is large, and so has large intersection with some litter. So we have a large subset of a litter in the preimage of \( C \setminus B \) mapped to a large subset of a litter. Choose two elements of the large subset in the preimage, not belonging to \( S \) and not mapped to elements of \( S \). A permutation interchanging their images and fixing all elements of \( S \) will exist and will move \( f \) but not move any element of \( S \), which is a contradiction.

It follows that it is reasonable to define \( |\text{litter}(p)| \) for any \( p \in P \) as the collection of subsets of \( \mathcal{A} \) with small symmetric difference from \( \text{litter}(p) \), as this is precisely the collection of subsets of \( \mathcal{A} \) with the same cardinality as the litter (in the internal sense of the FM interpretation).

We have shown that the power set of \( \mathcal{A} \) in the FM interpretation is extremely impoverished. In particular, it certainly conveys no set theoretical information about the structure of \( P \) in the ground interpretation. We show that \( \mathcal{P}^2(\mathcal{A}) \), on the other hand, contains a subset the same size as \( \mathcal{P}(P) \) in the sense of the FM interpretation (under the further assumption that \( P \) remains a set in the FM interpretation, which is again clearly true if \( P \) is a pure set and may be true under other conditions). This is unsurprising if we consider the size of this set in the ambient ZFA, but one must note that neither \( \mathcal{A} \) nor \( \mathcal{P}(\mathcal{A}) \) should be expected to contain a set the same size as \( P \) in the sense of the FM interpretation.

The crucial result is that the map \( (p \in P \mapsto |\text{litter}(p)|) \) is a set of the FM interpretation. To see this, observe that any pair \( (p, |\text{litter}(p)|) \) in this set is actually fixed by any \( \pi \in G \), since elements of \( P \) are fixed, and sets with small symmetric difference from elements of \( \text{litter}(p) \) are mapped exactly to
sets with small symmetric difference from \texttt{litter}(p). This implies further that the map \((B \subseteq P \mapsto \bigcup_{p \in B} |\texttt{litter}(p)|)\) is a set: to see that the correct invariance holds, it is useful to recall that the cardinalities of litters are pairwise disjoint sets. The map \((B \subseteq P \mapsto \bigcup_{p \in B} |\texttt{litter}(p)|)\) is the promised injection from \(\mathcal{P}(P)\) into \(\mathcal{P}^2(A)\). It is a set in the FM interpretation because it is invariant under all permutations in \(G\).

The reason that this construction is interesting is that it shows how to allow structure not visible to the FM interpretation but concealed in type 0 of a model of TST (visible to the ground interpretation) to unfold not in type 1 (completely nondescript here) but in type 2. This technique can be used to cause unexpected structure to unfold at any desired type level in a model of TST, as the reader may discern in the construction that follows.
8 The simple typed theory of sets, with an enhancement (main argument)

The definitions of certain theories under consideration are restated here as we begin the main argument, supporting our intention that the sections describing the main argument are self-contained, except for occasional motivational remarks.

The simple typed theory of sets, which we will abbreviate TST, is the sorted first-order theory with equality and membership as primitive predicates, with sorts (called types) indexed by the natural numbers (each variable has a type (index) which is a natural number, not necessarily explicitly shown), with well-formedness rules for atomic formulas as follows: \( x = y \) is well-formed iff \( x \) and \( y \) have the same type, and \( x \in y \) is well formed iff the type of \( y \) is the successor of the type of \( x \).

There are two axiom schemes: an axiom scheme of extensionality,

\[
(\forall xy.x = y \leftrightarrow (\forall z.z \in x \leftrightarrow z \in y))
\]

is an axiom for each assignment of types to variables that gives a well formed formula, and an axiom scheme of comprehension consisting of all formulas

\[
(\exists A. (\forall x.x \in A \leftrightarrow \phi)),
\]

where \( \phi \) is any formula not containing the variable \( A \), for any sensible assignment of types to \( x \) and \( A \). The witness to each specific formula \( (\exists A. (\forall x.x \in A \leftrightarrow \phi)) \) is denoted by \( \{x \in A : \phi\} \), which is of type one higher than that of \( x \).

Objects of type 0 are called individuals. Objects of positive type \( i + 1 \) are sets of type \( i \) objects. A model of this theory in our metatheory (some fragment of ZFC) is readily obtained by implementing type 0 as an arbitrary set \( X_0 \) and implementing each type \( i + 1 \) as the set \( X_{i+1} = \mathcal{P}(X_i) \), with the equality and membership relations of the model with first argument in type \( i \) being implemented respectively as restrictions of the equality relation of the metatheory to \( X_i \times X_i \) and the membership relation of the metatheory to \( X_i \times X_{i+1} \).

For our purposes, we modify the theory to what we call enhanced TST by (possibly) increasing the supply of sorts. Fix a limit ordinal \( \lambda \) for the rest of the paper and provide sorts indexed by ordinals less than \( \lambda \). Use the same well-formedness rules for formulas and provide the same axiom schemes, using the larger pool of sorts. Notice that each limit ordinal less than \( \lambda \) will index a type of individuals or atoms, and each successor ordinal will index a type of sets. Models of the enhanced theory in the metatheory are obtained in essentially the same way. Of course if \( \lambda = \omega \) we obtain the same theory described above. Enhanced TST is not very interesting in itself as there is no evident relation at all between types whose indices do not have finite difference.

