Name : ______________________________

Midterm #2

Math 301, Spring 2013

Monday, April 8rd, 2013

You are allowed one page of notes (front and back). Otherwise, no books, calculators or neighbors! Be sure to answer questions as completely as you can. Convince me that you know why you get the answers you do.

1. (15 points) Consider the linear system below

$$\begin{bmatrix} -3 \\ -1 \end{bmatrix} [x] = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

   (a) Show that $\mathbf{b}$ is not in the column space of A.

   (b) Find the projection $\mathbf{p}$ of $\mathbf{b}$ onto the column space of A.

   (c) Find the error $\mathbf{e} = \mathbf{b} - \mathbf{p}$ associated with this projection

   (d) Show that $\mathbf{e}$ is orthogonal to the column space of A.

   (e) Illustrate on a graph the subspaces $C(A)$ and $N(A^T)$ and the vector $\mathbf{b}$. Show that the vector $\mathbf{b}$ can be written as $\mathbf{b} = \mathbf{p} + \mathbf{e}$, where $\mathbf{p} \in C(A)$ and $\mathbf{e} \in N(A^T)$.

2. (15 points) A matrix $A \in \mathcal{R}^{12\times 3}$ has rank 3.

Fill in the following blanks with an integer value or a choice from an indicated set of values.

(a) The row space is a subspace of $\mathcal{R}$_____ .

(b) The left null space is in $\mathcal{R}$_____ and has dimension __________ .

(c) The get a complete basis for $\mathcal{R}$_____ , $C(A)$ can be extended with __________ additional vectors from __________ (one of the four fundamental subspaces).

(d) The matrix $A^T A$ is a matrix in $\mathcal{R}$_____ $\times$ _____ and is (singular/non-singular).

(e) To project the vector $\mathbf{b} \in \mathcal{R}$_____ onto the column space of A, multiply $\mathbf{b}$ by the projection matrix

$$\mathbb{P} = \_\_\_\_\_\_\_\_\_\_ ,$$

where $\mathbb{P}$ is in $\mathcal{R}$_____ $\times$ _____ .

3. (15 points) Find the complete solution to the following linear system :

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned}$$

4. (10 points) Below are four possible row-reduced echelon forms for a matrix $A \in \mathcal{R}^{m \times n}$ with rank r.

$$R_1 = \begin{bmatrix} \mathbf{I} \end{bmatrix}, \qquad R_2 = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}, \qquad R_3 = \begin{bmatrix} \mathbf{I} & \mathbf{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad R_4 = \begin{bmatrix} \mathbf{I} & \mathbf{F} \end{bmatrix}$$

where $\mathbf{I}$ is an $r \times r$ identity matrix and $\mathbf{F}$ is in $\mathcal{R}^{r \times (n-r)}$.

Match these four forms with the number of possible solutions to $A\mathbf{x} = \mathbf{b}$.

(a) 0 or 1, depending on $\mathbf{b}$

(b) ∞, regardless of $\mathbf{b}$

(c) 0 or ∞, depending on $\mathbf{b}$

(d) 1, regardless of $\mathbf{b}$

5. (10 points) Mickey Mouse and Donald Duck were arguing one day about a problem in their linear algebra class. Mickey was arguing that a vector $\mathbf{b} \in \mathcal{R}^m$ must either be in the column space of a matrix $A \in \mathcal{R}^{m \times n}$, or in the left null space of A. Donald argued that it was possible that $\mathbf{b}$ might not be in either subspace. Who is right? Give a counter example illustrating why one of the debaters is wrong.

6. (20 points) Give the dimensions and find a basis for each of the four fundamental subspaces associated with the matrix

$$A = \begin{bmatrix} 1 & 5 & -3 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Show the orthogonality relationships between the subspaces.

7. (15 points) The automated music recommendation service Pandora allows you to create music stations based on your music preferences. For example, if you indicate that you like the hit single Thrift Shop, Pandora will try to find other tracks that have similar musical attributes such as a "dance-able beat", "house roots", "west coast rap influences" or "electronica influences".

   Behind the scenes, Pandora is storing each track in its database as a vector (a "list" of attributes). Each entry in the vector is a number indicating of how strongly that track represents that attribute. To make its recommendations, Pandora finds tracks in its data base that are close to the songs you have already indicated that you like.

   Suppose we want to create a mini-Pandora. We have 10 tracks (songs) in our database, 20 attributes per track, and a single user who has indicated a preference for two of the tracks. What might be a good way to provide song recommendations to our user?

   Let our database be represented by the vectors (or tracks) be $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{10}\}$, $\mathbf{v}_i \in \mathcal{R}^{20}$. Our user has indicated that she likes tracks $\mathbf{v}_3$ and $\mathbf{v}_7$. Describe an approach you would take to find the best recommendation for our user from the remaining 8 tracks in our database.

8. (10 points extra credit). The vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ form an orthonormal set. Show that these three vectors must then be linearly independent.