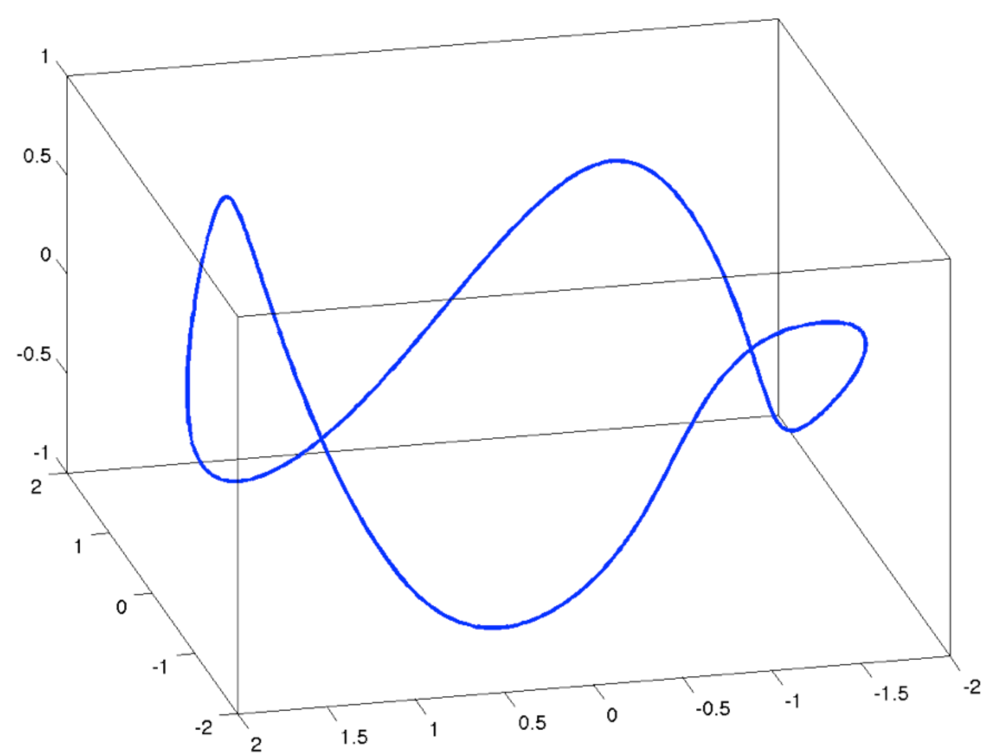
Motion and Geometry in 3d

Parameterized curves in 3d

We can represent curves in three dimensional space with the parameterization

$$\mathbf{r}(\alpha) = (x(\alpha), y(\alpha), z(\alpha))$$

where α is a monotonically increasing parameter. Examples include *distance* along the curve, time traveled along the curve, or some other parameter.



Tangent and normal vectors

We can compute *tangent* and *normal* vectors using the following simple formulas:

$$\mathbf{T}(\alpha) = \frac{\mathbf{r}'(\alpha)}{g(\alpha)}$$

Unit tangent vector

and

$$\mathbf{N}(\alpha) = \frac{g(\alpha)\mathbf{r}''(\alpha) - g'(\alpha)\mathbf{r}'(\alpha)}{g(\alpha)^3\kappa(\alpha)}$$

Unit normal vector

where

$$\begin{aligned} \mathbf{r}'(\alpha) &= (x'(\alpha), y'(\alpha), z'(\alpha)) \\ \mathbf{r}''(\alpha) &= (x''(\alpha), y''(\alpha), z''(\alpha)) \end{aligned}$$

and

$$g(\alpha) = \|\mathbf{r}'(\alpha)\| \quad \text{and} \quad \kappa(\alpha) = \frac{\|\mathbf{r}''(\alpha) \times \mathbf{r}'(\alpha)\|}{g(\alpha)^3}$$

Tangent and normal vectors

We compute 2-norm $\|\mathbf{r}\|$ for $\mathbf{r} = (r_1, r_2, r_3)$ as

$$\|\mathbf{r}\| = \sqrt{r_1^2 + r_2^2 + r_3^2}$$

The 2-norm is the length of vector r.

The cross product $\mathbf{r} \times \mathbf{s}$ is computed as

$$\mathbf{r} \times \mathbf{s} = (r_2 s_3 - r_3 s_2, -(r_1 s_3 - r_3 s_1), r_1 s_2 - r_2 s_1)$$

The cross product is perpendicular to both r and s.

The 2-norm is a *scalar* whereas the cross product results in a *vector*.

In Matlab, we can use **`norm(r,2)`** and **`cross(r,s)`**.