It will be convenient for us to adopt ZFA (ZFC with extensionality weakened to allow a set of atoms) as the metatheory in our further work. In ZFA, there
is a natural way to define an inner model of (enhanced) TST. In such a model, the sets implementing distinct types are disjoint. In such a model, each type indexed by a non-successor ordinal is implemented as a set of atoms. For each $\tau < \lambda$, the set implementing type $\tau + 1$ is a subset of the power set of the set implementing type $\tau$, not containing the empty set, plus an atom $\emptyset^{\tau+1}$. Membership of type $\tau$ objects in type $\tau + 1$ objects is then implemented as the natural subclass of the membership relation of the metatheory, and equality on each type is the equality relation on the set implementing that type. Note that any model of enhanced TST can be converted to a model of this kind by an extensional collapse if there are enough atoms to cover the base types and the empty sets in each successor type.
9 Construction of the model (main argument)

We begin the description of a particular model of our enhanced TST.

Fix a regular uncountable ordinal $\kappa$ for the rest of the paper. We refer to all sets of the metatheory with cardinality less than $\kappa$ as small and all other sets of the metatheory as large.

Let $\mu$ be a strong limit cardinal with cofinality greater than either $|\lambda|$ or $\kappa$.

We modify our metatheory to ZFA (ZFC with extensionality weakened to allow atoms) with $\mu$ atoms.

The model is an inner model in the sense outlined at the end of the previous section. Each type is implemented as a set in the metatheory. Each type indexed by 0 or a limit ordinal is implemented as a collection of $\mu$ atoms. The set implementing the type indexed by $\tau + 1$ ($\tau$ any ordinal less than $\lambda$) is a subset of the power set of the set implementing type $\tau$ [exactly what subset will be revealed below], with the empty set replaced by an atom $\emptyset^{\tau+1}$. The sets implementing the types are pairwise disjoint. The equality relation on each type is the metatheoretic equality restricted to the set implementing that type. The membership relation of the model is the restriction of the membership relation of the metatheory to the union of the types (this relation does not hold between objects taken from sets implementing non-successive types; these inappropriately typed facts are actually never considered).

We will henceforth call the set implementing type $\tau$ simply “type $\tau$".
9.1 Description of junk objects (main argument)

Each of the non-successor types is partitioned into disjoint sets of size $\kappa$ called litters. Each litter with type $\rho$ elements ($\rho$ non-successor) is an element of type $\rho + 1$. A subset of a non-successor type $\rho$ with small symmetric difference from a litter is called a near-litter. Each near-litter with type $\rho$ elements ($\rho$ non-successor) is an element of type $\rho + 1$.

If $L$ is a litter, with type $\rho$ elements ($\rho$ non-successor) we define the pseudo-cardinal of $L$ as the collection of all near-litters with small symmetric difference from $L$. We denote the pseudo-cardinal of $L$ by $|L|$. Each pseudo-cardinal $|L|$ is an element of type $\rho + 2$. Note that distinct pseudo-cardinals are disjoint sets.

We now show how to define pseudo-cardinals in each type $\tau + 3$ ($\tau$ arbitrary). The pseudo-cardinals of type $\tau + 2$ are partitioned into two sets, both of size $\mu$, the collection of “litter-elements” (atoms in non-successor types are also regarded as litter-elements) and the collection of “loose junk”. The collection of litter-elements is partitioned into disjoint collections of size $\kappa$ called “litters”. A set of litter-elements of the same type with small symmetric difference from a litter is termed a near-litter. Each litter and near-litter with type $\tau + 2$ elements is an element of type $\tau + 3$. Each set union of a litter or near-litter (we will more briefly call these “near-litter unions”) is an element of type $\tau + 2$. We assume as an inductive hypothesis that distinct pseudo-cardinals of type $\tau + 2$ are disjoint sets, so that near-litters are uniquely determined by their set unions. For each litter $L$ with type $\tau + 2$ elements, we define the pseudo-cardinal $|L|$ as the set of all near-litter unions $\bigcup N$ where $L \Delta N$ is small. Each pseudo-cardinal $|L|$ will be an element of type $\tau + 3$. Notice that distinct pseudo-cardinals are disjoint as desired.

The individuals/atoms of limit types $\rho$ and the pseudo-cardinals of types $\tau + 2$ for arbitrary $\tau$ are referred to as junk objects. We define “junk-elements” as elements of junk objects, which will be near-litters with atom elements or set unions of near-litters with set elements.

We partition the loose junk objects of each type $\tau + 2$ into $|\lambda|$ disjoint stages each of cardinality $\mu$, indexed by the ordinals $\tau + 2$ for $\tau$ less than $\lambda$. 
9.2 Setting up the inductive construction of successor types
(main argument)

We have already described the sets implementing types $\rho$ for $\rho$ not a successor.

Fix an ordinal $\tau$ for the rest of this section and suppose that we have constructed all sets of types $\leq \tau$, and further assume various inductive hypotheses which we will state. Our aim is to specify the collection implementing type $\tau + 1$ and to verify that the inductive hypotheses continue to hold.

Note that we have already constructed some objects of all types, namely junk objects, near-litters, and near-litter unions.

We stipulate (an inductive hypothesis) that all singletons of elements of type $\sigma$ are elements of type $\sigma + 1$, for any $\sigma$.

We stipulate (an inductive hypothesis) that each type $\sigma < \tau + 1$ is implemented as a set of size $\mu$.

