Homework #2

Math 567

1. An approach to finding finite difference stencil coefficients that does not require a matrix inversion is to differentiate an interpolating polynomial. To see how this works, use the Lagrange Interpolation Formula to find the coefficients a_i, $i = 0, 1, 2$ of the second degree interpolating polynomial

$$p_2(x) = a_2 x^2 + a_1 x + a_0$$

through points $(x_i, u(x_i))$, $i = 0, 1, 2$ for some function $u(x)$. The coefficients you get will be in terms of x_i and $u_i = u(x_i)$. Then, differentiate this polynomial and evaluate the result at x_1 to show that for equally spaced points x_i, you recover the expected finite difference formulas for $u(x_1)$, $u'(x_1)$ and $u''(x_1)$.

List at least two advantages and two disadvantages of this approach.

2. Show that the eigenvalues that the $m \times m$ matrix A in equation 2.10 from your book has eigenvalues given by

$$\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1), \quad p = 1, 2, \ldots, m \tag{1}$$

where $h = 1/(m+1)$.

3. Show that $\| A\mathbf{x} \| \leq \| A \| \| \mathbf{x} \|$ for matrix norm defined as

$$\| A \| = \max_{\mathbf{x} \in \mathcal{R}^n} \frac{\| A\mathbf{x} \|}{\| \mathbf{x} \|}$$

4. Show that for matrix A,

$$\| A \|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|, \qquad \text{the "max column sum"}$$

and that

$$\| A \|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|, \qquad \text{the "max row sum"}$$

The point of this problem is to show that our definition of the "operator norm", i.e.

$$\| A \|_p = \max_{\mathbf{x} \in \mathcal{R}^n} \frac{\| A\mathbf{x} \|_p}{\| \mathbf{x} \|}$$

can in fact, at least for the 1-norm and the ∞-norm, be easily computed using the formulas above.

Hint: For the 1-norm, find a vector $\mathbf{x}$ for which

$$\| A\mathbf{x} \|_1 = \left(\max_j \sum_{i=1}^{n} |a_{ij}| \right) \| \mathbf{x} \|_1$$

Then show that for any other $\mathbf{x}$, we have

$$\| A\mathbf{x} \|_1 < \left(\max_j \sum_{i=1}^{n} |a_{ij}| \right) \| \mathbf{x} \|_1$$

Use a similar approach for the ∞-norm.

5. When we solve to the linear system $A\mathbf{x} = \mathbf{b}$ using Gaussian elimination, we do not expect to satisfy the linear system exactly. Instead, we will get a non-zero *residual* vector $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ Let $\widehat{x}$ be the exact solution to the linear system so that $A\widehat{\mathbf{x}} = \mathbf{b}$ is satisfied exactly. Define the *error* in Gaussian elimination as $\mathbf{e} = \mathbf{x} - \widehat{\mathbf{x}}$.

   (a) Show that $A\mathbf{e} = -\mathbf{r}$

   (b) Show the error $\mathbf{e}$ satisfies

   $$\frac{\| \mathbf{e} \|}{\| \widehat{\mathbf{x}} \|} \le \kappa(A) \frac{\| \mathbf{r} \|}{\| \mathbf{b} \|}$$

   where $\kappa(A)$ is the condition number, defined as

   $$\kappa(A) = \| A \| \| A^{-1} \|$$

   Hint: Here are the first few steps in the proof.

   $$\| A\mathbf{e} \| \le \ldots \le \| A \| \; \| A^{-1} \| \; \| \mathbf{r} \|$$

   It can be shown that Gaussian elimination produces small residuals, so that $\| \mathbf{r} \| \,/\, \| \mathbf{b} \| \approx \epsilon$. But if the matrix is ill-conditioned, a small residual may not lead to a small error in the solution $\mathbf{x}$.

6. (**Condition number vs. error**) Consider the BVP given by

   $$u''(x) = -4\pi^2 \sin(2\pi x)$$

   on the domain $[0, 1]$, subject to boundary conditions $u(0) = u(1) = 0$. Solve this BVP on sequence of meshes with mesh width $h = 1/N$, where $N = 8 \times 2^n$, $n = 0, 1, \ldots, 15$. Then, create a log-log plot with the following four quantities plotted verses N.

   - Plot the error in your solution, using the ∞-norm. Use the exact solution to the BVP to compute the error.
   - Plot $\epsilon\kappa(A)$ vs. N, where $\kappa(A)$ is the condition number of A and ϵ is "machine epsilon" and can be computed as $\epsilon = 2^{-52}$ or using the Matlab command `eps(1)`.
   - Plot a line showing the best fit line to the error data. Include the slope of your best fit line in a legend entry.
   - Plot a line show the best fit line to the scaled condition number. Include the slope of this line in a legend entry.
   - Add a title, axis labels and a legend to your plot.

   Question : How does the result of Problem 5 explain the results from this problem?

   For this problem, construct A as sparse matrix using `spdiags`. Solve the system using the backslash \ operator. To compute the condition number, use `condest`.

7. One way to avoid inverting the ill-conditioned linear system that arises from discretizing the BVP in Problem 6 is to evaluate the integral formulation of the solution directly. This equation, given in equation 2.42 from the course textbook, can be discretized using a trapezoidal rule for the integrals to obtain

$$U = BF$$

where B is given in the text. Construct matrix B and the solution to the BVP from Problem 6 on a sequence of meshes. Be careful - B is a dense matrix so you will not be able to create matrices nearly as large as those in Problem 6.

- Plot the resulting global error $E = U - \widehat{U}$ on a loglog plot. You should observe that you can reduce the error far below that which was achievable with solving the linear system from Problem 6.
- Show how the matrix A and B are related.
- Speculate on what you think the advantages and disadvantages of this approach to solving the BVP

This approach to solving BVPs is an example of an *integral equation* formulation.