Vector operations

One additional operator that is useful is the *dot* product $\mathbf{r} \cdot \mathbf{s}$, defined as

$$\mathbf{r} \cdot \mathbf{s} = r_1 s_1 + r_2 s_2 + r_3 s_3$$

The 2-norm can be written in terms of the dot product as

$$\|\mathbf{r}\| = \sqrt{\mathbf{r} \cdot \mathbf{r}}$$

We also have

$$\mathbf{r} \cdot (\mathbf{r} \times \mathbf{s}) = \mathbf{s} \cdot (\mathbf{r} \times \mathbf{s}) = 0$$

The cross product is perpendicular to both r and s.

In Matlab, we can use **`dot(r,s)`**.

Example

$$x(\theta) = \cos(2\theta)(\cos(5\theta) + 3)$$
$$y(\theta) = \sin(2\theta)(\cos(5\theta) + 3)$$
$$z(\theta) = \sin(5\theta), \qquad 0 \leq \theta \leq 2\pi$$

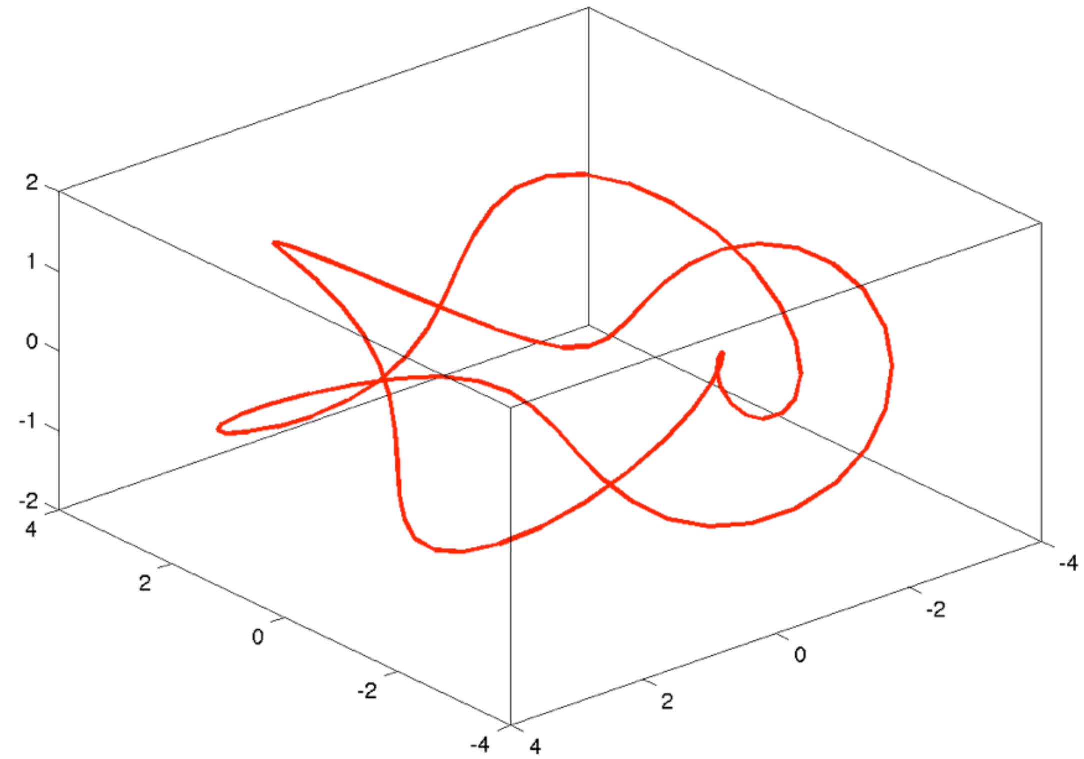
We have

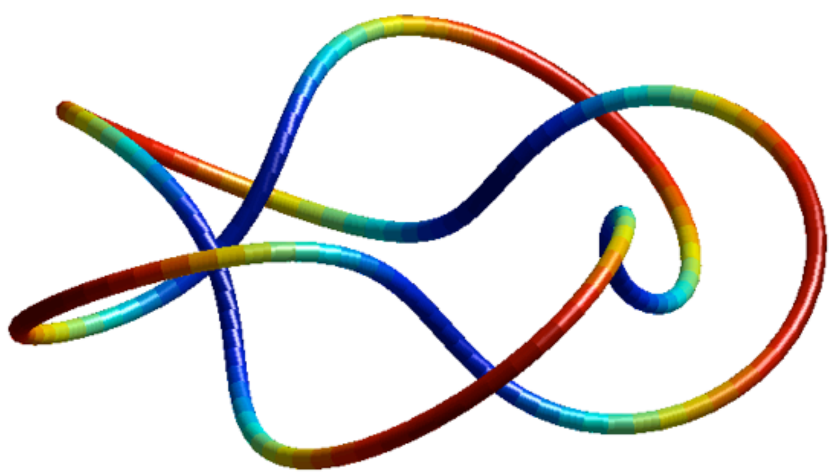
$$\mathbf{r}(\theta) = (x(\theta), y(\theta), z(\theta))$$
$$\mathbf{r}'(\theta) = (x'(\theta), y'(\theta), z'(\theta))$$