The next subsection contains further inductive hypotheses.
9.3 Functions which transfer information between types
(main argument)

For each pair of types $\sigma_1 + 2 < \sigma_2 + 2 \leq \tau + 1$ there is an injection $\text{skip}_{\sigma_1 + 2, \sigma_2 + 2}$ which has domain all of type $\sigma_1 + 2$, satisfies the conditions that $\text{skip}_{\sigma_1 + 2, \sigma_2 + 2}(A) = \bigcup \{ \text{skip}_{\sigma_1 + 2, \sigma_2 + 2}(\{a\}) : a \in A \}$ for each $A$ (the value being the appropriate empty set for the type if $A$ is empty) and $\text{skip}_{\sigma_1 + 2, \sigma_2 + 2}(A)$ belongs to type $\sigma_2 + 2$ for each $A$, with each image $\text{skip}_{\sigma_1 + 2, \sigma_2 + 2}(\{a\})$ being a loose junk item of type $\sigma_2 + 2$, taken from stage $\sigma_1 + 2$ in the loose junk of that type. The ranges of distinct skip maps do not share any nonempty elements. This is an inductive hypothesis.

We stipulate that for each pair of types $\sigma_1 + 2 < \sigma_2 + 2 \leq \tau + 1$ there is a map $\text{implode}_{\sigma_1 + 2, \sigma_2 + 2}$ which is an injection from type $\sigma_2 + 1$ into stage $\sigma_2 + 2$ in the loose junk in type $\sigma_1 + 2$. Again, distinct implode maps have distinct ranges. Note that no image under an implode map can be an image under a skip map. We define $\text{explode}_{\sigma_1 + 2, \sigma_2 + 2}(A)$ as

$$\bigcup \{ \text{skip}_{\sigma_1 + 2, \sigma_2 + 2}(\text{implode}_{\sigma_1 + 2, \sigma_2 + 2}(a)) : a \in A \} :$$

$\text{explode}_{\sigma_1 + 2, \sigma_2 + 2}$ is an injection from type $\sigma_2 + 2$ into the set of unions of subcollections of the range of $\text{skip}_{\sigma_1 + 2, \sigma_2 + 2}$ (an image under the explode map is not necessarily an image under a skip map as its expected preimage may not be a member of the lower type): we stipulate that all values of this map belong to type $\sigma_2 + 2$. This is an inductive hypothesis.
9.4 Our intentions described (motivational)

The previous subsection implements the core idea of the construction.

We assume (and will be able to show in the course of the main argument) that the collections of junk objects in each type have combinatorics analogous to that in the “relevant FM construction” above in the model of enhanced TST we are constructing: that is, we can freely add sets of junk objects without being forced to add new sets of lower type by the axiom of set union.

The skip map \( \text{skip}_{\sigma_1+2, \sigma_2+2} \) provides an embedding of each type \( \sigma_1 + 1 \) into loose junk in type \( \sigma_2 + 2 \) which is extended to an embedding of type \( \sigma_1 + 2 \) into type \( \sigma_2 + 2 \) by adding unions of elementwise images of sets of type \( \sigma_1 + 2 \) to type \( \sigma_2 + 2 \), as our underlying combinatorial idea suggests that we should be able to do.

We further require that the collection of unions of images of type \( \sigma_1 + 1 \) objects under the skip map in type \( \sigma_2 + 2 \) should be as large as the entire type \( \sigma_2 + 2 \). We arrange this by providing a map \( \text{explode}_{\sigma_1+2, \sigma_2+2} \), which we will arrange to be an internal function of the eventual model of enhanced TST, mapping all of type \( \sigma_2 + 2 \) to unions of elements of the range of the skip map. This is done by providing the external map \( \text{implode}_{\sigma_1+2, \sigma_2+2} \) which sends elements of type \( \sigma_2 + 1 \) to loose junk objects in type \( \sigma_1 + 2 \): the explode map then sends any type \( \sigma_2 + 2 \) object to the union of the images under the skip map of the images under the implode map of its elements. Notice that the union of the images under its elements under the implode map is not necessarily an element of type \( \sigma_1 + 2 \): once again we are exploiting the idea that we are free to add arbitrary new collections of loose junk.

We can now suppose a bijective map \( \beta_{\sigma_1+2, \sigma_2+2} \) from type \( \sigma_2 + 2 \) into the collection of set unions of collections of images under the skip map in type \( \sigma_2 + 2 \) to be definable in a uniform way, using the Schröder-Bernstein theorem internally to enhanced TST.

The membership of the eventual model of TTT will be defined only between types whose indices are of the form \( \tau + 2 \). If \( \sigma_1 + 3 < \sigma_2 + 2 \), we will define the membership relation \( \in_{\sigma_1+2, \sigma_2+2} \) for use in our eventual model of TTT by \( x \in_{\sigma_1+2, \sigma_2+2} y \leftrightarrow \text{skip}_{\sigma_1+3, \sigma_2+2}(\{x\}) \subseteq \beta_{\sigma_1+3, \sigma_2+2}(y) \). The membership of successive types (for which the last formula would not make sense) will be the usual membership.

Showing that all of this works will take considerable effort, but it may be useful to have this laid out in order to understand what follows.
9.5 Coding of symmetric sets (main argument)

The sets in type $\tau + 1$ will be the sets which are codable in a sense we now explain. The sets which are codable turn out to be those which are symmetric in a specific sense to be revealed.

A set is codable iff it is of the form $f(L)$ where $f$ is a coding function and $L$ is an argument list. We need to define the notions of “coding function” and “argument list”.

A coding function takes as input argument lists which have a fixed argument list type (“argument list type” a notion to be defined) as input type and has output in a fixed type $\tau < \lambda$ in the structure we are building.

