Homework #4

Math 567

1. (**Finite difference stencils**) For finite difference stencils on equally spaced points, the algebra required to compute truncation errors can be simplified considerably. As an example, consider the second order approximation to the second derivative. Let the coefficients $\mathbf{c}$ be given by $\mathbf{c} = [1, -2, 1]$. Without loss of generality, we can define this on a set of equally spaced points $\mathbf{x} = [-1, 0, 1]$, which we can scale by a mesh width h. Then, we can show the expected order of accuracy by carrying out the following computations.

$$
\begin{aligned}
u(x) \sum_{i=-1}^{1} \frac{c_i}{h^2}(x_i h)^0 &= \frac{u(x)}{h^2}\left(\sum_{i=-1}^{1} c_i\right) = 0 \\
u'(x) \sum_{i=-1}^{1} \frac{c_i}{h^2} x_i h &= \frac{u'(x)}{h}\left(\sum_{i=-1}^{1} c_i x_i\right) = 0 \\
\frac{u''(x)}{2!} \sum_{i=-1}^{1} \frac{c_i}{h^2}(x_i h)^2 &= \frac{u''(x)}{2!}\left(\sum_{i=-1}^{1} c_i x_i^2\right) = u''(x) \\
\frac{u'''(x)}{3!} \sum_{i=-1}^{1} \frac{c_i}{h^2}(x_i h)^3 &= \frac{u'''(x)}{3!} h \left(\sum_{i=-1}^{1} c_i x_i^3\right) = 0 \\
\frac{u''''(x)}{4!} \sum_{i=-1}^{1} \frac{c_i}{h^2}(x_i h)^4 &= \frac{u''''(x)}{4!} h^2 \left(\sum_{i=-1}^{1} c_i x_i^4\right) = \frac{1}{12} u''''(x) h^2
\end{aligned}
$$

The last non-zero term gives us the expected leading order term in the truncation error

$$
\tau = \frac{1}{12} u''''(x) h^2 + O(h^3)
$$

More generally, we can check the accuracy of a p^{th} order accurate stencil to the k^{th} derivative with weights given by computing $\mathbf{c} = [c_{-m}, \ldots, c_{-1}, c_0, c_1, \ldots, c_m]$ over points $\mathbf{x} = [-m, -m+1, \ldots, -1, 0, 1, \ldots, m]$

$$
\frac{u^{(n)}(x)}{n!} \sum_{i=-m}^{m} \frac{c_i}{h^k}(x_i h)^n = \frac{u^{(n)}(x)}{n!} h^{n-k} \left(\sum_{i=-m}^{m} c_i x_i^n\right) = \frac{u^{(n)}(x)}{n!} h^{n-k} S_n
$$

where

$$
S_n = \sum_{i=-m}^{m} c_i x_i^n = \begin{cases} 0 & n < k \\ n! & n = k \\ 0 & k < n < p + k \\ \sigma \neq 0 & n = p + k. \end{cases}, \qquad n = 0, 1, 2, \ldots
$$

To compute accuracy of a finite difference stencil, all we need to compute is the value of S_n for $n = 0, 1, \ldots$. The first $n > k$ for which we have $S_n \neq 0$ gives us the order of accuracy of the scheme as $p = n - k$. Then the leading order term in the truncation error is given by

$$
\tau = \frac{\sigma}{n!} u^{(p+k)}(x) h^p + O(h^{p+1})
$$

(a) Using only values of $\mathbf{c} = [-1/12, 4/3, -5/2, 4/3, -1/12]$ and $\mathbf{x} = [-2, -1, 0, 1, 2]$, compute values of S_n to show that these can be used to compute $u''(x)$ to fourth order accuracy. (You did this problem on Homework #1).

(b) Show that the coefficients $\mathbf{c} = [-1/6, 2, -13/2, 28/3, -13/2, 2, -1/6]$ and $\mathbf{x} = [-3, -2, -1, 0, 1, 2, 3]$ can be used to compute $u''''(x)$. What is the leading order in the truncation error? **Hint :** You can do these calculations easily in Matlab or Python.

