The Falling Ladder Paradox

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Anyone who has studied calculus has probably solved the classic *falling ladder* problem of related rates fame:

> A ladder L feet long leans against a vertical wall. If the base of the ladder is moved outwards at the constant rate of k feet per second, how fast is the tip of the ladder moving downward?

The standard solution model for this problem is to assume that the tip of the ladder slips downward, maintaining contact with the wall until impact at ground level, so that if the base and tip of the ladder at any time t have coordinates $(x(t), 0)$ and

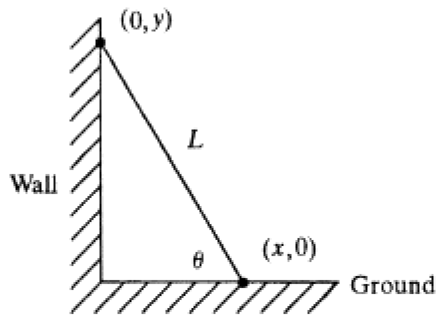

Figure 1. The standard falling ladder model.

$(0, y(t))$, respectively, the Pythagorean theorem gives $x^2 + y^2 = L^2$; see Figure 1. Differentiating with respect to time t yields the formula

$$\dot{y} = \frac{-kx}{y}. \tag{1}$$

The paradox in this solution is that as the ladder nears the ground, $\dot{y}$ attains astronomical proportions. In fact, in [**5**] the student is lightheartedly asked to find (for a particular k and L) at what height y the ladder's tip is moving at light speed.

Of course, the resolution of this paradox is that the ladder's tip leaves the wall at some point in its descent. A few classroom experiments using a yardstick lend observational support for this explanation, for as the base of a stick or ladder is moved away from the wall at constant speed, at the moment of impact it appears as if the tip lands some small distance from the wall, although the action transpires so quickly and catastrophically that it is hard to be certain about what happens. A paper [**2**] in the physics literature points out the flaw of using (1) and demonstrates the correct model for the falling ladder. Our approach is somewhat simpler, making no use of the force exerted by the wall on the ladder's tip; we furthermore show how to numerically plot the path of the ladder's tip, from the time it leaves the wall until its crash landing.

Let's determine y_c, the critical height at which the ladder leaves the wall and (1) ceases to be valid. We will do this by examining the differential equations governing these two different physical situations: the moving ladder supported by the wall and the unsupported ladder behaving as a stick pendulum.[1]

For the pendulum, recall that the rotational version of Newton's second law of motion states that if a rigid body rotates in a plane about an axis that moves with uniform velocity, then the total torque exerted by all the external forces on the body equals the product of the moment of inertia and the angular acceleration, where the torque and the moment of inertia are computed with respect to this moving axis. We apply this principle for the axis which passes through the point of contact (the *pivot*) of the ladder with the ground and which is perpendicular to the plane in which the ladder falls, since this pivot moves with constant velocity.

The only forces on the freely falling ladder are the upward force from the ground at the pivot point, which produces no torque, and the gravitational force, which produces the same torque τ as a force of magnitude mg acting downward at the center of mass of the ladder, as indicated in Figure 2. That is,

$$\tau = mg\frac{L\cos\theta}{2},$$

a positive value since this torque is counterclockwise. Finding the moment of inertia I of a uniform rod with mass m and length L about its endpoint is a standard exercise in calculus or physics, namely,

$$I = \int_0^L x^2\frac{m}{L}\,dx = \frac{1}{3}mL^2.$$

The angular acceleration is simply $-\ddot{\theta}$, being the second derivative with respect to time of the angle $\pi - \theta$ between the ground and the ladder, measured counterclockwise. Thus Newton's law for the falling ladder is $\frac{1}{3}mL^2(-\ddot{\theta}) = \frac{1}{2}mgL\cos\theta$, or

$$\ddot{\theta} = -\frac{3g}{2L}\cos\theta, \qquad (2)$$

which is valid after the ladder loses contact with the wall.