Recall that we define a junk-element as an element of a junk item: this will be either a near-litter of atoms or a near-litter union.

An argument list is a function with domain a small ordinal and values taken from litter-elements and junk-elements possibly of many types. Further, an argument list must have an argument list type.

An extended type index is a nonempty finite subset of $\lambda$ which does not contain the predecessor of its largest element. An extended type index $A$ correlates with the actual type $\min(A)$. The successor of an extended type index $A$ with two or more elements is $A \setminus \{\min(A)\}$. The successor of a singleton extended type index $\{\alpha\}$ is $\{\alpha + 1\}$. We write the successor of an extended type index $A$ as $A^+$.

Motivational comment: The extended type indices can be regarded as indexing types expanded to a tree as in the “tangled webs” approach in the background sections. However, we are here constructing a model of tangled type theory rather than a tangled web: if one does think of distinct types being indexed by extended type indices $A$, one should think of types $A$ and $B$ as being externally isomorphic if $\min(A) = \min(B)$.

Argument list types determine the length of argument lists, the type and variety (litter-element or junk-element) of each item in the list, and additional internal structure. An argument list type $T$ is a function from a small ordinal to argument list type components, where these components are of kinds listed below:

1. $(1, \alpha, A)$: if $T$ is an argument list type and $T(\beta) = (1, \alpha, A)$, then $\alpha < \beta$ and if $L$ is of argument list type $T$, then $L(\beta)$ is an atom and so a litter-element belonging to the near-litter $L(\alpha)$, or $\alpha = \lambda$ and $L(\beta)$ is an atom of the highest type appearing in the range of $L$. $A$ is an extended type index, always either a singleton or having its two smallest elements successive: $L(\beta)$ is of type $\min(A)$, a non-successor, and the extended type index component of $L(\alpha)$ is $A^+$, if $\alpha \neq \lambda$.

2. $(2, f, M, A)$: if $T$ is an argument list type and $T(\beta) = (2, f, M, A)$, then $f$ is a coding function, $M$ is a strictly increasing function from a small
ordinal into $\beta$ and if $L$ is of argument list type $T$, then $L(\beta)$ is a junk-element and

$$L(\beta) \in \text{skip}_{\sigma_1+2,\sigma_2+2}(\{f(L \circ M)\}),$$

where $\sigma_1 + 1$ is the output type of $f$ and $\sigma_2 + 1$ is the type of $L(\beta)$. The type of $L(\beta)$ is $\text{min}(A)$, and the two bottom elements of $A$ are successive. Each $T(M(\alpha))$ has extended type component $A^+ \cup B$, where $B$ is the extended type component of $U(\alpha)$, where $U$ is the input type of $f$.

3. $(3, f, M, A)$: if $T$ is an argument list type and $T(\beta) = (3, f, M, A)$, then $f$ is a coding function, $M$ is a strictly increasing function from a small ordinal into $\beta$ and if $L$ is of type $T$, then $L(\beta)$ is a junk-element and

$$L(\beta) \in \text{implode}_{\sigma_1+2,\sigma_2+2}(f(L \circ M)),$$

where $\sigma_2 + 1$ is the output type of $f$ and $\sigma_1 + 1$ is the type of $L(\beta)$. The extended type component of $L(\beta)$ will be of the form $A = B \cup \{\sigma_1 + 1\}$, where $\text{min}(B) = \sigma_2 + 2$. $T(M(\alpha))$ for each $\alpha$ is of the form $B \cup C$, where the maximum of $C$ is one less than the minimum of $B$, and $C$ is the extended type component of $U(\alpha)$, where $U$ is the input type of $f$.

4. Closure conditions on lists $M$: where $(2, f, M, A)$ or $(3, f, M, A)$ occurs in the range of $T$, if $T(M(\alpha))$ is of the form $(2, g, N, B)$ or $(3, g, N, B)$, then the range of $N$ is included in the range of $M$. Similarly, if $L(M(\alpha)) = (1, \gamma, B)$ then $\gamma$ is in the range of $M$.

5. $(4, A)$: if $T$ is an argument list type and $T(\beta) = (4, A)$, and $L$ is a list of type $T$, then $L(\beta)$ is a junk-element of type $\text{min}(A)$, and not an element of any image under any skip operator, and $A$ either has one element or has its two smallest elements successive.

6. Every element of the range of any argument list type is of one of the four kinds listed above.

7. If $L$ is of argument list type $T$, and $T(\alpha)$ and $T(\beta)$ have distinct extended type components, then $L(\alpha)$ and $L(\beta)$ are distinct, and if they are junk-elements their associated near-litters have small intersection.

8. If $L$ is in the domain of a coding function $f$ with output type $\tau$, the type of every range element of $L$ is $\leq \tau$.

An argument list $L$ has argument list type $T$ iff its domain is the same as the domain of $T$, $T$ satisfies all conditions on argument list types stated above, and $L$ and $T$ satisfy the conditions above for each $\beta$ in the common domain of $L$ and $T$.

If $L$ is an argument list of type $T$ with output type $\tau + 1$, the argument list $L^{-}$ is obtained by restricting $L$ to types $\leq \tau$, and will have argument list type $T^{-}$ determinable by appropriately changing indices in each $T(\alpha)$ to correct for omitted positions in the list. A position of type $(3, f, M, A)$ where $f$ has output
type $\tau$ can be retyped as $(4,A)$. A position typed $(1,\tau,A)$ can be retyped $(1,\lambda,A)$.