(c) The vector

$$\mathbf{c} = \left[-\frac{801}{80}, \frac{349}{6}, -\frac{18353}{120}, \frac{2391}{10}, -\frac{1457}{6}, \frac{4891}{30}, -\frac{561}{8}, \frac{527}{30}, -\frac{469}{240}\right]$$

are the coefficients for a one-sided approximation to $u^{(k)}(0)$. Use the positions $\mathbf{x} = [0, 1, \ldots, 8]$ to find the value of k and the order of accuracy of the scheme using these coefficients.

(d) (**Extra Credit - 10 points**) Show that $\sigma = S_{p+k}$ is always an integer, so that a computer code can be used to obtain the leading order term in the truncation error exactly.

2. (**Runge-Kutta Method**) In class, it was suggested by averaging the Forward and Backward Euler schemes, the first-order truncation errors would cancel, and we could obtain a second order method. This scheme can be written as the following Runge-Kutta type method.

$$\begin{aligned} Y_1 &= U^n \\ Y_2 &= U^n + kf(Y_2) \\ U^{n+1} &= U^n + \frac{k}{2}\left(f(Y_1) + f(Y_2)\right) \end{aligned}$$

- Verify that this scheme is the result of averaging the Forward Euler and Backward Euler steps. That is,

$$U^{n+1} = \frac{1}{2}\left(\text{Forward Euler step} + \text{Backward Euler step}\right)$$

- Use the Dalquist test problem $u'(t) = \lambda u$ to show that this approach is second order accurate, at least for a model problem.
- Derive the local truncation error for the method by computing

$$\tau = \frac{u(t+k) - u(t)}{k} - \frac{1}{2}\left(f(Y_1) + f(Y_2)\right)$$

- Compare this method with Heun's Method and the Trapezoidal method.

3. (**Adam's methods**) Derive the second and fourth order Adam-Bashforth and Adam-Moulton methods. **Hint.** Use the Lagrange Interpolation Formula, and then integrate the resulting polynomial approximation.

4. (**Adam's Bashforth**) Solve the following ODE using the Adam's Bashforth Method.

$$u'(t) = t - 2u, \qquad u(0) = 1$$

over the time interval $[0, 2]$. The, compute the error at time $t = 2$, for each of the following sets of starting values.

- Use the exact solution for the starting values
- Use Forward Euler

- Use the 4th order Runge-Kutta method

Show the convergence rate for each of the three cases above.

Hint. Use Duhamel's Principle to get the exact solution to the ODE.

5. (**Stability regions for Adam's methods**) Plot absolute stability regions for the 4th order Runge-Kutta method, and compare it to the region for the 4th order Adams- Bashforth and Adams- Moulton methods. Can you use the Runge-Kutta method to get starting values for the Adams methods?

6. (**Adams-Bashforth**) Use the Adams-Bashforth method to solve the model problem

$$u'(t) = \lambda u(t)$$

for $\lambda = -10$ and $u(0) = 1$, over the interval $t \in [0, 3]$. Compute starting values using the 4^{th} order Runge-Kutta method. Show that the method is stable for 100 equally spaced steps over this interval, i.e. for $\Delta t = 0.03$, but that is unstable for 99 steps over the same interval. Provide convincing analytic, numerical and graphical evidence for your argument.

7. (**Adam's Moulton methods**) Use the Adams-Moulton methods to solve the

$$u'(t) = \lambda u(t)$$

for $\lambda = -10$ and $u(0) = 1$, over the interval $t \in [0, 9]$. For this problem, compute starting values using the 4^{th} order Runge-Kutta method. Show the method is stable for 49 equally spaced time intervals over $[0, 9]$, but that is unstable for 48 steps. Provide convincing analytic, numerical and graphical evidence for your argument.