Homework #1

Math 567

1. The following three tables of errors were produced from different finite difference approximations to $f(x)$, $f'(x)$, for some function $f(x)$. The first column of each table shows the size h used in the difference approximation, and the second column is error in that approximation, as a function of h.

   Assume that the error can be modeled as

   $$e(h) \approx Ch^p.$$

   Approximate the order of accuracy p for each sequence. You can find the data for these sequences in the file `test_errors.m` on the course website, under Hmwk1.

h	err(h)
7.8125000e-03	2.0844e-02
3.9062500e-03	1.1118e-02
1.9531250e-03	5.3455e-03
9.7656250e-04	2.7049e-03
4.8828125e-04	1.3469e-03

h	err(h)
7.8125000e-03	1.9059e-03
3.9062500e-03	4.3086e-04
1.9531250e-03	1.0318e-04
9.7656250e-04	2.6007e-05
4.8828125e-04	6.5716e-06

h	err(h)
1.00000e+00	1.3829e-02
5.00000e-01	1.8805e-03
2.50000e-01	1.3742e-04
1.25000e-01	8.9170e-06
6.25000e-02	5.6252e-07
3.12500e-02	3.5239e-08
1.56250e-02	2.2037e-09
7.81250e-03	1.3775e-10
3.90625e-03	8.6098e-12

2. Find the leading term in the truncation error τ of the following finite difference approximations.

   (a)
$$\frac{u(x-\frac{h}{2})+u(x+\frac{h}{2})}{2} \approx u(x)+\tau$$

   (b)
$$\frac{u(x+h)-u(x)}{h} \approx u'(x)+\tau$$

   (c)
$$\frac{u(x)-u(x-h)}{h} \approx u'(x)+\tau$$

   (d)
$$\frac{u(x+\frac{h}{2})-u(x-\frac{h}{2})}{h} \approx u'(x)+\tau$$

   (e)
$$\frac{u(x+h)-2u(x)+u(x-h)}{h^2} \approx u''(x)+\tau$$

   (f)
$$\hat{u}(x)=\frac{1}{h}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} u(\xi)\,d\xi \approx u(x)+\tau$$

   Hint : Expand $u(\xi)$ about x and integrate the terms in the resulting Taylor series.

3. Search for “finite difference coefficient” in Wikipedia to find the coefficients for a centered fourth order approximation to $u''(x)$. Compute the leading order term in the truncation error.

4. Using a centered, second order approximation to $u''(x)$ (see Problem 2e) derive a second order approximation to a fourth derivative. What is the trunction error of your approximation? *Note*: You should be able to do this without solving any linear systems.

5. Derive a second order accurate centered approximation to a third derivative. What is the leading order term in the truncation error of your finite difference formula? Hint : Combine second order accurate centered approximations to the first and second derivatives.

6. Higher order finite difference approximations used in solving hyperbolic equations can involve computing a "one-sided approximation" to the first derivative of a function. Given give values $u(x_0)$, $u(x_1)$, $u(x_2)$, $u(x_3)$, $u(x_4)$ for equally spaced points $x_j = (j - 0.5)h, j = 0, 1, 2, 3, 4$, derive a fourth order finite difference approximation to $u'(0)$.

   - Sketch the location of the equally spaced points.
   - Sketch the location of where the approximation should be made (i.e. at $x = 0$).
   - Write down a Taylor series expansion for each value of $u(x_j)$ about the point $x = 0$.
   - Describe the linear system you would need to solve to construct the stencil for the fourth order approximation to $u'(x)$.
   - Solve the linear system to get the coefficients for the approximation. Ideally, you can obtain the exact rational expressions for the coefficients. Hint : Use the "rat" function in Matlab.
   - Verify numerically that your approximation is fourth order. Hint : You can show that it gives the exact derivative of the functions $u(x) = x^n$ for $n = 0, 1, 2, \ldots, 4$.
   - Use your approximation to approximate the derivative of $u(x) = \sin(x)$ at $x = 0$ for a sequence of h values, e.g. $h = 1, h = 0.5, h = 0.25, \ldots, 2^{-N}$ for some large N (maybe $N = 8$?). Show that your approximation approaches the expected convergence rate by using the method you used in (1).

7. In finite volume schemes, we only have a cell *average values* $\widehat{u}(x)$, rather than the pointwise values $u(x)$. Suppose you used $\widehat{u}(x)$, defined in Problem 2f above, in the formula for the second derivative given in Problem 2e. What is the resulting truncation error? Assume that you know $\widehat{u}(x)$ exactly, so you can use Problem 2f to write $u(x) = \widehat{u}(x) - \tau$.