A coding function $f$ with input type $T_f$ and output type $\rho_f$, not a successor, will be determined by $T_f$ and an ordinal $\alpha_f$ in the domain of $T_f$ such that $T(\alpha_f)$ is of the form $(1,\gamma,\{\rho_f\})$: for any list $L$ with argument list type $T_f$, $f(L) = L(\alpha_f)$. These functions are generalized projection functions. Every such pair of an argument list type $T_f$ and such an ordinal $\alpha_f$ determines a coding function.

A coding function $f$ with input type $T_f$ and output type $\tau_f + 1$ will be determined by $T_f$ and a set $A_f$ of coding functions with output type $\tau_f$ and input type extending $T_f$: $f(L)$, for $L$ with argument list type $T$, will be defined as the set of all $g(M)$ such that $g \in A_f$ and $M$ extends $L^-$, where $L^-$ is the restriction of $L$ to types $\leq \tau_f$ described above. Every pair of a type $T_f$ and such a collection $A_f$ determines a coding function.

The sets of type $\tau + 1$ are exactly the subsets of type $\tau$ which are codable.

We need to verify that the size of the collection of codable sets of size $\tau + 1$ is $\mu$. It is straightforward to establish by induction on set-theoretical rank that the set theoretical rank of each coding function and argument list type is less than the maximum of $\lambda$ and $\kappa$, which establishes that there are no more than $\mu$ coding functions. There are no more than $\mu$ litter-elements, junk-elements, or argument lists. Each coded object is determined by a coding function and an argument list, so there are no more than $\mu$ coded objects. That there are at least $\mu$ coded objects is readily established by considering, e.g., singletons of litter-elements of the appropriate type, which are easily seen to be coded.
9.6 Allocation of loose junk at each stage (main argument)

We then need to define the skip_{σ^2,τ^2} maps for each σ + 2 < τ + 2, which merely requires that we choose injections from type σ + 1 into stage σ + 2 of the loose junk in type τ + 2 for each σ. This has the effect of stipulating the existence of many sets in type τ + 2 before the official beginning of the next stage (unions of loose junk items correlated with type σ + 1 objects belonging to a given type σ + 2 object). It is straightforward to establish that all such sets are codable (and that the singleton sets whose existence is stipulated earlier are codable).

We further need to define implode_{σ^2,τ^2} for each σ, an injection from type τ + 1 into stage τ + 2 of the loose junk in type σ + 2. The images under new explode maps are readily verified to be codable.

This completes the recursive construction of the entire model. Of course, we still need to verify that it actually is a model of TST (as enhanced with types below λ), and then show that it has special properties which allow us to deduce Con(NF).
10 Allowable permutations and symmetry: the codable sets are exactly the symmetric sets (main argument)

A permutation \( \pi \) of a non-successor type \( \tau \) extends to each type \( \tau + n \), \( n \) a natural number, by the convention \( \pi(A) = \pi^"A \). Every type is part of the extended domain of permutations of a uniquely determined non-successor type. An allowable permutation of order \( \tau + 1 \) is a permutation \( \pi \) of a non-successor type \( \leq \tau + 1 \) extended to act on further types as above, satisfying further conditions:

1. \( \pi \) sends all near-litters in its extended domain to near-litters.

2. For any \( \text{skip}_{\tau_1+2,\tau_2+2} \) and \( A \) for which \( \text{skip}_{\tau_1+2,\tau_2+2}(A) \) is in the extended domain of \( \pi \), \( \pi(\text{skip}_{\tau_1+2,\tau_2+2}(A)) = \text{skip}_{\tau_1+2,\tau_2+2}(\pi_{\tau_1+2,\tau_2+2}(A)) \), where \( \pi_{\tau_1+2,\tau_2+2} \) is an allowable permutation of order \( \tau + 1 \), not necessarily the same one even if it has the same extended domain (\( \sigma_2 + 2 \leq \tau + 1 \)).

3. \( \pi(\text{explode}_{\tau_1+2,\tau_2+2}(A)) = \text{explode}_{\tau_1+2,\tau_2+2}(\pi(A)) \) for all appropriate \( A, \tau_1, \tau_2 \) (\( \sigma_2 + 2 \leq \tau + 1 \)).

Note that this is equivalent to

\[
\pi(\bigcup \{\text{skip}_{\tau_1+2,\tau_2+2}(\text{implode}_{\tau_1+2,\tau_2+2}(a)) : a \in A\})
\]

\[
= \bigcup \{\text{skip}_{\tau_1+2,\tau_2+2}(\text{implode}_{\tau_1+2,\tau_2+2}(\pi(a))) : a \in \pi(A)\}
\]

which can be seen to be equivalent to the condition

\[
\text{implode}_{\tau_1+2,\tau_2+2}(\pi(a)) = \pi_{\tau_1+2,\tau_2+2}(\text{implode}_{\tau_1+2,\tau_2+2}(a))
\]

(let \( A = \{a\} \) in the previous equation, and apply the previous condition).

The set of permutations (of order \( \tau + 1 \)) derived from \( \pi \) is the smallest collection of allowable permutations (of order \( \tau + 1 \)) containing \( \pi \) and containing each \( \pi'_{\tau_1+2,\tau_2+2} \) if it contains \( \pi' \).

A support set is a small set of litter-elements and junk-elements. A subset \( A \) of a type \( \sigma \) has support \( S \) of order \( \sigma \) iff for any allowable permutation \( \pi \) of order \( \sigma \) such that every element \( s \) of \( S \) is fixed by all permutations derived from \( \pi \) which have \( s \) in their extended domain, \( \pi \) also fixes \( A \). If a set \( A \) has a support of order \( \sigma \), we say it is symmetric of order \( \sigma \).

