Name : ______________________________

# Homework #4

## Math 566, Spring 2014

Due Monday, April 28

1. Show that that region of absolute stability for the trapezoidal method is the half plane Re $z < 0$. Use both the boundary locus method, described in Section 7.6.1 of your text, and the method for one-step schemes, described in 7.6.2. Both methods should produce the same results.

2. This problem concerns the result

$$\pi(e^z; z) = O(z^{p+1}) \tag{1}$$

for a $p^{th}$ order accurate linear multistep method (LMM).

   (a) Show that for a first order, consistent LMM,

$$\pi(e^z; z) = O(z^2). \tag{2}$$

   (b) How does the general result (1) help determine the stability region for an LMM, if we only know the boundary of the region?

   (c) Show how you might go about demonstrating the more general result in (1). Work through the details for a second order method.

3. For this problem, you will use what you now understand about stability regions to modify your pursuit curve code from Homework #2 so that you can take a variable step size based on the solution at each time step. Carry out the following steps:

   (a) Plot the stability region for the fourth order Runge-Kutta scheme.

   (b) Write down the Jacobian of the pursuit curve system, and determine its eigenvalues.

   (c) Modify your code from Homework #2 to allow for a variable time step size. That is, at each step of your one-step scheme, determine a stable time step that is appropriate for the state of the system.

   (d) Produce a plot of the time step sizes your were able to achieve verses time. Do this for several different starting locations for the fox. How is the time step size related to the speed of the fox?

4. Use your code for the elliptic equation from Homework #2 to solve the two dimensional heat equation

$$q_t = \nabla^2 q$$

using (1) Forward Euler, (2) Backward Euler and (3) Trapezoidal Method. Solve the problem on the domain $[-1, 1] \times [-1, 1]$. Use as initial conditions the unit disk

$$u(r, 0) = \begin{cases} 1 & 0 \le r \le 0.3 \\ 0 & \text{otherwise} \end{cases}$$

where $r = \sqrt{x^2 + y^2}$. Impose *no flux* boundary conditions on the boundary. That is,

$$u_n(x, y, t) = 0$$

on the boundary of the domain, where $u_n(x, y, t)$ is the outward normal derivative at the boundary. Run the simulation until you see an approximate constant solution.

(a) Plot the solution at a few of the early time steps. What you should observe is that while both forward Euler and Backward Euler produce solutions with relatively smooth error, the trapezoidal method shows oscillations in the error. This is an illustration as to why *L-stability* is often a desirable property to have in a numerical scheme. You can read more about L-stable methods in section 8.3.2 in your course text.

(b) The steady state solution is a constant. What should this constant be approximately equal to? How close is your value?

(c) (**Extra credit**). Solve the heat equation in Problem 4 using a TR-BDF2 method, described on page 175 (Section 8.5) of the course text. This method is a second order L-stable method which has the desirable properties of backward Euler, but is second order.