Name : ______________________________

Homework #5

Math 566, Spring 2014

Due Wednesday May 7th,

1. Do a "grid refinement" study on the code `hyper_demo.m` from the course website. Confirm that the upwind method is first order accurate, and the Lax Wendroff method is second order accurate. You don't have to turn in any code for this problem, but do report the errors that you found, and demonstrate the order of accuracy.

2. The wave equation is a second order PDE given by

$$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty \tag{1}$$

subject to $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Show that this is equivalent to the system of two equations, given by

$$\begin{aligned} u_t + v_x &= 0 \\ v_t + c^2 u_x &= 0 \end{aligned} \tag{2}$$

subject to $u(x,0) = f(x)$, $v(x,0) = G(x) = -\int g(x)dx$ for functions $u(x,t)$ and $v(x,t)$. This is a linear, constant coefficient hyperbolic system of the form

$$q_t + Aq_x = 0 \tag{3}$$

where A is a 2×2 matrix and $q(x,t) = (u(x,t), v(x,t))$.

We can find the solution of this system in the following way.

(a) Find the eigenvalues and eigenvectors of the matrix A. Assume that $\lambda^1 < \lambda^2$.

(b) Let R be the 2×2 matrix of eigenvectors $[r^1, r^2]$, for 2×1 eigenvectors r^1, r^2 associated with λ^1 and λ^2. Show that by setting $\omega(x,t) = R^{-1}q(x,t)$, the system in (3) be written as a decoupled system of two equations

$$\omega_t + \Lambda \omega_x = 0$$

where $\omega(x,t) = (\omega^1(x,t), \omega^2(x,t))$ and $\Lambda = \text{diag}(\lambda^1, \lambda^2)$. From this, we can recover the classic D'Alembert solution for $u(x,t)$.

(c) Write down the exact solution to the wave equation (1) in terms of the initial conditions specified by $f(x)$ and $G(x)$. What is the exact solution $v(x,t)$ to the second equation in (2)?

(d) The code `wave_eqn.m` on the course website solves the wave equation as a coupled system of two one-wave equations. Initial conditions for both $u(x,t)$ and $v(x,t)$ are supplied in subroutines in the file, along with a subroutine to supply the exact solution. Use this code to solve the following two problems. For each problem, supply the initial conditions, and the exact solution. Produce a plot showing the agreement between your exact solution and the numerical solution.

i.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= \cos(8\pi x) \\ u_t(x,0) &= 0 \end{aligned}$$

over the time interval $[0, 2]$.

ii.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= 0 \\ u_t(x,0) &= \begin{cases} 2 & 0.4 \le x \le 0.6 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Check that your exact solution is correct by checking the convergence of your solution for a range of N values (i.e. number of intervals on the mesh). You should see first order convergence for the upwind method and second order for Lax-Wendroff, at least for the first problem.