Induction establishes that application of an allowable permutation \( \pi \) to a codable set \( f(L) \) will yield a codable set \( f(L_\pi) \), where \( L_\pi \) is obtained by applying to each argument in \( L \) an allowable permutation derived from \( \pi \) in a way determined by the extended type component of the argument list type of \( L \) at that position. This further establishes that every \( f(L) \) has the range of \( L \) as a support, so all codable sets are symmetric. We exhibit the details.
We present the inductive definition of $L_\pi$, where $L$ is an argument list and $\pi$ is an allowable permutation. For each extended type index $A$ we will define a permutation $\pi_A$ just below. If $A$ is the extended type component of $T(\alpha)$, $L_\alpha(\alpha) = \pi_A(L(\alpha))$.

We now define $\pi_A$ for each $A$. $\pi_{\{\alpha\}} = \pi$. If the two smallest elements of $A$ are successive, $\pi_A = \pi_{A^+}$. Otherwise, if the two smallest elements of $A$ are $\sigma < \tau$, $\pi_A = (\pi_{A^+})_{\sigma + 1, \tau}$. It can be shown from this that if $A = B \cup C$, with all elements of $B$ greater than all elements of $C$, that $\pi_A = ((\pi_B)_{\max(C) + 1, \min(B)+1})_C$, or just $(\pi_B)_C$ if the maximum of $C$ and minimum of $B$ are successive.

We verify that $L_\pi$ is an argument list. Let $T$ be the argument list type of $L$. We verify that $L_\pi$ has the same argument list type.

1. If $T(\alpha) = (1, \gamma, A)$, then $A$ has its smallest two elements successive and since $T(\gamma)$ has extended type component $A^+$, $\pi_\alpha = \pi_{\gamma}$, and $\pi_\alpha(L(\alpha))$ will be an atom belonging to the near-litter $\pi_\gamma(L(\gamma))$. If $\gamma = \lambda$ there is nothing that must be enforced.

2. $(2, f, M, A)$: if $T$ is an argument list type and $T(\beta) = (2, f, M, A)$, then $f$ is a coding function, $M$ is a strictly increasing function from a small ordinal into $\beta$ and if $L$ is of argument list type $T$, then $L(\beta)$ is a junk-element and

$$L(\beta) \in \text{skip}_{\sigma_1 + 2, \sigma_2 + 2}(\{f(L \circ M)\}),$$

where $\sigma_1 + 1$ is the output type of $f$ and $\sigma_2 + 1$ is the type of $L(\beta)$. The type of $L(\beta)$ is $\min(A)$ and the two smallest elements of $A$ are successive. Each $T(M(\alpha))$ has extended type component $A^+ \cup B$, where $B$ is the extended type component of $U(\alpha)$, where $U$ is the input type of $f$. Note in the following calculation that $\pi_A = \pi_{A^+}$.

$$L_\pi(\beta) = \pi_A(L(\beta)) \in \pi_A(\text{skip}_{\sigma_1 + 2, \sigma_2 + 2}(\{f(L \circ M)\}))$$

$$= (\text{skip}_{\sigma_1 + 2, \sigma_2 + 2}(\{(\pi_A)_{\sigma_1 + 2, \sigma_2 + 2}(f(L \circ M))\}))$$

$$= (\text{skip}_{\sigma_1 + 2, \sigma_2 + 2}(\{(f(L \circ M)(\pi_{\sigma_1 + 2, \sigma_2 + 2})\}))$$

It completes the verification of this case to show that $(L \circ M)(\pi_{\sigma_1 + 2, \sigma_2 + 2}) = (\pi_{A^+})_{\sigma_1 + 2, \sigma_2 + 2}B((L \circ M)(\alpha)) = \pi_{A^+ \cup B}(L(M(\alpha))) = (L_\pi)(M(\alpha))$ for each $\alpha$ in the domain of $M$, where $T(M(\alpha)) = A^+ \cup B$. This follows directly from attention to the way that subscripted permutations are defined.

3. if $T$ is an argument list type and $T(\beta) = (3, f, M, A)$, then $f$ is a coding function, $M$ is a strictly increasing function from a small ordinal into $\beta$ and if $L$ is of type $T$, then $L(\beta)$ is a junk-element and

$$L(\beta) \in \text{implode}_{\sigma_1 + 2, \sigma_2 + 2}(f(L \circ M)),$$

where $\sigma_2 + 1$ is the output type of $f$ and $\sigma_1 + 1$ is the type of $L(\beta)$. The extended type component of $L(\beta)$ will be of the form $A = B \cup \{\sigma_1 + 1\}$,
where \( \min(B) = \sigma_2 + 2 \). \( T(M(\alpha)) \) for each \( \alpha \) is of the form \( B \cup C \), where the maximum of \( C \) is one less than the minimum of \( B \); and \( C \) is the extended type component of \( U(\alpha) \), where \( U \) is the input type of \( f \).

Now

\[
L_\pi(\beta) = \pi_A(L(\beta)) \in \\
\pi_A(\text{implode}_{\sigma_1 + 2, \sigma_2 + 2}(f(L \circ M))) = \\
(\pi_B)_{\sigma_1 + 2, \sigma_2 + 2}(\text{implode}_{\sigma_1 + 2, \sigma_2 + 2}(f(L \circ M))) = \\
\text{implode}_{\sigma_1 + 2, \sigma_2 + 2}(\pi_B(f(L \circ M))).
\]

Again, this follows on attention to the way that indexed permutations are defined.

