Homework #3

Math 567

1. (**Neumann boundary conditions**) Consider the problem

$$u''(x) = -4\pi^2 \cos(2\pi x) \tag{1}$$

subject to $u'(0) = 0$, $u(1) = 1$. Discretize the equation using the standard second order discretization for the second derivative, and the second order one-sided difference approximation

$$u'(0) \approx \frac{-3u_0 + 4u_1 - u_2}{2h} \tag{2}$$

to approximate the Neumann boundary condition at the left endpoint. Show that the matrix you get is identical to the matrix you get when using the first order approximation to $u'(0)$ (done in class), but that the minor modification to the right hand side is enough to get a second order accurate solution. Do a grid convergence study to show that you get second order accuracy.

2. (**Neumann boundary conditions**) Repeat the Problem 1, but use a centered approximation to the Neumann boundary condition,

$$u'(0) \approx \frac{u_1 - u_{-1}}{2h} \tag{3}$$

Show that again, you get the same matrix, but that the right hand side differs. Do a grid convergence study to show that you get second order accuracy.

3. (**Linear algebra**) Consider the following linear system

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

(a) Show that this system is singular.

(b) Show that the system does not have a solution for the right hand side vector $\mathbf{b}_1 = [1, 3, 1]^T$ but has an infinite number of solutions for the right hand side $\mathbf{b}_2 = [-1, 0, 1]^T$. For the right hand side $\mathbf{b}_2$, describe all solutions.

(c) Show that if you augment the matrix A by adding a fourth column consisting of all ones (so that the resulting matrix is 3×4), the system defined as

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \alpha \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has a solution for any right hand side vector $\mathbf{b}$. Explain the significance of the value of α.

(d) We can augment the matrix A from above with a linearly independent row to get the symmetric system

$$A = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \alpha \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix}$$

Show that this system is non-singular. **Hint:** Show that the eigenvalues of A are all non-zero. To do this, write the system as

$$\begin{bmatrix} A_{11} & \mathbf{z} \\ \mathbf{z}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \alpha \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u} \\ \alpha \end{bmatrix}$$

where A_{11} is the original 3×3 matrix and $\mathbf{z}$ is a column vector of 1s. Write out the system in equation form to get

$$\begin{aligned} A\mathbf{u} + \alpha\mathbf{z} &= \lambda\mathbf{u} \\ \mathbf{z}^T\mathbf{u} &= \lambda\alpha \end{aligned}$$

Then, consider the case $\alpha \neq 0$ and show that you get two non-zero eigenvalues. Then consider the case $\alpha = 0$ and show that you must have $\lambda \neq 0$.

4. (**Neumann boundary conditions**) Use what you learned from Problem 3 to discretize the equation

$$u''(x) = -4\pi^2 \cos(2\pi x) \tag{4}$$

on $[0, 1]$ subject to $u'(0) = u'(1) = 0$. Use the second order centered difference approximation from Problem 2 to discretize the boundary conditions. Augment the resulting matrix as in Problem 3 to construct a non-singular system. Do a grid convergence study to show that you get a second order accurate solution to the Neumann problem.

Modify the boundary condition at $x = 1$ to $u'(1) = 3$. In what significant way does the solution to this problem differ from the original problem? **Hint:** The answer depends on your value of α.

5. (**Variable coefficient problem**) A common problem in engineering is to solve for a steady state temperature field in a medium with variable conductivity. We can model a two dimensional version of this problem with the 2d *variable coefficient* Poisson problem

$$\nabla\beta(x, y)\nabla u(x, y) = f(x, y)$$

where $\beta(x, y)$ represents the conductivity of the material. Heat will flow faster in regions of the domain where β is relatively high, and slower in regions with low β. The right hand side represents a heat source, such as a steady flame used to heat the material.

A particularly simple choice for β is a piecewise constant function. For example, we might divide the domain in half, and assign the left half a β^- value and the right half a β^+ value. On the domain $[0, 1] \times [0, 1]$, our β might take the form

$$\beta(x, y) = \begin{cases} \beta^- & x < 0.5 \\ \beta^+ & x \geq 0.5 \end{cases} \tag{5}$$

Another way to model this is with a smooth function

$$\beta(x, y) = \beta^- + (\beta^+ - \beta^-)\frac{1 + \tanh((x - 0.5)/\varepsilon)}{2} \tag{6}$$

As $\varepsilon \to 0$, the smooth function (6) approaches the piecewise constant function (5).

For this problem, you will solve the variable coefficient Poisson problem on the domain $[0, 1] \times [0, 1]$ using the right hand side function

$$f(x, y) = -100 \exp(-100((x - 0.5)^2 + (y - 0.5)^2))$$

β^+/ε	0.5	0.1	0.05	0.01	0.005	0
1	488	488	488	488	488	488
10	1104	1659	1725	–	–	–
100	1249	–	–	–	–	–
1000	1338	–	–	–	–	–

Table 1: Table of convergence rates for Problem 2 as function of β^+ (left column) and ε (top row). For this problem, $N = 160$ and $\beta^- = 1$. For $\varepsilon = 0$, use the piecewise constant defintion of $\beta(x, y)$ given in (5). For non-zero ε, use the smooth definition given in (6).

Impose Dirichlet boundary conditions $u(x, y) = 1$ on the boundary of the domain and discretize the equation on a 160×160 mesh using the discretization approach described in equation 2.72 of your book (page 36).

(a) Check that you have coded the matrix-vector multiply for the variable coefficient problem correctly by evaluating the truncation error in your discretization and showing that for the smooth $\beta(x, y)$, your truncation error is second order. Recall that the truncation error can be computed as

$$\tau \approx A\mathbf{u} - \nabla\beta(x, y)\nabla u(x, y)$$

for some function $u(x, y)$. You might choose $u(x, y)$ from Problem 1. Or another a simple polynomial which you can easily differentiate. For example,

$$u(x, y) = Ax^p + By^q$$

for integers p and q. Then

$$\nabla\beta(x, y)\nabla u(x, y) = \beta(x, y)\nabla^2 u(x, y) + \nabla\beta(x, y) \cdot \nabla u$$

Use the inf-norm to plot the truncation error vs. N, for a range of N values, and show that you are getting second order accuracy. **Note:** Just because the truncation error is second order isn't a guarantee in general that your global error will be second order. Recall that we need some sort of stability for this. But for this problem, we have stability, and so checking the truncation error does guarantee us global accuracy.

(b) One goal of this problem is to see how convergence of the conjugate gradient algorithm depends on both ϵ and on the jump $[\beta] = \beta^+ - \beta^-$. Set $\beta^- = 1$. To explore this dependence, vary ε and β^+, using the values

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ewvec = [0.5 0.1 0.05 0.01 0.005 0];   % epsilon
bpvec = [1 10 100 1000];               % beta_plus
```

and complete Table 1 of convergence rates.

(c) Draw conclusions about how the convergence rate is affected by both the smoothness of $\beta(x, y)$ (as measured by ε) and the strength of the discontinuity, as measured by the ratio β^+/β^-.

(d) Plot a few representative figures showing the solution for $\varepsilon = 0$, and different values of β^+. Can you explain the resulting plots using your physical intuition about how heat flow behaves through materials of varying conductivity?

(e) The temperature field $u(x, y)$ obtained from the smooth $\beta(x, y)$ is second order accurate. However, the solution obtained using the piecewise constant $\beta(x, y)$ is only first order accurate. What goes wrong in this case?