A less mutually recursive approach to forcing in set theory (with typos and mental failures fixed, thanks to the audience)

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I will talk about a different way of defining forcing for equality and membership statements. The intention is to avoid the elaborate mutual recursion between equality and membership in the usual treatment. The original occasion for this was that I was defining forcing in systems related to New Foundations, where recursion on membership is actually impossible because of the nature of the comprehension axioms of the theory, so a different approach needed to be taken. I will not mention NF (or only incidentally) in this talk. The same strategy can be adapted to the usual set theory and may be seen to have advantages.

I think about forcing somewhat differently so my notation may be unfamiliar. We begin as always with a partial order $\leq$ with domain P. The elements of P, called *conditions* represent states of information. I write $p \leq q$ when p and q are compatible and q has as much or more information as p, which I believe is the reverse of the usual practice, but not unique to me.

With sentences ϕ of the language of set theory I associate subsets $[\phi]$ of P: for "p forces ϕ" I write $p \in [\phi]$. The sets $[\phi]$ belong to but do not exhaust a class of subsets of P called "truth value sets".

Truth value sets are the subsets of P which satisfy the conditions (1) and (2) stated and motivated on this slide.

If $p \leq q$ and $p \in [\phi]$, we have $q \in [\phi]$ as well: this is clearly motivated by our interpretation of what $\leq$ means: if we know more at q than at p, and we know ϕ at p, then we know ϕ at q. So (1) for any conditions p, q and truth value set τ, $p \leq q$ and $p \in \tau$ implies $q \in \tau$.

We give part of the definition of the sets $[\phi]$ by recursion on formulas, in order to motivate the second condition: $[\neg\phi] = \{p \in P \mid (\forall q \geq p : q \notin [\phi]\}$.

The second restriction on truth value sets is imposed by the desire to enforce double negation. We want $[\neg\neg\phi]$ to be the same set as $[\phi]$, so we require that $(\forall q \geq p : (\exists r \geq q : r \in [\phi]))$, which the reader may check is the requirement for $p \in [\neg\neg\phi]$, must imply that $p \in [\phi]$. Thus we require that (2) $(\forall q \geq p : (\exists r \geq q : r \in \tau]))$ holds iff $p \in \tau$ for any p and truth value set τ.

Conjunction and universal quantification are easy.

$[\phi \wedge \psi] = [\phi] \cap [\psi]$.

$[(\forall x : \phi[x])] = \bigcap_{x \in D}[\phi[x]]$ where D is our universe of discourse.

Disjunction and universal quantification are trickier. Of course, their definitions in terms of negation, conjunction, and the universal quantifier will give the intended sets, but what they are may be a little surprising.

$[\phi \vee \psi]$ is the smallest truth value set which contains $[\phi] \cup [\psi]$ as a subset. The double negation condition may force the set to be larger: for example, $[\phi \vee \neg\phi]$ is P, and it is certainly not the case that $[\phi] \cup [\neg\phi]$ is P.

Similarly $[(\exists x : \phi[x])]$ is the smallest truth value set such that it includes $\bigcup_{x \in D}[\phi[x]]$. This may include conditions p such that there is no $x \in D$ such that $p \in [\phi[x]]$.

All of this is too abstract to be useful without a domain D and some atomic sentences. We begin with the domain, or rather with a collection of names for elements of the domain. The names are generated by an iterative process similar to that which generates our set theoretical universe.

1. $N_0 = \emptyset$

2. $N_{\alpha+1} = \mathcal{P}(N_\alpha \times P)$

3. $N_\lambda = \bigcup_{\beta<\lambda} N_\beta$, for λ limit.

The intention is that for any names x, y, $(y, p) \in x$ ensures $p \in [x \in y]$, but this cannot be the whole story. Our domain of quantification is now stated to be the class N of all names.

We present the usual horror briefly.

We assert that $[x \in y]$ is the smallest truth value set which contains each p such that for some name z and some $q \leq p$, $p \in [x = z]$ and $(z, q) \in y$.

We assert that $[x = y]$ is the smallest truth value which contains every p such that for every name z and every $q \geq p$, $q \in [z \in x]$ iff $q \in [z \in y]$. This is the same as the definition of $[(\forall z \in N : z \in x \leftrightarrow z \in y)]$.

This *does* work, but all arguments from it are by mutual recursion on the two definitions, relying on the well-founded structure of N.

We start our own development by defining a notation we will require: for any name x and condition p, the name x_p is defined as the set of all $(y_p, q) \in x$ such that q is compatible with p (conditions p, q are compatible iff there is r such that $p \leq r$ and $q \leq r$). This is a definition by transfinite recursion: when x_p is being defined for $p \in N_\alpha$, we may suppose that y_p has already been assigned for each $y \in N_\beta$ $(\beta < \alpha)$ and moreover $y_p \in N_\beta$. From these inductive hypotheses we can conclude that x_p is defined successfully and belongs to N_α.

The intention is to strip out of the name x all information which is irrelevant at stage p.

We now define a weak membership relation on names.

We define the set $[`x' \in_0 y]$ as the smallest truth value set which contains each p such that for some z with $z_p = x_p$ and some $q \leq p$, we have $(z, q) \in y$.