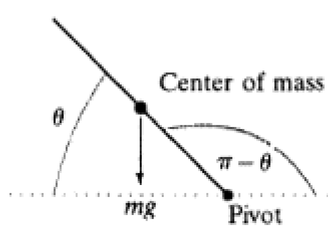

Figure 2. A straight stick pendulum of length L.

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[1] Some texts presents an alternative falling ladder problem in which an unfortunate fellow clings to the top of a ladder. Such a problem can be modeled by the motion of a standard pendulum, by neglecting the mass of the (lightweight) ladder and taking the mass of the pendulum to be the mass of the man. This analysis would be a suitable project for students.

On the other hand, when the ladder is in contact with the wall, $y = L \sin\theta$ and differentiation yields $\dot{y} = L \cos\theta\dot{\theta} = x\dot{\theta}$. By equation (1)

$$\dot{\theta} = -\frac{k}{L \sin\theta}, \tag{3}$$

and another differentiation yields

$$\ddot{\theta} = \frac{k \cos\theta}{L \sin^2\theta}\,\dot{\theta} = -\frac{k^2 \cos\theta}{L^2 \sin^3\theta}, \tag{4}$$

which is valid while the ladder maintains contact with the wall.

Given specific values of L and k, we can determine the critical angle θ_c at which the ladder loses contact with the wall by finding the point of intersection of the graphs of (2) and (4), plotting $\ddot{\theta}$ versus θ. Figure 3 (page 52) illustrates this idea using the values $L = 41$ ft, $k = 10$ ft/s, $g = 32$ ft/s^2, from [**1**]. From the graph we see that as θ decreases the ladder falls according to equation (4) until the two curves meet at $\theta_c \approx 0.38$, the critical angle, and thereafter the ladder falls according to equation (2). That is, up until the critical angle the ladder is held up by the wall, but after θ_c it is free to behave as a stick pendulum.

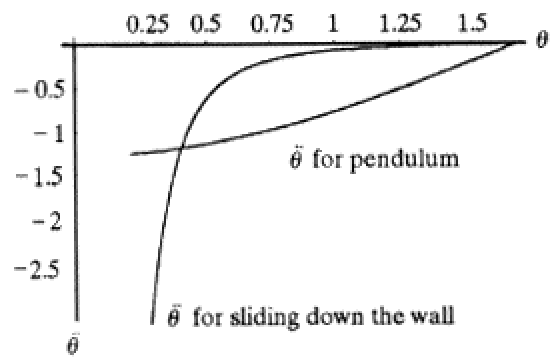

Figure 3. The transition between sliding and swinging.

Leaving L and k as parameters, we equate the right sides of (2) and (4), then simplify, yielding

$$\sin^3\theta_c = \frac{2k^2}{3gL}. \tag{5}$$

If $2k^2/(3gL) \geq 1$, that is, if $k \geq \sqrt{\frac{3}{2}gL}$, equation (5) is impossible, and we conclude that the tip of the ladder pulls away from the wall immediately when the bottom begins to move away with speed k. Otherwise, since $y_c = L \sin\theta_c$,

$$y_c = \sqrt[3]{\frac{2k^2L^2}{3g}} \tag{6}$$

It is interesting to find $\ddot{y}_c$, the acceleration of the tip of the ladder at the critical height. Differentiating (1) and simplifying yields $\ddot{y} = -(k^2L^2)/y^3$, which is valid while the ladder stays in contact with the wall. By (6), then, the acceleration at the moment of separation is

$$\ddot{y}_c = -\frac{3}{2}g. \tag{7}$$

To find the path of the ladder's tip after it leaves the wall, first observe that at the moment of separation the base is at $x_c = L\cos\theta_c$. Since the base moves away at constant speed k, its distance from the wall t seconds later will be $x_c + kt$, so the distance from the wall to the upper end of the ladder will be $d = x_c + kt - L\cos\theta$ at this time. Thus, if we solve the differential equation (2) to find $\theta(t)$, the path of the ladder's tip will be given by the parametric equations