We then need to confirm that \( (L \circ M)\pi_B(\alpha) = (\pi_B((L \circ M)(\alpha)) = \pi_{B \cup C}(L(M(\alpha))) = (L_\pi)(M(\alpha)) \) for each \( \alpha \) in the domain of \( M \), which again follows by attention to the way subscripted permutations are defined.

4. if \( T(\beta) = (4, A) \), then \( L(\beta) \) is a junk-element of type \( \min(A) \) which does not belong to an image under a skip map, and so is \( L_\pi(\beta) = \pi_A(L_\beta) \).

We then show by induction that \( \pi(f(A)) = f(L_\pi) \).

If \( f \) is a projection function, then \( f(L) = L(\alpha) \) and the extended type component of \( T(\alpha) \) is a singleton \( \{\gamma\} \), so \( f(L_\pi) = L_\pi(\alpha) = \pi_{\{\gamma\}}(L(\alpha)) = \pi(L(\alpha)) \) as desired.

Otherwise, \( \pi(f(A)) = \pi\{g(M) : g \in A_f \land L^- \subseteq M\} = \pi(g(M)) : g \in A_f \land L^- \subseteq M\} = \{g(M) : g \in A_f \land L^- \subseteq M\} = f(L_\pi) \), by induction on type.

From these results it follows that every symmetric set is codable. This is proved by induction on type. If \( S \) is a type of type \( \tau + 1 \) with support \( \Sigma \), let \( L \) be an argument list containing the elements of \( \Sigma \) of type \( \leq \tau + 1 \) in its range [note that we are proving, and are assuming inductively, that support elements of type greater than that of the set itself are redundant], and for each element of \( S \) choose a code \( g(M) \) with \( M \) extending \( L^- \). The inductive hypothesis required, that every element of a symmetric set has a code with argument list extending any fixed argument list given in advance, clearly holds for non-successor types as the coding functions are simply projections. The component \( A_f \) of the desired code is then the collection of all \( g \)'s taken from codes \( g(M) \) chosen for elements of \( S \), and any other \( g(M') \) with \( M' \) extending \( L^- \) is seen to belong to \( S \) because it is the image of \( g(M) \) under an allowable permutation fixing the elements of \( \Sigma \) (including those not in the range of \( L \) because of too high a type) and so respecting the support of \( S \). This last point remains to be established.

To establish the existence of such a map from such a \( g(M) \) to such a \( g(M') \) we first establish that any collection of small maps \( \pi_A^* \) on litter-elements intended to be restrictions of the maps \( \pi_A \) for a \( \pi \) and suitable merely by reason of type can indeed be extended to be the maps \( \pi_A \) for a fixed \( \pi \). First extend each \( \pi_A^* \) so that it is bijective on each type of litter-element that it acts on, and still small. Then choose for each pair \( L, M \) of litters in each appropriate type a
bijection from elements of $L$ not in the field of $\pi_A^*$ as extended to elements of $M$ not in the field of $\pi_A^*$ as extended, intended to serve as a subset of $\pi_A$ in case $\pi_A(|L|) = |M|$. This allows us to uniformly define each $\pi_A$ on any litter as soon as we define $\pi_A$ on its pseudo-cardinal. We then define the desired $\pi$ and all maps $\pi_A$ by induction on the structure of codes: each litter-element of type the order of the permutation to be constructed can be handled arbitrarily; each litter-element belonging to a junk-element in the list is handled by ind hyp (the pseudo-cardinal of that junk-element being handled); each junk-element in the list either has a pseudo-cardinal which is image under a skip or implode map of something with a simpler code, handled by ind hyp, or an item not an image under a skip map which can be handled arbitrarily. Now given $g(M)$ and $g(M')$, we can determine the partial maps to be extended by considering litter-elements in corresponding positions in the lists $M$ and $M'$ and applying the identity map to elements of $\Sigma$ which were omitted from the range of $L$, if they are relevant.

It follows that the structure we have built is a model of TST, for it is straightforward to show that any set $\{x : \phi\}$ where $\phi$ is a formula of the language of enhanced TST with symmetric parameters is symmetric.
11 Interpreting tangled type theory in the structure and showing \( \text{Con}(\text{NF}) \) (main argument)

For any pair of ordinals \( \sigma + 1 < \tau + 2 < \lambda \) we define a relation \( \epsilon_{\sigma+1,\tau+2} \) from type \( \sigma+2 \) to type \( \tau+2 \). If \( \tau = \sigma \), \( \epsilon_{\sigma+1,\tau+2} \) is the intersection of the membership relation with the cross product of type \( \sigma+1 \) and type \( \tau+2 \).

Otherwise we observe that \( \text{explode}_{\sigma+2,\tau+2} \) is a symmetric injection from type \( \tau+2 \) into the range of \( \text{skip}_{\sigma+2,\tau+2} \), and so by the internal version of the Schröder-Bernstein theorem there is a symmetric bijection \( \beta_{\sigma+2,\tau+2} \) from type \( \tau+2 \) onto the range of \( \text{skip}_{\sigma+2,\tau+2} \) which we may suppose implemented in a uniform way. We define \( x \in_{\sigma+1,\tau+2} y \) as holding iff \( \text{skip}_{\sigma+2,\tau+2}(\{x\}) \subseteq \beta_{\sigma+2,\tau+2}(y) \).

For any formula \( \phi \) of the language of TST with natural number types, define \( \phi^+ \) as resulting by raising the type of each variable in \( \phi \) by one without introducing identifications of variables.