The reason for the odd form of the notation can be telegraphed: if y, y' are names for the same object, $[`x' \in_0 y]$ and $[`x' \in_0 y']$ are the same set, but it is not necessarily the case that if x and x' are the same object that $[`x' \in_0 y]$ and $[`x'' \in_0 y]$ are the same set. This will be immediately evident when equality is defined on the next slide. Weak elements are in quotes, because they are names, and the weak extension of a name might distinguish between names for the same object.

Equality on names is straightforward: names are equal (regarded as referring to the same object in our domain) iff they have the same weak extension.

The set $[x = y]$ is defined as

$$[(\forall z \in N : `z' \in_0 x \leftrightarrow `z' \in_0 y].$$

Notice that this has been *defined*. There are no dangling recursions. But how do we fix the scandal of a membership relation which doesn't respect this equality? [Well, it does respect equality on the right of the $\in_0$ symbol, but not on the left].

We define [$\mathtt{set}(x)$] as

$$[(\forall zw \in N : z = w \rightarrow (`z\text{'}\in_0 x \leftrightarrow `w\text{'}\in_0 x))].$$

We define [$x \in y$] as [$x \in_0 y \wedge \mathtt{set}(y)$].

A name is the name of a set if its weak extension respects equality. A condition forces $x \in y$ just in case it forces x to belong to the weak extension of y and forces y to be a set.

The (strong) membership relation respects equality on both sides. The oddity is that names with weak extensions which do not respect equality become atoms.

We can close up defective names to get names for sets.

Given a name x, we want to show how to fatten it uniformly so that it becomes the name of a set. For any $(y, q) \in x$ and any $p \geq q$ such that $p \in [y = z]$, we want to ensure that $(z_p, p) \in x$. Notice that we do not need $(z, p) \in x$ for $z \in_0 x$ to hold: this would be hopeless, as names of arbitrary high rank can belong to x under conditions which strip out information about possible members of rank higher than that of x. In fact, all the (z_p, p) that we need to add for this closure process are of rank no higher than that of y and lower than that of x, so the resulting "name closure" of x is a set (in the metatheoretic sense). And it is the name of a set by construction.

This slide (which is too long) was added to make a point after the original talk.

There are proof obligations which have to be met re the previous slide.

The fattening process is clearly defined: we add to a name x each (z_p, p) such that there is $(y, q) \in x$ such that $q \leq p$ and $p \in [y = z]$ to get the fattening x^*.

Show that for each $q \in [y \in_0 x]$ and $p \geq q$ such that $q \in [y = z]$ we also have $p \in [z \in_0 x^*]$. This is handled, beause we add (z_p, p) to x^*, and this witnesses $p \in [z \in_0 x^*]$.

Show that for each $p \in [z \in_0 x^*]$ we have w such that $p \in [w = z \wedge w \in_0 x]$: the fattening process adds no more than it needs to add. There must be w and z' such that $p \in [w = z' \wedge w \in_0 x]$, and $z'_p = z$. The additional thing to be shown is that $p \in [w = z]$ as well, that is that $p \in [y = z]$ iff $p \in [y = z_p]$. This follows if we can say that $p \in [(\forall u : u \in_0 y \leftrightarrow u \in_0 z]$ iff $p \in [(\forall u : u \in_0 y \leftrightarrow u \in_0 z_p]$, which looks highly reasonable but should be chased down.

Show that for each $q \in [y \in_0 x]$ and each $p \geq q$ with $p \in [y = z]$, we have z_p of rank no higher than that of y and so strictly less than that of x. This is needed to ensure that x^* is a set in the usual non-forcing sense, and in fact a name of the same rank as x.

Passing to an actual model in which the various forcing conditions are collapsed to actual truth values is handled in the usual way by stipulating in advance that we are working in a countable transitive model of ZFC (or of enough of ZFC for local purposes) and invoking a generic filter G. There is nothing remarkable about what happens when we use G to "decide" truth conditions (though I may talk about this when we get to this point).

Each name gets a definite weak extension when G is used to settle the issue of which forcing conditions are true. Names with the same weak extensions are viewed as the same objects in our forcing model, but their weak elements are *names*, with the stronger identity conditions of names (in quotes, as it were). Those objects of the theory whose weak extensions respect equality of referents become sets. Notice that x_p will collapse to the same object as x if $p \in G$, so the role of x_p in the definition of weak membership is covered.

This all has to be cashed out.

Usual arguments that desired sets exist (witnessing each of the axioms) go as usual with one exception and one systematic modification. The world we have constructed satisfies weak extensionality rather than extensionality: it actually contains a proper class of atoms!

Each of the axioms which asserts the existence of desired sets is proved in the same way as usual, except that name closures have to be taken to ensure that names built are actually names of sets. With that modification, the arguments go in the usual way.

The fact that an object is an atom is expressible in our language: $(\forall z : z \not\in x) \wedge z \neq \emptyset)$ asserts that x is an atom. So we can express the notion that an object x is a pure set (has no atoms in its transitive closure) and relativize all statements to the domain of pure sets, which will be a model of ZFC.