and

$$\mathbf{r}''(\theta) = (x''(\theta), y''(\theta), z''(\theta))$$

A Trefoil knot





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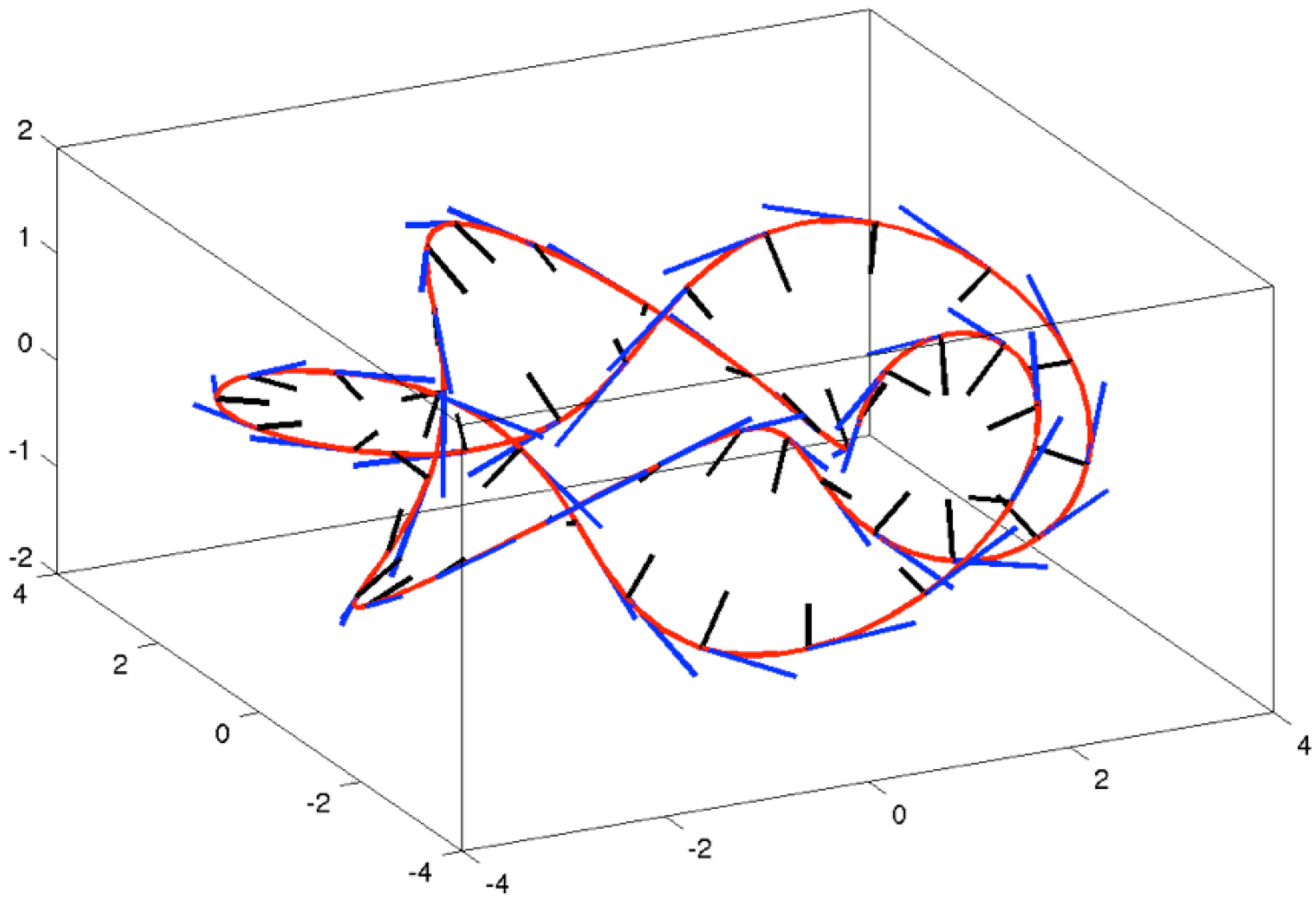
Trefoil knot with tangent and normals



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