Name : ______________________________

Homework #1

Math 566, Spring 2014

Due Wednesday February 26th

1. Write a Matlab function that takes an array of values $x_j \in [0,1]$, $j = 0, 1, \ldots, N$, (not necessarily equally spaced), a value $\bar{x} \in [0,1]$ a desired order of accuracy p, and a derivative $k = 0, 1, 2$. This function should then return coefficients c_j that can be used to approximate the $u^{(k)}(\bar{x})$ to order p, using the formula

$$u^{(k)}(\bar{x}) = \sum_{j=0,N} c_j u(x_j) + O(h^p)$$

where h is some measure of the width of the stencil. You will first have to decide which m stencil points $J, J+1, J+2, \ldots, J+m-1$ to use. Note that the choice is not unique, but some choices are better than others.

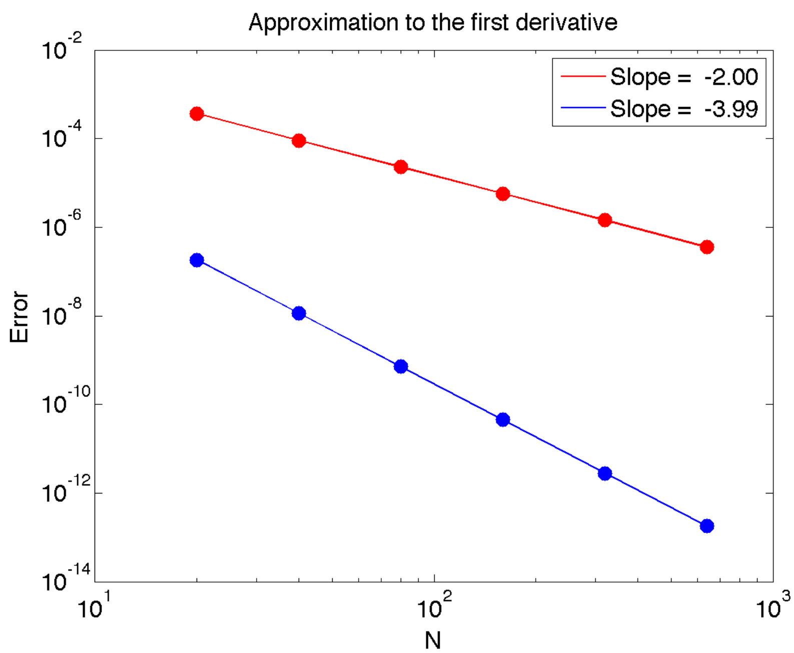

Figure 1: Plot of the second and fourth order approximations in the error in the first derivative.

(a) Compute second and fourth order approximations to the first and second derivatives of the function $u(x) = \sin(x)$ at the point $\bar{x} = 0.5$ on a set of equally spaced points $x_j = jh$, $h = 1/N$. For each derivative, construct a log-log plot of the error in your second and fourth order approximations verses N for a range of N values, $N = 20, 40, \ldots, 640$. Show that the results you get confirm second order and fourth order accuracy by including in your plots a best fit line to the error, and showing that the slope of these lines is close to -2 or -4. That is, confirm that your error $e(N)$ satisfies

$$e(N) \approx CN^{-p}$$

See Figure 1 for a sample of the kind of plot you should get.

(b) Demonstrate that your function returns accurate results even for randomly distributed points. To construct an array of random points in $[0,1]$, use

```
x = [0 sort(rand(1,N-1)) 1];
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To show accuracy for the random distribution, you may want to include more N values. For example, you could consider using

```
nvec = round(logspace(log10(20),log10(640),20);
```

(c) Use what you found above to argue that even on a random grid, the finite difference approximation to the second derivative is still *consistent*.

2. Another way to approximate the second derivative is to use the divergence theorem (or, in 1d, the Fundamental Theorem of Calculus) to approximate $u''(x_{j+1/2})$ at *cell centered points* $x_{j+1/2}$. This approximation is given by

$$u''(x_{j+1/2}) \approx \frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}} u''(x)\,dx = \frac{u'(x_{j+1})-u'(x_j)}{x_{j+1}-x_j}, \qquad x_{j+1/2} = \frac{x_j + x_{j+1}}{2}$$

where x_j are the usual finite difference nodes, and $x_{j+1/2}$ is the point halfway between x_j and x_{j+1}. If we approximate the first derivative as

$$u'(x_j) \approx \frac{u(x_{j+1/2}) - u(x_{j-1/2})}{x_{j+1/2} - x_{j-1/2}}$$

we obtain the discretization

$$D(u(x_{j+1/2})) = \frac{1}{x_{j+1}-x_j}\left[\frac{u(x_{j+3/2}) - u(x_{j+1/2})}{x_{j+3/2} - x_{j+1/2}} - \frac{u(x_{j+1/2}) - u(x_{j-1/2})}{x_{j+1/2} - x_{j-1/2}}\right].$$

(a) Show that if we assume a uniform grid, i.e. $x_j = jh$, $h = 1/N$, the stencil you obtain for the approximation to $u''(x_{j+1/2})$ at the cell centers $x_{j+1/2}$ is essentially the same second order approximation we obtained in class for $u''(x_j)$.

(b) Show that on a random grid, however, this approximation is no longer consistent.

(c) Numerically approximate the truncation error $\tau(x_{j+1/2})$ by

$$\tau(x_{j+1/2}) \approx \left|u''(x_{j+1/2}) - D(u(x_{j+1/2}))\right|$$

using the function $u(x) = \sin(x)$. At the left and right endpoints, use the exact value of the derivative, i.e. $u'(x_{-1/2}) = u'(0)$ and $u'(x_{N+1/2}) = u'(1)$. Plot the 1-norm truncation error verses N on both the uniform grid and the random grid. The 1-norm definition you should use is given by

$$\|\tau(x)\|_1 = \int_0^1 \tau(x)\,dx \approx \sum_{j=0}^{N-1} \tau(x_{j+1/2})h_j, \qquad h_j = x_{j+1} - x_j.$$

You should see that while $\tau(x)$ on the uniform grid converges nicely to 0, the error on the random grid does not.

This approximation is called a "finite volume" approximation, since we are using (2) to approximate the *average* value of $u''(x_{j+1/2})$ over the grid cell $[x_j, x_{j+1}]$. The surprising fact is that even though the discretization is not formally consistent, we can still use it to obtain second order approximations to the PDE $u''(x) = f$. On the other hand, the discretization of the system of equations given by

$$\int_{x_j}^{x_{j+1}} u''(x)\,dx = u'(x_{j+1}) - u'(x_j)$$

is formally consistent.

3. The following are questions about the material in Section 2.11 of your textbook.

   (a) Verify that Equation 2.41 satisfies the PDE $u''(x) = f(x)$ on $(0, 1)$, subject to Dirichlet boundary conditions $u(0) = \alpha$, $u(1) = \beta$.

   (b) Using 2.46, numerically construct the matrix B and show how B is related to the matrix A from equation 2.43.

   (c) Numerically confirm the main result of this section, namely, that

   $$\| B \|_{\infty} < 3$$

   from page 28. How does this help us show stability in the ∞-norm?