$$d(t) = x_c + kt - L\cos\theta(t), \tag{8}$$

$$y(t) = L\sin\theta(t).$$

Figure 4 shows the trajectory generated by *Mathematica*, which numerically solves (2) and plots the parametric curve, with $L = 41$ ft, $k = 10$ ft/s, $g = 32$ ft/s^2. The initial values are

$$\theta(0) = \theta_c = \arcsin\sqrt[3]{\frac{2k^2}{3gL}} \approx 0.379428, \qquad \text{from (5), and}$$

$$\dot{\theta}(0) = \dot{\theta}_c = -\frac{k}{L\sin\theta_c} \approx -0.658503, \qquad \text{from (3).}$$

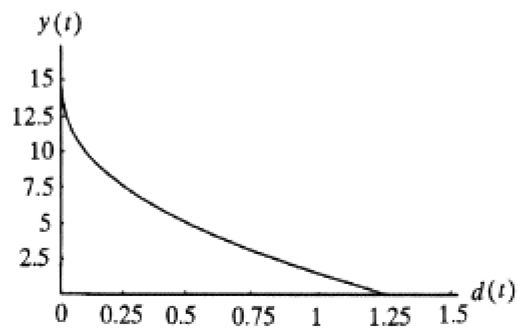

Figure 4. The path of the ladder's tip.

Note that $y_c = L\sin\theta_c \approx 15.19$ ft in this example. The solution is computed as long as $\theta(t) \geq 0$, which turns out to be about 0.42 second, and at this moment of impact the distance of the tip of the ladder from the wall is $d \approx 1.32$ ft.

To contrast these results with a typical textbook solution, consider the problem in [**1**], where the student is asked to find $\dot{y}$ at the instant when $y = 9$ ft, with L and k as before. Since the ladder separates from the wall when $y = y_c \approx 15.19$ ft, we can use *Mathematica's* numerically generated solution of the differential equation (2) to find the correct value $\dot{y} \approx -35.49$ ft/s when $y = 9$, rather than the value of -44.44 as given by (1).

So what should be the status of the falling ladder problem in introductory calculus texts? Here are a few possibilities:

- Remove such problems from the textbooks [**4**].
- Instead of asking for $\dot{y}$, ask for $\dot{x}$ for a ladder falling under the force of gravity, with no friction at either end. But this is a classic mechanics problem, probably best left for a physics course.
- Leave the problems in the text, but ensure that the exercises have $k \leq \sqrt{\frac{3}{2}gL}$ and ask for $\dot{y}$ when y is larger than the y_c of (6), so that the standard approach rings true physically. Mention as a marginal note that the standard model breaks down once $y < y_c$ or if $k \geq \sqrt{\frac{3}{2}gL}$.

An interesting variant that avoids the separation pathology has a 15-foot ladder sliding down a wall while its base slides at 4 ft/s across a 9-foot-wide alley, bounded on the other side by another wall [**3**]. Equation (1) faithfully models this situation, and would do so up to an alley width of 14.4 feet.

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References

1. C. H. Edwards and D. Penney, *Calculus with Analytic Geometry,* Prentice Hall, Englewood Cliffs, NJ, 1994, exercise 45, p. 171.
2. M. Freeman and P. Palffy-Muhoray, On mathematical and physical ladders, *American Journal of Physics* 53:3 (1985) 276–277.
3. P. Gillett, *Calculus and Analytic Geometry*, Heath, Lexington, MA, 1984, pp. 194–195.
4. D. Hughes-Hallett et al., *Calculus,* Wiley, New York, 1994.
5. G. Strang, *Calculus*, Wellesley Cambridge Press, Wellesley, MA, 1991, p. 164.