Let \( s \) be any strictly increasing sequence of types which are double successors. Let \( \phi \) be a formula of the language of TST with natural number types. Let \( \phi^s \) be the formula of enhanced TST obtained by replacing each type \( i \) in \( \phi \) with \( s(i) \) and replacing each instance of \( \epsilon \) with the appropriate \( \epsilon_{\sigma+1,\tau+2} \) to make it well-typed. It is straightforward to establish that all sets \( \{x : \phi^s\} \) are symmetric and so belong to the model of enhanced TST, and that for any \( \phi \) which is an axiom of TST, \( \phi^s \) holds in the model of enhanced TST.

Now consider any finite set \( \Sigma \) of sentences of the language of TST with natural number types, mentioning no type of index \( \geq n \). Partition the \( n \) element subsets \( A \) of double successors in \( \lambda \) into no more than \( 2 |\Sigma| \) partitions by considering the truth values of the sentences \( \phi^s \) in the enhanced model for sequences \( s \) such that the range of \( s|n \) is \( A \). This partition has an infinite homogeneous set \( H \) by Ramsey’s theorem. Any \( n+1 \)-element set \( B \) of \( H \) determines, via a strictly increasing sequence whose first \( n+1 \) elements are the elements of \( B \), a model of TST in which \( \phi \leftrightarrow \phi^+ \) holds for each sentence in \( \Sigma \). It follows by compactness that the full scheme \( \phi \leftrightarrow \phi^+ \) is consistent with TST. Specker showed that TST with this scheme is equiconsistent with NF.
12 Conclusions to be drawn about NF (conclusions and further questions, beyond the main argument)

The conclusions to be drawn about NF are rather unexciting ones.

By choosing the parameter $\lambda$ to be larger (and so to have stronger partition properties) one can show the consistency of a hierarchy of extensions of NF similar to extensions of NFU known to be consistent: one can replicate Jensen’s construction of $\omega$- and $\alpha$-models of NFU to get $\omega$- and $\alpha$-models of NF (details given above). One can show the consistency of NF + Rosser’s Axiom of Counting (see [12]), Henson’s Axiom of Cantorian Sets (see [4]), or the author’s axioms of Small and Large Ordinals (see [6], [7], [14]) in basically the same way as in NFU.

It seems clear that this argument, suitably refined, shows that the consistency strength of NF is exactly the minimum possible on previous information, that of TST + Infinity, or Mac Lane set theory (Zermelo set theory with comprehension restricted to bounded formulas). Actually showing that the consistency strength is the very lowest possible might be technically tricky, of course. I have not been concerned to do this here. It is clear from what is done here that NF is much weaker than ZFC.

By choosing the parameter $\kappa$ to be large enough, one can get local versions of Choice for sets as large as desired, using the fact that any small subset of a type of the structure is symmetric. The minimum value $\omega_1$ for $\kappa$ already enforces Denumerable Choice (Rosser’s assumption in his book) or Dependent Choices. It is unclear whether one can get a linear order on the universe or the Prime Ideal Theorem: that would require major changes in this construction. But certainly the question of whether NF has interesting consequences for familiar mathematical structures such as the continuum is answered in the negative: set $\kappa$ large enough and what our model of NF will say about such a structure will be entirely in accordance with what our original model of ZFC said. It is worth noting that the models of NF that we obtain are not $\kappa$-complete in the sense of containing every subset of their domains of size $\kappa$: it is well-known that a model of NF cannot contain all countable subsets of its domain. But the models of TST from which its theory is constructed will be $\kappa$-complete, so combinatorial consequences of $\kappa$-completeness will hold in the model of NF (which could further be made a $\kappa$-model by making $\lambda$ large enough).

The consistency of NF with the existence of a linear order on the universe or the Prime Ideal theorem is not established: questions about many weak versions of Choice remain.

The question of Maurice Boffa as to whether there is an $\omega$-model of TNT (the theory of negative types, that is TST with all integers as types, proposed by Hao Wang ([17])) is settled: an $\omega$-model of NF yields an $\omega$-model of TNT instantly. This work does not answer the question, very interesting to the author, of whether there is a model of TNT in which every set is symmetric under permutations of some lower type.
The question of the possibility of cardinals of infinite Specker rank (in TST at least, but with a little additional care also in ZFA) is answered, and we see that the existence of such cardinals doesn’t require much consistency strength. For those not familiar with this question, the Specker tree of a cardinal is the tree with that cardinal at the top and the children of each node (a cardinal) being its preimages under $\alpha \mapsto 2^\alpha$. It is a theorem of Forster (a corollary of a well-known theorem of Sierpinski) that the Specker tree of a cardinal is well-founded (see [2], p. 48), so has an ordinal rank, which we call the Specker rank of the cardinal. NF + Rosser’s Axiom of Counting proves that the Specker rank of the cardinality of the universe is infinite; it was unknown until this point whether the existence of a cardinal of infinite Specker rank was consistent with any set theory in which we had confidence.

This work does not answer the question as to whether NF proves the existence of infinitely many infinite cardinals (discussed in [2], p. 52). A model with only finitely many infinite cardinals would have to be constructed in a totally different way. We conjecture on the basis of our work here that NF probably does prove the existence of infinitely many infinite cardinals, though without knowing what a proof will look like.

A natural general question which arises is, to what extent are all models of NF like the ones indirectly shown to exist here? Do any of the features of this construction reflect facts about the universe of NF which we have not yet proved as theorems, or are there quite different models of NF as well?
13 References and Index

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